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Whitehead Aspherical Conjecture via Ribbon Sphere-Link

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Abstract
Whitehead aspherical conjecture says that every connected subcomplex of every aspherical 2-complex is aspherical. By an argument on ribbon sphere-links, it is confirmed that the conjecture is true for every contractible finite 2-complex. In this paper, by generalizing this argument, this conjecture is confirmed to be true for every aspherical 2-complex.

Keywords: Infinite Ribbon Sphere-Link, Aspherical 2-Complex, Contractible 2-Complex, Whitehead Aspherical Conjecture

1. Introduction

A *finite* or *infinite* 2-complex is a finite or countably-infinite CW 2-complex constructed from a connected finite or countably-infinite graph by attaching a finite or at most countably-infinite system of 2-cells with attaching maps, respectively. A 2-complex is homotopy equivalent to a simplicial 2-complex constructed from a simplicial graph by attaching 2-cells with simplicial approximations of the attaching maps. By this homotopy equivalence, every subcomplex of a 2-complex is also homotopy equivalent to a simplicial subcomplex of the simplicial 2-complex [1]. A path-connected space X is *aspherical* if the universal cover $\tilde{X}$ of X is contractible (i.e., homotopy equivalent to a point). In particular, a connected 2-complex P is aspherical if and only if the second homotopy group $\pi_2(P, v) = 0$. The Whitehead asphericity conjecture is the following conjecture [2,3].

Conjecture 1. Every connected subcomplex of any aspherical 2-complex is aspherical.

The purpose of this paper is to show that Conjecture 1 is yes. That is,

Theorem 1.1. Whitehead Aspherical Conjecture is true.

A 2-complex P is *locally finite* if every 1-cell of P attaches only to a finite number of 2-cells of P. Conjecture 1 reduces to the following conjecture for every contractible locally finite 2-complex.

Conjecture 2. Every connected subcomplex of every contractible locally finite 2-complex is aspherical.

In Section 2, the claim of Conjecture 2 ⇒ Conjecture 1 is shown. Conjecture 2 for every contractible finite 2-complex is confirmed [4]. In this paper, the argument for an infinite 2-complex becomes the main argument. The *2-complex* of a group presentation

$$GP = < x_1, x_2, \dots, x_n, \dots \,|\, r_1, r_2, \dots, r_m, \cdots >$$

is the connected 2-complex constructed from a graph with fundamental group isomorphic to the free group $< x_1, x_2, \dots, x_n, \cdots >$ on the generators x_i ($i = 1, 2, \dots, n, \dots$) by attaching 2-cells with attaching maps given by the relators r_j ($j = 1, 2, \dots, m, \dots$), where note that this 2-complex is a connected graph for the empty relator. Up to cellular homotopy equivalences, every connected 2-complex P and the connected subcomplexes of P can be uniquely considered as the 2-complex and the subcomplexes of a group presentation GP, where a subcomplex of GP is the 2-complex of the group presentation of a sub-presentation

$$< x_{i_1}, x_{i_2}, \dots, x_{i_s}, \dots \,|\, r_{j_1}, r_{j_2}, \dots, r_{j_t}, \cdots > .$$

A group presentation GP is *locally finite* if every generator x_i appears only in a finite number of the relators r_j ($j = 1, 2, \ldots, m, \ldots$). The 2-complex of a locally finite group presentation GP can be taken as a connected locally finite 2-complex. A group presentation GP is a *homology-trivial unit-group presentation* if GP is a presentation of the unit group $\{1\}$ and the relator word r_j is equal to the generator x_j for every j in the abelianized free abelian group $< x_1, x_2, \ldots x_n, \cdots >^a$ of the free group $< x_1, x_2, \ldots x_n, \cdots >$ with as basis the generators x_i ($i = 1, 2, \ldots, n, \ldots$) of GP. Note that the 2-complex of a homology-trivial unit-group presentation is always contractible.

In Section 3, Conjecture 2 reduces to the following conjecture for the 2-complex of every homology-trivial unit-group presentation.

Conjecture 3. Every subcomplex of every homology-trivial unit-group presentation is aspherical.

The claim that Conjecture 3 ⇒ Conjecture 2 is shown there. For this purpose, after observations on base changes of a free group and a free abelian group of possibly infinite ranks, it is shown that if the 2-complex of a locally finite group presentation GP is contractible, then there is an iteration of base changes x'_i ($i = 1, 2, \ldots, n, \ldots$) in the free group $< x_1, x_2, \ldots, x_n, \cdots >$ with as basis the generators x_i($i = 1, 2, \ldots, n, \ldots$) of GP so that the resulting group presentation

$$GP' =< x'_1, x'_2, \ldots, x'_n, \ldots \,|\, r'_1, r'_2, \ldots, r'_n\ , \cdots >$$

is a homology-trivial unit-group presentation (see Lemma 3.2). This means that there is a cellular-homotopy equivalence from every contractible locally finite 2-complex P to the 2-complex P' of a homology-trivial unit-group presentation GP' inducing a cellular-homotopy equivalence from the subcomplexes of P to the subcomplexes of P' (see Corollary 3.3).

In Section 4, a (possibly infinite) sphere-link (namely, an S^2-link) L in the 4-space $\mathbf{R}^4$ is discussed. The closed complement of L in $\mathbf{R}^4$ is denoted by $E(L)$. It is shown there that for every homology-trivial unit-group presentation GP, a ribbon S^2-link L in $\mathbf{R}^4$ is constructed so that the fundamental group $\pi_1(E(L), v)$ is isomorphic to the free group $< x_1, x_2, \ldots, x_n, \cdots >$ of the generators x_i ($i = 1, 2, \ldots, n, \ldots$) of GP by an isomorphism sending a meridian system of L in $\pi_1(E(L), v)$ to the relator word system r_i ($i = 1, 2, \ldots, n, \ldots$) of GP (see Lemma 4.1). It is also observed there that a ribbon S^2-link L in $\mathbf{R}^4$ contains canonically a ribbon disk-link L^D in the upper-half 4-space $\mathbf{H}^4$ so that the fundamental group $\pi_1(E(L), v)$ is canonically identified with the fundamental group $\pi_1(E(L^D), v)$ for the closed exterior $E(L^D)$ of L^D in $\mathbf{H}^4$ (see Lemma 4.2 and Corollary 4.3).

In Section 5, it is shown that $E(L^D)$ is always aspherical and every 1-full subcomplex P' of the 2-complex P of a homology-trivial unit-group presentation GP is homotopy equivalent to the closed exterior $E(L^D)$ of a ribbon disk-link L^D in $\mathbf{H}^4$, where a 1-full subcomplex P' of P is a subcomplex of P containing the 1-skelton P^1 of P. Then Conjecture 3 is confirmed to be true and the proof of Theorem 1.1 is completed.

The author mentions here that there is a preprint by Pasku claiming the same result, which is a purely group-theoretic argument much different from the current argument [5].

2. Reducing to the Conjecture for a Contractible Locally finite 2-Complex

In this section, it is explained that Conjecture 1 (Whitehead Asphericity Conjecture) is obtained from the following conjecture.

Conjecture 2

Every connected subcomplex of every contractible locally finite 2-complex is aspherical.

For this reduction, the following three lemmas are used.

Lemma 2.1

If every connected finite subcomplex of a contractible 2-complex P is aspherical, then every connected subcomplex of P is aspherical.

Lemma 2.2

If every connected subcomplex of every contractible 2-complex is aspherical, then every connected subcomplex Q of every aspherical 2-complex P is aspherical.

Lemma 2.3

Every connected finite subcomplex of a connected infinite 2-complex P is a subcomplex of a connected locally finite 2-complex P′ homotopy equivalent to P.

Proof of Lemma 2.1 is done as follows.

Proof of Lemma 2.1
Let Q be any given connected subcomplex of a contractible 2-complex P. Let $f : S^2 \to |Q|$ be a map from the 2-sphere S^2 to the polyhedron |Q|. For a simplicial 2-complex Q, the topology of $|Q|$ is the topology coherent with the simplexes of Q so that the image $f(S)$ is in the polyhedron Q of a connected finite subcomplex Q^f of Q p.111 [1]. By assumption, Q^f is aspherical, so that the map $f : S^2 \to |Q^f|$ defined by the original map f is null-homotopic in $|Q^f|$ and hence in $|Q|$, so that Q is aspherical.

Proof of Lemma 2.2 is done as follows.

Proof of Lemma 2.2
Let P be an aspherical 2-complex, and Q any connected subcomplex of P. Since the universal cover $\tilde{P}$ of P is a contractible 2-complex, the subcomplex Q lifts to a subcomplex $\tilde{Q}$ of the contractible 2-complex $\tilde{P}$ and any connected component $\tilde{Q}_1$ of the subcomplex $\tilde{Q}$ is aspherical by assumption. Since the second homotopy group is independent of a covering by the lifting property, Q is aspherical [1].

Proof of Lemma 2.3 is done as follows.

Proof of Lemma 2.3. Let P be a connected infinite 2-complex, and P_0 any given connected finite subcomplex of P. Let

$$P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n \subset \ldots$$

be a sequence of connected finite subcomplexes P_i ($i = 0, 1, 2, \ldots, n, \ldots$) of P such that $P = \cup_{i=0}^{+\infty} P_i$. Let $P_i = P_{i-1} \cup J_i$ for a subcomplex J_i of P_i with $\gamma_i = P_{i-1} \cap J_i$ a graph for all i. Triangulate the rectangle $a \times [0, 1]$ for every 1-simplex a of γ_i by introducing a diagonal and regard the product $\gamma_i \times [0, 1]$ as a 2-complex. To construct a desired 2-complex P', make the finite subcomplexes J_i ($i = 1, 2, 3, \ldots, n, \ldots$) disjoint. Let $P'_i = P_{i-1} \cup \gamma_i \times [0, 1]$ be the 2-complex obtained from the subcomplexes P_{i-1} and $\gamma_i \times [0, 1]$ by identifying γ_i in P_{i-1} with $\gamma_i \times 0$ and $\gamma_i \times 1$ with γ_i in J_i in canonical ways. The sequence

$$P_0 = P'_0 \subset P'_1 \subset P'_2 \subset \cdots \subset P'_n \subset \ldots$$

of connected finite subcomplexes P'_i ($i = 0, 1, 2, \ldots, n, \ldots$) is obtained. By construction $P' = \cup_{i=0}^{\infty} P'_i$ is a connected locally finite 2-complex containing P_0 as a subcomplex and homotopy equivalent to P.

Conjecture 1 is obtained from Conjecture 2 as follows.

2.4: Proof of Conjecture 2 ⇒ Conjecture 1. By assuming Conjecture 2, it suffices to show that every connected finite subcomplex Q of every contractible 2-complex P is aspherical. Because this claim means by Lemma 2.1 that every connected subcomplex of every contractible 2-complex P is aspherical, which also means by Lemma 2.2 that every connected subcomplex of every aspherical 2-complex is aspherical, confirming Conjecture 1. If Q is a connected finite subcomplex of a contractible 2-complex P, then Q is a subcomplex of a contractible locally finite 2-complex P' homotopy equivalent to P by Lemma 2.3, so that Q is aspherical by Conjecture 2. This completes the proof of Conjecture 2 ⇒ Conjecture 1.

3. Reducing to the Conjecture for the 2-Complex of a Homology-Trivial Unit-Group Presentation
In this section, it is explained that Conjecture 2 is obtained from the following conjecture.

Conjecture 3. Every subcomplex of every homology-trivial unit-group presentation is aspherical.

A *base change* of a free group **F** with basis x_i ($i = 1, 2, \ldots, n, \ldots$) is a consequence of a finite number of the following operations, called *Nielsen transformations* [6].

(1) Exchange two of x_i ($i = 1, 2, \ldots, n, \ldots$),
(2) Replace an x_i by x_i^{-1} ,
(3) Replace an x_i by $x_i x_j$ ($i \neq j$).

A *base change* of a free abelian group **A** on a basis a_i ($i = 1, 2, \ldots, n, \ldots$) is a consequence of a finite number of the following operations:

(1) Exchange two of a_i ($i = 1, 2, \ldots, n, \ldots$),
(2) Replace an a_i by $-a_i$,
(3) Replace an a_i by $a_i + a_j$ ($i \neq j$).

The following lemma is well-known for a finite rank free abelian group **A**.

Lemma 3.1. Let **A** be a free abelian group with a countable basis a_i ($i = 1, 2, \ldots, n, \ldots$). Let b_i ($i = 1, 2, \ldots, n, \ldots$) be a countable basis of A such that every column vector and every row vector of the base change matrix C given by

$$(b_1 b_2 \ldots b_n \ldots) = (a_1 a_2 \ldots a_n \ldots)C$$

have only a finite number of non-zero entries. Then there is a base change of **A** on a_i ($i = 1, 2, \ldots, n, \ldots$) such that C is the block sum $(1) \oplus C'$ for a matrix C'.

Proof of Lemma 3.1. For every j ($j = 1, 2, \ldots, n, \ldots$), let

$$b_j = c_{1j}a_1 + c_{2j}a_2 + \cdots + c_{nj}a_n + \ldots$$

be a linear combination with (i, j) entries c_{ij} of C which are 0 except for a finite number of i ($i = 1, 2, \ldots, n, \ldots$). Note that for every j, the non-zero integer system of $c_{1j}, c_{2j}, \ldots, c_{nj}, \ldots$ is a coprime integer system. By base changes (1) and (2), assume that c_{11} is the smallest positive integer in the integers $|c_{i1}|$ for all i. For $i > 1$, write $c_{i1} = c_{i1}c_{11} + d_{i1}$ for $0 \le d_{i1} < c_{11}$. By a base change on a_i ($i = 1, 2, \ldots, n, \ldots$), assume that

$$b_1 = c_{11}a_1 + d_{21}a_2 + \cdots + d_{n1}a_n + \ldots.$$

By continuing this process, it can be assumed that $b_1 = a_1$. Note that there is a positive integer $m \ge 2$ such that $c_{1j} = 0$ for all $j > m$. Consider the linear combination

$$b_2 = c_{12}a_1 + c_{22}a_2 + \cdots + c_{n2}a_n + \ldots.$$

Note that the non-zero integer system of $c_{22}, c_{32}, \ldots, c_{n2}, \ldots$ is coprime. Otherwise, there is a prime common divisor $p > 1$, so that b_1 and b_2 would be $\mathbf{Z}_p$-linearly dependent in the $\mathbf{Z}_p$-vector space $\mathbf{A}\otimes\mathbf{Z}_p$ which contradicts that b_i ($i = 1, 2, \ldots, n, \ldots$) form a basis of $\mathbf{A}\otimes\mathbf{Z}_p$, where $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$. By a base change on a_i ($i = 2, 3, \ldots, n, \ldots$), it can be assumed that $b_2 = c_{12}a_1 + a_2$. By an inductive argument, it can be assumed that

$$b_j = c_{1j}a_1 + c_{2j}a_2 + \cdots + c_{j-1\,j}a_{j-1} + a_j \; (j = 3, 4, \ldots, m).$$

By a base change replacing a_j to $a_j - c_{1j}a_1 - c_{2j}a_2 - \cdots - c_{j-1\,j}a_{j-1}$ ($j = 2, 3, \ldots, m$), the identities $b_j = a_j$ ($1 \le j \le m$) are obtained. Then the entries c_{ij} of the matrix C are written as

$$c_{11} = 1 \quad c_{1j} = c_{i1} = 0 \, (1 < i < +\infty, 1 < j < +\infty).$$

This completes the proof of Lemma 3.1.

The proof of the following lemma uses Lemma 3.1.

Lemma 3.2. If the 2-complex P of a locally finite group presentation

$$GP =< x_1, x_2, \ldots, x_n, \ldots \,|\, r_1, r_2, \ldots, r_m, \cdots >$$

is contractible, then there is a basis x'_i (i=1,2,...) of the free group $<x_1,x_2,\ldots,x_n,\ldots>$ obtained from the basis x_i (i=1,2,...) by an iteration of base changes such that the resulting group presentation

$$GP' =< x'_1, x'_2, \ldots, x'_n, \ldots \,|\, r'_1, r'_2, \ldots, r'_n, \cdots >$$

is a homology-trivial unit-group presentation.

Proof of Lemma 3.2. Since the 2-complex P is a contractible locally finite 2-complex, every generator x_i appears only in a finite number of the relators $r_1, r_2, \ldots, r_m, \ldots$ and the inclusion homomorphism

$$< r_1, r_2, \ldots, r_m, \cdots > \to < x_1, x_2, \ldots x_n, \cdots >$$

on the free groups $< r_1, r_2, \ldots, r_m, \cdots >$ and $< x_1, x_2, \ldots x_n, \cdots >$ induces an isomorphism on the abelianized groups $< r_1, r_2, \ldots, r_m, \cdots >^a$ and $\mathbf{A} =< x_1, x_2, \ldots x_n, \cdots >^a$ which are free abelian groups with a base change matrix C given in Lemma 3.1.

Do Nielsen transformations on the free group $< x_1, x_2, \ldots x_n, \cdots >$ induced from base changes on the free abelian group **A** of Lemma 3.1. Then the word r_1 is changed into x_1 in **A**. This base change is done by using only finitely many letters in x_i ($i = 1, 2, \ldots, n, \ldots$) belonging to the word r_j except for re-indexing of the letters x_i ($i = 1, 2, \ldots, n, \ldots$). By continuing this process, the conclusion of Lemma 3.2 is obtained.

The following corollary means that a contractible locally finite 2-complex may be considered as the 2-complex of a homology-trivial unit-group presentation.

Corollary 3.3. There is a cellular-homotopy equivalence from every contractible locally finite 2-complex P to the 2-complex P' of a homology-trivial unit-group presentation GP' inducing a cellular-homotopy equivalence from the connected subcomplexes of P to the subcomplexes of GP'.

Proof of Corollary 3.3. Let P be a contractible locally finite 2-complex obtained from the 1-skelton P^1 with $\pi_1(P^1, v) =< x_1, x_2, \ldots, x_n, \cdots >$ for a basis x_i ($i = 1, 2, \ldots, x_n, \ldots$) by attaching 2-cells with attaching maps given by relators r_j ($j = 1, 2, \ldots, m, \ldots$). Then the inclusion homomorphism

$$< r_1, r_2, \ldots, r_m, \cdots > \to < x_1, x_2, \ldots x_n, \cdots >$$

induces an isomorphism from the abelianized group $< r_1, r_2, \ldots, r_m, \cdots >^a$ to the abelianized group $< x_1, x_2, \ldots, x_n, \cdots >^a$. Let

$$g :< x_1, x_2, \ldots, x_n, \cdots > \to < x'_1, x'_2, \ldots, x'_n, \cdots >$$

be an isomorphism given by Lemma 3.2, sending the word r_j to a word r'_j such that r'_j is equal to x'_j in the abelianized group $< x'_1, x'_2, \ldots x'_n, \cdots >^a$ for all j. Let P' be the 2-complex of the homology-trivial unit-group presentation

$$GP' =< x'_1, x'_2, \ldots, x'_n, \ldots \,|\, r'_1, r'_2, \ldots, r'_n, \cdots > .$$

The isomorphism g induces a desired cellular homotopy equivalence $P \to P'$.

Conjecture 2 is obtained from Conjecture 3 as follows.

3.4 Proof of Conjecture 3 ⇒ Conjecture 2

By Corollary 3.3, every connected subcomplex of every contractible locally finite 2-complex is homotopy equivalent to a subcomplex of a homology-trivial unit-group presentation. This completes the proof of Conjecture 3 ⇒ Conjecture 2.

4. A Ribbon Sphere-Link and a Ribbon Disk-Link Constructed from a Homology-Trivial Unit-Group Presentation

Let X be an open connected oriented smooth 4D manifold. A countably-infinite system of disjoint compact sets X_i ($i = 1, 2, \ldots, n, \ldots$) in X is *discrete* if the set $\{p_i| \, i = 1, 2, \ldots, n, \ldots\}$ constructed from any one point $p_i \in X_i$ for every i is a discrete set in X. An S^2*-link* in X is the union L of a discrete (finite or countably-infinite) system of disjoint 2-spheres smoothly embedded in X. An S^2*-link* in X is *trivial* if it bounds a discrete system of mutually disjoint 3-balls smoothly embedded in X, and *ribbon* if it is obtained from a trivial S^2-link O by surgery along a discrete system of disjoint 1-handles embedded in X. An S^2*-link* L in X is *finite* if the number of the components of L is finite. Otherwise, L is *infinite*. Let $\mathbf{R}^4 = \{(x,y,z,t)|-\infty<x,y,x,t<+\infty\}$ be the 4-space, and $\mathbf{H}^4 = \{(x,y,z,t)| -\infty<x,y,x<+\infty, 0\leq t\}$ the upper-half 4-space of $\mathbf{R}^4$ with boundary $\partial\mathbf{H}^4= \{(x,y,z,0)|-\infty<x,y,z<+\infty\}$ identified with the 3-space $\mathbf{R}^3=\{(x,y,z)|-\infty<x,y,x<+\infty\}$. The *open 4D handlebody*

$$Y^O = \mathbf{R}^4 \#_{i=1}^{+\infty} S^1 \times S^3_i$$

denotes the connected sum of the 4-space $\mathbf{R}^4$ and a discrete system of $S^1 \times S^3_i$ ($i = 1, 2, \ldots, n, \ldots$). The following lemma is basic to our purpose.

Lemma 4.1. For every homology-trivial unit-group presentation

$$GP =< x_1, x_2, \ldots, x_n, \ldots \,|\, r_1, r_2, \ldots, r_n, \cdots >,$$

there is a ribbon S^2-link L with components K_i ($i = 1, 2, \ldots, n, \ldots$) in $\mathbf{R}^4$ such that there is an isomorphism

$$\pi_1(E(L), v) \to < x_1, x_2, \ldots x_n, \cdots >$$

sending a meridian system of K_i ($i = 1, 2, \ldots, n, \ldots$) to the relator system r_i ($i = 1, 2, \ldots, n, \ldots$).

Proof of Lemma 4.1. In the open 4D handlebody $Y^O = \mathbf{R}^4 \#_{i=1}^{+\infty} S^1 \times S^3_i$, let γ^O be a legged loop system with loop system k_i^O

$= S^1 \times 1_i$ ($i = 1, 2, \ldots, n, \ldots$) representing a basis x_i ($i = 1, 2, \ldots, n, \ldots$) of the free group $\pi_1(Y^O, v)$. Let k_j ($j = 1, 2, \ldots, n, \ldots$) be a simple loop system k_* in Y^O representing the relator system r_j ($j = 1, 2, \ldots, n, \ldots$). By assumption of the homology-trivial unit-group presentation GP, the loop k_j for every j meets transversely $1 \times S^3_i$ in Y^O with intersection number +1 for $j = i$ and with intersection number 0 for $j \neq i$. Further, the loop k_j does not meet $1 \times S^3_i$ except for a finite number of i. Let X be the smooth open 4D manifold obtained from Y^O by surgery along the loops k_j ($j = 1, 2, \ldots, n, \ldots$) replacing a normal D^3-bundle $k_j \times D^3$ of k_j in Y^O with the D^2-bundle $D_j \times S^2$ of S^2 for a disk D_j with $\partial D_j = k_j$. Then the S^2-link $L = \cup_{j=1}^{+\infty} K_j$ with $K_j = 0_j \times S^2$ is obtained in X.

(4.1.1) The open 4D manifold X is contractible.

Proof of (4.1.1). By van Kampen theorem, X is simply connected because the loops k_j ($j = 1, 2, \ldots, n, \ldots$) normally generate the free fundamental group $\pi_1(Y^O, v) = < x_1, x_2, \ldots, x_n, \cdots >$. Thus, if $H_q(X;\mathbf{Z}) = 0$ ($q = 2, 3$), then X is contractible since X is an open 4D manifold. Since the loop system k_* meets transversely $1 \times S^3_i$ in a finite number of points in Y^O with intersection number $\mathrm{Int}(k_j, 1 \times S^3_i) = +1$ ($j = i$), 0 ($j \neq i$), there is an arc system Is ($s = 1, 2, \ldots, u$) in the 1D manifold system obtained from k_* by cutting along the set $k * \cap 1 \times S^3_i$ such that I_s attaches to $1 \times S^3_i$ with opposite signs for all s and the 3D orientable manifold Z_i obtained from $1 \times S^3_i$ by piping along I_s ($s = 1, 2, \ldots, u$) meets k_i with just one point and does not meet k_j ($i \neq j$). By the construction of X, the component K_i of L bounds a once-punctured 3D manifold V_i of Z_i in X not meeting $L \backslash K_i$, for every i. This means that the inclusion homomorphism $H_2(D_i \times S^2;\mathbf{Z}) \to H_2(X;\mathbf{Z})$ is the zero map for all i. Then

$$H_2(X;\mathbf{Z}) \cong H_2(X, D_* \times S^2;\mathbf{Z}) \cong H_2(Y^O, k_* \times D^3;\mathbf{Z}) = 0$$

by the homology long exact sequence of the pair $(X, D_* \times S^2)$ and the excision isomorphism theorem. On the other hand, by construction, the 3D manifolds V_i ($i = 1, 2, \ldots, n, \ldots$) represent a basis for $H_3(Y^O, k_* \times D^3;\mathbf{Z})$ and hence a basis for $H_3(X, D_* \times S^2;\mathbf{Z})$ by the excision isomorphism theorem. This means that the boundary homomorphism $\partial_* : H_3(X, D_* \times S^2;\mathbf{Z}) \to H_2(D_* \times S^2;\mathbf{Z})$ is an isomorphism. Since $H_3(D_* \times S^2;\mathbf{Z}) = 0$, the homology long exact sequence on the pair $(X, D_* \times S^2)$ shows that $H_3(X;\mathbf{Z}) = 0$. Thus, X is a contractible open 4D manifold. This completes the proof of (4.1.1).

The proof of Free Ribbon Lemma means that the 2-sphere component K_i of L is isotopic to a ribbon S^2-knot in X obtained from a finite trivial S^2-link O_i split from L by surgery along a finite disjoint 1-handle system $\mathbf{h}_i$ such that $\cup_{i=1}^{+\infty} O_i$ is a trivial link and $\mathbf{h}_i$ ($i = 1, 2, \ldots, n, \ldots$) are disjoint discrete systems. Thus, the S^2-link L is a ribbon S^2-link in X. By taking the upper-half 4-space $\mathbf{H}^4$ near the end of the connected summand $\mathbf{R}^4$ of Y^O, consider $\mathbf{H}^4$ in X so that $X \setminus \mathbf{H}^4$ is diffeomorphic to X. For a 4-space $\mathbf{R}^4$ in $\mathbf{H}^4$, the ribbon S^2-link L in X can be moved into $\mathbf{R}^4$, since L is obtained from a discrete trivial S^2-link which is movable into $\mathbf{R}^4$ by surgery along disjoint discrete 1-handle systems $\mathbf{h}_i$ ($i = 1, 2, \ldots, n, \ldots$) which are also movable into $\mathbf{R}^4$. By construction, there is an isomorphism from $\pi_1(X \setminus L, v) \cong \pi_1(\mathbf{R}^4 \setminus L, v)$ to the free fundamental group $\pi_1(Y^O, v) = < x_1, x_2, \ldots, x_n, \cdots >$ sending a meridian system of K_i ($i = 1, 2, \ldots, n, \ldots$) to the relator system r_i ($i = 1, 2, \ldots, n, \ldots$) of GP. This completes the proof of Lemma 4.1.

Let α be the reflection in $\mathbf{R}^4$ sending (x, y, z, t) to $(x, y, z, -t)$. The image $\alpha(\mathbf{H}^4)$ of the upper-half 4-space $\mathbf{H}^4$ by α is given by the lower-half 4-space

$$\{(x, y, z, t) | -\infty < x, y, z < +\infty,\ t \leq 0\}.$$

A *disk-link* L^D in $\mathbf{H}^4$ is a discrete (finite or countably-infinite) system of disjoint disks smoothly and properly embedded in $\mathbf{H}^4$. A (possibly infinite) disk-link L^D in H^4 is *trivial* if it is obtained from a discrete system of disjoint disks in $\mathbf{R}^3$ by pushing up the interiors of the disks into the interior of $\mathbf{H}^4$. A disk-link L^D in $\mathbf{H}^4$ is *ribbon* if it is obtained from a disjoint discrete embedded disk system $\mathbf{D} \cup \mathbf{b}$ in $\mathbf{H}^4$ which is the union of a trivial disk-link $\mathbf{D} = \{D_i | i = 1, 2, \ldots, n, \ldots\}$ in $\mathbf{H}^4$ and a disjoint spanning band system $\mathbf{b} = \{b_j | j = 1, 2, \ldots, m, \ldots\}$ on the trivial link $\partial\mathbf{D}$ in $\mathbf{R}^3$ by pushing up the interiors of the disk system $\mathbf{D} \cup \mathbf{b}$ into the interior of $\mathbf{H}^4$. Thus,

$$L^D = \tilde{\mathbf{D}} \cup \tilde{\mathbf{b}}$$

for a pushing up disk system $\tilde{\mathrm{D}} = \{\tilde{D}_i | i = 1, 2, \ldots, n, \ldots\}$ of $\mathbf{D}$ and a pushing up band system $\tilde{\mathbf{b}} = \{\tilde{b}_j | j = 1, 2, \ldots, m, \ldots\}$ of $\mathbf{b}$. The *closed exterior* of a ribbon disklink L^D in $\mathbf{H}^4$ is the 4D manifold $E(L^D) = \mathrm{cl}(\mathbf{H}^4 \setminus N(L^D))$ for a regular neighborhood of L^D in $\mathbf{H}^4$. Every ribbon S^2-link L in $\mathbf{R}^4$ is isotopically deformed into an α-invariant position for the reflection α in $\mathbf{R}^4$ so that L is obtained from a ribbon disk-link L^D in $\mathbf{H}^4$ by doubling of $\mathbf{H}^4$ by α II [7]. The following lemma is shown by the same method as Lemma 3.1 [4].

Lemma 4.2. For a ribbon disk-link L^D in $\mathbf{H}^4$ in a (possibly infinite) ribbon S^2-link L in $\mathbf{R}^4$, the inclusion $(\mathbf{H}^4, L^D) \to (\mathbf{R}^4, L)$ induces an isomorphism

$$\pi_1(E(L^D), v) \to \pi_1(E(L), v).$$

The following corollary is obtained from Lemmas 4.1 and 4.2.

Corollary 4.3. For every homology-trivial unit-group presentation

$$GP =< x_1, x_2, \ldots, x_n, \ldots | r_1, r_2, \ldots, r_n, \cdots >,$$

there is a ribbon disk-link L^D with components K_i^D (i = 1, 2, . . . , n, . . .) in $\mathbf{H}^4$ such that there is an isomorphism

$$\pi_1(E(L^D), v) \to < x_1, x_2, \ldots, x_n, \cdots >$$

sending a meridian system of K_i^D (i = 1, 2, . . . , n, . . .) to the relator system r_i (i = 1, 2, . . . , n, . . .).

5. A Ribbon Disk-Link Corresponding to a 1-Full Subcomplex of a Homologytrivial Unit-Group Presentation

A ribbon disk-link L^D in $\mathbf{H}^4$ is *free* if the fundamental group $\pi_1(E(L^D), v)$ is a free group. The following lemma contains an infinite version of the results of Theorem 1.4 and Lemma 3.2 [4].

Lemma 5.1. The closed exterior $E(L^D)$ of every (possibly infinite) ribbon disklink L^D in $\mathbf{H}^4$ is aspherical. In particular, for every (possibly infinite) free ribbon disk-link L^D in $\mathbf{H}^4$ with $\pi_1(E(L^D), v) \cong < x_1, x_2, \ldots, x_n, \cdots >$, there is a strong deformation retract $r : E(L^D) \to \omega x$ for a locally finite graph ωx with $\pi_1(\omega x, v) \cong < x_1, x_2, \ldots, x_n, \cdots >$.

Proof of Lemma 5.1. The proof is done for an infinite ribbon disk-link L^D in the upper-half 4-space $\mathbf{H}^4$, because the finite ribbon disk-link case is given in [4]. Let $L^D = \widetilde{\mathbf{D}} \cup \widetilde{\mathbf{b}}$. Divide the upper-half 4-space $\mathbf{H}^4$ along the upper-half 3-space

$$\mathbf{H}_0^3 = \{(x, y, 0, t)| -\infty < x, y < +\infty,\ 0 \leq t\}$$

into the 2-parts

$$\mathbf{H}_+^4 = \{(x, y, z, t)| -\infty < x, y < +\infty,\ 0 \leq z,\ 0 \leq t\}$$

and

$$\mathbf{H}_-^4 = \{(x, y, z, t)| -\infty < x, y < +\infty,\ z \leq 0,\ 0 \leq t\}.$$

Assume that the trivial disk-link $\mathbf{D} = \{D_i| i = 1, 2, \ldots, n, \ldots\}$ in $\mathbf{H}^4$ is disjoint from $\mathbf{H}_0^3$ and splits into two disk systems $\mathbf{D}_\pm$ so that $\mathbf{D}_+$ is a finite trivial disk-link in $\mathbf{H}_+^4$ and $\mathbf{D}_-$ is an infinite trivial disk-link in $\mathbf{H}_-^4$. Let $\widetilde{\mathbf{D}}_\pm$ be the pushing up disk systems of $\mathbf{D}_\pm$ into $\widetilde{\mathbf{D}}$. The spanning band system b in $\partial\mathbf{H}^4$ meets H_0^3 with a disjoint simple arc system consisting of an arc parallel to an arc attaching to the trivial link $\partial\mathbf{D}$. The band system $\mathbf{b}_\pm = \mathbf{b} \cap \mathbf{H}_\pm^4$ consists of a band system $\mathbf{b}_\pm^1$ of bands with no end or one end in $\mathbf{H}_0^3$ and a band system $\mathbf{b}_\pm^2$ of bands with both ends in $\mathbf{H}_0^3$. Let $\widetilde{\mathbf{b}}_\pm = \widetilde{\mathbf{b}}_\pm^1 \cup \widetilde{\mathbf{b}}_\pm^2$ be the pushing up band systems of $\mathbf{b}_\pm$. Note that the band system $\mathbf{b}_+ = \mathbf{b}_+^1 \cup \mathbf{b}_+^2$ is a finite band system. Let $f : S^q \to \mathrm{Int}\, E(L^D)$ be a map from the q-sphere S^q for $q \geq 2$. By a slide of the upper-half 3-space $\mathbf{H}_0^3$, it can be assumed that the image $f(S^q)$ is in the interior of $\mathbf{H}_+^4$ and does not meet $L_+^D = \widetilde{\mathbf{D}}_+ \cup \widetilde{\mathbf{b}}_+^1$ and $\widetilde{\mathbf{b}}_+^2$. Let $D(\mathbf{b}_+^2)$ be a trivial disk system in $\mathbf{H}_+^4 \setminus \mathbf{H}_0^3$ obtained from $\widetilde{\mathbf{b}}_+^2$ by sliding the attaching part of $\widetilde{\mathbf{b}}_+^2$ to $\mathbf{H}_0^3$ down along $\mathbf{H}_0^3$. From construction, $D(\mathbf{b}_+^2)$ is disjoint from L_+^D. Let $L_+^{Db} = L_+^D \cup D(\mathbf{b}_+^2)$. Let $\overline{L}_+^{Db}$ be a finite ribbon disk-link in $\mathbf{H}^4$ obtained from L_+^{Db} in $\mathbf{H}_+^4$ by taking the double along $\mathbf{H}_0^3$. Let $E(L_+^{Db}) = \mathrm{cl}(\mathbf{H}_+^4 \setminus N(L_+^{Db}))$ for a regular neighborhood $N(L_+^{Db})$ of L_+^{Db} in $\mathbf{H}_+^4$. Since $E(\overline{L}_+^{Db})$ is aspherical and there is a retraction $r : E(\overline{L}_+^{Db}) \to E(L_+^{Db})$, the inclusion $E(L_+^{Db}) \to E(\overline{L}_+^{Db})$ induces a monomorphism $\pi_q(E(L_+^{Db}), v) \to \pi_q(E(\overline{L}_+^{Db}), v)$ and the map $f : S^q \to \mathrm{Int}E(L_+^{Db})$ defined by $f : S^q \to \mathrm{Int}E(L^D)$ extends to a map $f^+ : D^{q+1} \to \mathrm{Int}E(L_+^{Db})$ from the (q+1)-disk D^{q+1} Lemma 3.2 [4]. The union $L_+^D \cup \widetilde{\mathbf{b}}_+^2$ is recovered from L_+^{Db} by a deformation keeping L_+^D and $f^+(D^{q+1})$ fixed. Thus, the image $f^+(D^{q+1})$ is in $\mathbf{H}_+^4 \setminus (L_+^D \cup \mathbf{b}_+^2)$ and hence in $E(L^D)$. This means that $E(L^D)$ is aspherical. Further, if $\pi_1(E(L^D), v)$ is isomorphic to $< x_1, x_2, \ldots, x_n, \cdots >$, then $E(L^D)$ is homotopy equivalent to ωx and there is a strong deformation retract $r : E(L^D) \to \omega x$. This completes the proof of Lemma 5.1.

For a free ribbon disk-link L^D in $\mathbf{H}^4$, let

$$Q(L^D) = E(L^D) \cup N(L^D)$$

be a decomposition of $\mathbf{H}^4$ into the closed complement $E(L^D)$ and the normal disk-bundle $N(L^D) = L^D \times D^2$. Let $p_*(L^D) = \{p_i | i = 1, 2, \ldots, n, \ldots\}$ be a discrete set made by taking one point from every component of L^D. The strong deformation retract $r : E(L^D) \to \omega x$ in Lemma 5.1 and the strong deformation retract $N(L^D) \to p_*(L^D) \times D^2$ shrinking L^D into $p_*(L^D)$ define a map $\rho : Q(L^D) \to P(L^D)$ for a connected locally finite 2-complex

$$P(L^D) = \omega x \cup p_*(L^D) \times D^2$$

with the attaching map $p_*(L^D) \times \partial D^2 \to \omega x$ defined by r. The map ρ is called a *ribbon disk-link presentation* for the 2-complex $P(L^D)$. For a sublink K^D of L^D, let $N(K^D) = K^D \times D^2$ be the subbundle of the disk-bundle $N(L^D)$. The union

$$Q(L^D, K^D) = E(L^D) \cup N(K^D)$$

is a decomposition of the closed complement $E(L^D \setminus K^D)$ of the sublink $L^D \setminus K^D$ of L^D in $\mathbf{H}^4$, which is a ribbon disk-link in $\mathbf{H}^4$. The ribbon disk-link presentation $\rho : Q(L^D) \to P(L^D)$ for $P(L^D)$ sends $Q(L^D, K^D)$ to the 1-full 2-subcomplex

$$P(L^D, K^D) = \omega x \cup p_*(K^D) \times D^2$$

of $P(L^D)$. Further, every 1-full 2-subcomplex of $P(L^D)$ is obtained from a sublink K^D of L^D in this way. The following theorem contains an infinite version of a ribbon disk-link L^D of Theorem 1.3 [4].

Theorem 5.2. For every free ribbon disk-link L^D in $\mathbf{H}^4$, the ribbon disk-link presentation $\rho : Q(L^D) \to P(L^D)$ induces a homotopy equivalence $Q(L^D, K^D) \to P(L^D, K^D)$ for every sublink K^D of L^D including $K^D = \emptyset$ and $K^D = L^D$. In particular, the 2-complex $P(L^D)$ is contractible. The 2-complex P of every homology-trivial unit-group presentation GP is taken as $P = P(L^D)$ for a free ribbon disk-link L^D in $\mathbf{H}^4$ so that for every 1-full subcomplex P' of P, there is just one sublink K^D of L^D with $P' = P(L^D, K^D)$.

Proof of Theorem 5.2. The homotopy equivalence of the ribbon disk-link presentation $\rho : Q(L^D) \to P(L^D)$ is similar to the proof of [2, Theorem 1.3]. Let $GP =< x_1, x_2, \dots, x_n, \dots \mid r_1, r_2, \dots, r_n, \cdots >$ be a homology-trivial unitgroup presentation By Corollary 4.3, there is a free ribbon disk-link L^D in $\mathbf{H}^4$ with an isomorphism $\pi_1(E(L^D), v) \cong < x_1, x_2, \dots, x_n, \cdots >$ sending a meridian system of L^D to the relator system r_i $(i = 1, 2, \dots, n, \dots)$. The 2-complexes P of GP and $P(L^D)$ are both obtained from the same graph ωx with $\pi_1(\omega x, v) =< x_1, x_2, \dots x_n, \cdots >$ by attaching 2-cells with attaching maps given by the relator words r_i $(i = 1, 2, \dots, n, \dots)$. Hence the 1-full subcomplexes of P coincide with the 1-full subcomplexes of $P(L^D)$.

The following corollary confirms that Conjecture 3 is true.

Corollary 5.3. Every subcomplex of every homology-trivial unit-group presentation is aspherical.

Proof of Corollary 5.3. Let P be the 2-complex of every homology-trivial unitgroup presentation, and P' a connected subcomplex of P. By Theorem 5.2, P is written as $P(L^D)$ for a free ribbon disk-link L^D in $\mathbf{H}^4$. If P' is a 1-full subcomplex of P, then P' is written as $P(L^D, K^D)$ for a sublink K^D of L^D in $\mathbf{H}^4$. The ribbon disk-link presentation $\rho : Q(L^D, K^D) \to P(L^D, K^D)$ is homotopy equivalent and $Q(L^D, K^D)$ is the closed exterior $E(L^D \setminus K^D)$, which is aspherical by Lemma 5.1. Thus, P' is aspherical. If P' is not 1-full, then a 1-full subcomplex P'' of P is constructed from P' by adding some loops in the 1-skelton $P^1 = \omega x$ to P', and P'' is aspherical if and only if P' is aspherical. Thus, P' is aspherical in this case. This completes the proof of Corollary 5.3.

The proof of Theorem 1.1 is now completed as follows.

Proof of Theorem 1.1. The proof of Theorem 1.1 is completed by Corollary 5.3 (a confirmation of Conjecture 3) and the proofs of Conjecture 3 ⇒ Conjecture 2 and Conjecture 2 ⇒ Conjecture 1. This completes the proof of Theorem 1.1.

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