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Vector-Valued Convex Functions

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Abstract
In this paper, we will extend some properties of the convex real functions to the valued functions in a Banach lattice: with adequate definitions, we will establish that an order convex function is continuous on a convex C if and only if it is continuous at a point of C (Theorem 1.2). We will show that order convex functions on a compact satisfy Bauer's maximal principle (Theorem 2.2). A fixed-point theorem is given for the contracting orders functions (Theorem 2.3).

Keywords: Order, Riesz Space, Banach Lattice, Convexity and Ordre Convexity.

1. Order Convexity of Vector-Valued Functions

Often in functional analysis, one needs local algebraic linearity. Thus, one of the interactions of the algebraic and topological structure of a topological vector is manifested in the important properties of the class of convex functions. So far, we have allowed the convex functions defined on the convex subsets of a vector space to be real valued. We will extend the definition of convexity to the valued functions in a Banach lattice [1-4].

Definition 1.1
Let E be a Banach lattice. A function f: C → E on a convex set C in a vector space X is:
(i) order convex (denoted by o-convex) if for all $x, y \in C$ *and all* $0 \leqslant \alpha \leqslant 1,\ f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha) f(y)$.

(ii) Strictly o-convex if for all x,y ∈ C with x 6= y and all 0 < α < 1, f (αx + (1 − α)y) < αf(x) + (1 − α)f(y).

(iii) O-concave (respectively, strictly o-concave) if −f is an o-convex (respectively, strictly o-concave) function.

It is easy to realize that; *f* is o-convex if and only if,

$$f\left(\sum_{k=1}^{n} \alpha_k x_k\right) \leqslant \sum_{k=1}^{n} \alpha_k f(x_k)$$

For every convex combination $\sum_{k=1}^{n} \alpha_k x_k$.

Example 1.1 Here are some familiar examples of o-convex mappings.
- Obviously, any convex real function is o-convex.
- Let *E* be a Banach lattice. The absolute value $x \longrightarrow |x|$ is an o-convex mapping from *E* to *E*.
- Let *A* be a commutative unital real Banach algebra. The set of all multiplicative linear functionals on *A* is denoted by Δ_A. It is well known that Δ_A, endowed with the the Gelfand topology, is compact and the Gelfand representation φ of A into $C(\Delta_A)$ is an homomorphism [3, Theorem 13]. Thus, φ is o-convex.

Proposition 1.1 *A function f: C → E on a convex subset of a vector space into a Banach lattice E is o-convex if and only if its epigraph, epi (f) = {(x,χ) ∈ C × E : χ > f(x)}, is convex. Similarly, f is o-concave if and only if its hypograph, {(x,χ) ∈ C × E : χ f(x)} , is convex.*

Proof. We prove the first part of this proposition, the remaining assertion is identical. Suppose that *f* is o-convex, then for (x_1, χ_1), (x_2, χ_2) ∈ epi*f* and $\alpha \in [0, 1]$ we have

$$\begin{aligned} \alpha_1 \chi_1 + (1-\alpha)\chi_2 &\geqslant \alpha_1 f(x_1) + (1-\alpha) f(x_2) \\ &\geqslant f(\alpha x_1 + (1-\alpha) x_2) \end{aligned}$$

So, $(\alpha_1 \chi_1 + (1-\alpha)\chi_2, \alpha_1 x_1 + (1-\alpha) x_2)$ ∈epi*f*. The 'only if' part stems from the fact that $(x_1, f(x_1))$ ∈ epi*f* and $(x_2, f(x_2))$ ∈ epi*f*.

Proposition 1.2 *The collection of o-convex functions on a fixed convex set C into a Banach lattice E has the following properties.*

1. Sums and nonnegative scalar multiples of o-convex functions are o-convex.
2. The (finite) pointwise order limit of a net of o-convex functions is o-convex.
3. The (finite) pointwise supremum of a family of o-convex functions is o-convex.

Proof. The first statement is trivial. For the second assertion, consider a net $\{f_i\}$ of o-convex functions (finite) pointwise order convergent to f, that is, for any finite part F of C, there is a net $\{\chi_i\}$ (with the same directed set) satisfying $\chi_i \downarrow 0$ and $|f_i(z) - f(z)|\ \chi_i$ for each i and every z F. Let $x,y \in C$ and $\alpha \in [0,1]$. For $F = \{x,y,\alpha x + (1-\alpha)y\}$ we have:

$$\begin{aligned}(1-\alpha)y) &\leqslant \chi_i + f_i\left(\alpha x + (1-\alpha)y\right)\\ &\leqslant \chi_i + \alpha f_i\left(x\right) + (1-\alpha)f_i\left(y\right)\\ &\leqslant \chi_i + \alpha\left[f\left(x\right) + f_i\left(x\right) - f\left(x\right)\right] + (1-\alpha)\left[f\left(y\right) + f_i\left(y\right) - f\left(y\right)\right]\\ &\leqslant \chi_i + \alpha f(x) + (1-\alpha)f(y) + \chi_i\\ &\leqslant \left[2\chi_i + \alpha f(x) + (1-\alpha)f(y)\right]\downarrow \alpha f(x) + (1-\alpha)f(y)\end{aligned}$$

So, f is o-convex.
Now, let $f_1,f_2,...,$fn are o-convex functions on a convex set C into a Banach lattice E. For all $x \in C$, we ask $f(x) = \bigvee_{1\leqslant k\leqslant n} f_k(x)$. It is easy to see that:

$$\begin{aligned}f\left(\alpha x + (1-\alpha)y\right) &= \bigvee_{1\leqslant k\leqslant n} f_k\left(\alpha x + (1-\alpha)y\right)\\ &\leqslant \bigvee_{1\leqslant k\leqslant n}\left[\alpha f_k(x) + (1-\alpha)f_k(y)\right]\\ &\leqslant \bigvee_{1\leqslant k\leqslant n}\left[\alpha f_k(x)\right] + \bigvee_{1\leqslant k\leqslant n}\left[(1-\alpha)f_k(y)\right]\\ &\leqslant \alpha\bigvee_{1\leqslant k\leqslant n} f_k(x) + (1-\alpha)\bigvee_{1\leqslant k\leqslant n} f_k(y)\\ &\leqslant \alpha f(x) + (1-\alpha)f(y).\end{aligned}$$

So, f is o-convex, what completes the proof.

Proposition 1.3 *Let f : C ! E be an o-convex function, where C is a convex subset of a vector space and E is a Banach lattice. Let x belong to C and suppose z satises x + z ∈ C and x ∈ z C. Let α ∈ [0; 1]: Then*

$$|f\left(x+\alpha z\right) - f(x)| \leqslant \alpha\left(\left[f\left(x+z\right) - f(x)\right] \vee \left[f\left(x-z\right) - f(x)\right]\right)$$

Proof. $f(x+\alpha z) \leqslant (1-\alpha)f(x) + \alpha f(x+z)$ because the hypothesis and the equality: $x + \alpha z = (1-\alpha)x + \alpha(x+z)$. Rearranging terms yields

$$f\left(x+\alpha z\right) - f\left(x\right) \leqslant \alpha\left[f(x+z) - f(x)\right] \quad (1.1)$$

$$\leqslant \alpha\left(\left[f\left(x+z\right) - f(x)\right] \vee \left[f\left(x-z\right) - f(x)\right]\right) \quad (1.2)$$

Replacing z by −z in (1.1) gives

$$f\left(x-\alpha z\right) - f\left(x\right) \leqslant \alpha\left[f(x-z) - f(x)\right] \quad (1.3)$$

Since $x = \frac{1}{2}(x+\alpha z) + \frac{1}{2}(x-\alpha z)$,

we have $f(x) \leqslant \frac{1}{2}f(x+\alpha z) + \frac{1}{2}f(x-\alpha z)$.

Multiplying by two and rearranging terms we obtain

$$f(x) - f(x+\alpha z) \leqslant f(x-\alpha z) - f(x) \quad (1.4)$$

(1.3) implies

$$\begin{aligned}f(x) - f(x+\alpha z) &\leqslant f(x-\alpha z) - f(x)\\ &\leqslant \alpha\left[f(x-z) - f(x)\right]\\ &\leqslant \alpha\left(\left[f\left(x+z\right) - f(x)\right] \vee \left[f\left(x-z\right) - f(x)\right]\right) \quad (1.5)\end{aligned}$$

With definition of the absolute value in mind, (1.2) in conjunction with (1.5) yields the conclusion of the proposition.

Recall that a subset A of a Riesz space X is order bounded, from above if there is a vector u (called an upper bound of A) that dominates each element of A, that is, satisfying $a \leq u$ for each $a \in A$. Sets order bounded from below are defined similarly. A box or an order interval, is any set of the form

$$[a,b] = \{x \in X : a \leqslant x \leqslant b\}$$

Definition 1.2 *A mapping $f : X \to E$ between Riesz spaces is o-bounded above (respectively, o-bounded) on a subset V of X, if f(V) is order bounded from above (respectively, if f(V) ⊂ [a,b] for some box [a,b] of E).*

We would have liked an o-convex function $f : X \to E$ to be order continuous, but this is not true even in the trivial case when E = R. Indeed, let $X = C\,[0,1]$, we emphasize: There is no nonzero σ–order continuous linear functional on the Riesz space X. However for the topological continuity we have the following, which generalizes a similar result previously proved for the convex (real) functions.

Theorem 1.1 *Let f : C → E be an o-convex function, where C is a convex subset of a normed space X, and E is a Banach lattice. If f is,o-bounded above on a neighborhood of an interior point of C, then f is continuous at that point.*

Proof. We may assume that for some $x \in C$ there exist an open ball V of radius η at 0 and some $\chi \in E$ satisfying $x + V \subset C$ and $f(y) < f(x) + \chi$ for each $y \in x+V$. Fix $\varepsilon > 0$ and choose some $0 < \alpha < 1$ so that $\alpha||\chi|| < \varepsilon$. From Proposition1.3, it follows that for each $y \in x+\alpha V$ we have $|f(y)-f(x)| < \alpha\chi$.

Now, the norm of E is lattice, then $||f(y) - f(x)|| < \alpha||\chi|| < \varepsilon$.

Remark 1.1 We shall say that a vector $e \in E$ is an order unit, if for each $x \in E$ there exists $\lambda > 0$ such that $x < \lambda e$. It is well known that, in an ordered topological vector space E, a positive vector e is an interior point of the cone $E+$ if and only if, the order interval $[-e,e]$ is a neighborhood of zero [2, Lemma 2.5]. From the fact that every neighborhood of zero is absorbing set, it follows that the interior points of E_+ are order units.

Provided that the interior $Int(E_+)$ of the cone E_+ is non-empty, semicontinuity can be generalized to vector functions as follows.

Definition 1.3 *A mapping $f : X \to E$ from a topological space X into a Banach lattice E is:*

* *Lower o-semicontinuous if for each $c \in E$ the set*
  $\{x \in X : f(x) - c \in Int(E_+)\}$
is open.
* *Upper o-semicontinuous if for each $c \in E$ the set*
  $\{X \in X: c - f(x) \in Int (E+)\}$
is open.

Obviously, a mapping *f* is lower o-semi continuous if and only $-f$ is upper
O-semi continuous, and vice versa.
The classic example of a lower (resp. upper) o-semi continuous mapping is given by the lower (resp. upper) semi continuous real functions. Now assume that E is a Banach lattice with an order unit e. It is well known that the principal ideal E_e generated by e coincides with E which when provided with the norm $\|x\|_\infty = \inf\{\lambda > 0: |x| < \lambda e\}$ becomes an AMspace with unit. Let $f: R \to R$ be a continuous real function. Then the mapping $f: E \to E$ defined by $fe(x) = f(\|x\|_\infty)$ e is a lower and upper o-semi continuous mapping.

The E-valued mapping on a Banach lattice is a useful device, but it needs to be handled with care. For example, the complement of the set $\{x \in X: f(x) > c\}$ in X is not at all the set $\{x \in X: f(x) < c\}$. However, the following lemma reduces this difficulty by reducing us to functions with real values.

Proposition 1.4 *If E_+ (respectively, E'_+) is the positive cone of a Banach lattice E (respectively, of E^0), then $x \,/\!\in E+$ if and only if it exists ‘$\in E'+$ such that $\phi(x) < 0$.*

Proof. The definition of the positive cone E'_+ gives a sense of lemma. Conversely, if $x \,/\!\in E+$, since E_+ is closed and convex, it follows from the HahnBanach theorem that there is *a* $\phi \in E'$ with $\phi(x) < \phi(y)$ for all $y \in E_+$.

Thus $\phi(x) < 0 = f(0)$ and $\phi(x) < \phi\ (ny)$ for all non-negative integer number n. So $\phi(x) < 0\ \phi(y)$ for all $y \in E+$.

As a first application of the above definitions, we have the following result.

Theorem 1.2 *For an o-convex mapping $f: C \to E$ on an open convex subset of a topological vector space X into a Banach lattice E, the following statements are equivalent.*
1. f is continuous on C.
2. f is upper o-semicontinuous on C.
3. f is o-bounded above on a neighborhood of each point in C.
4. f is o-bounded above on a neighborhood of some point in C.
5. f is continuous at some point in C.

Proof. (1) ⇒ (2) is obvious.
(2) ⇒ (3); Assume that f is upper o-semicontinuous. Let $x \in C$ and $a \in Int(E_+) - \{0\}$. Then the set $\{y \in E: f(x) + a - f(y) \in Int(E_+)\}$ is an open neighborhood of x on which f is o-bounded above.
(3) ⇒ (4) Obvious.
(4) ⇒ (5) This is Theorem 1.1.
(5) ⇒ (1) Suppose f is continuous at the point x, and let y be any other point in C. Since C is open and convex, therefore C does not contain extreme points. This implies that there exist $z \in C$ and $0 < \lambda < 1$ such that $y = \lambda x+(1-\lambda)z$. Fix $\varepsilon > 0$ and choose some circled neighborhood V of zero so that for all $v \in V$. We claim that, $\|f(y) - f(y+v)\| < \varepsilon$ for all $v \in \lambda V$. Indeed, let $v \in V$, Then $y + \lambda v = \lambda(x + v) + (1 - \lambda)z \in C$ and the o-convexity of f implies

$$\begin{aligned} f(y+\lambda v) &= f(\lambda(x+v)+(1-\lambda)z) \\ &\leqslant \lambda f(x+v)+(1-\lambda)f(z) \end{aligned}$$

and

$$\begin{aligned} f(y) &= f(\lambda x+(1-\lambda)z) \\ &\leqslant \lambda f(x)+(1-\lambda)f(z) \end{aligned}$$

Thus

$$\begin{aligned} f(y+\lambda v)-f(y) &\leqslant \lambda(f((x+v)-f(x))) \\ &\leqslant \lambda|f((x+v)-f(x))| \end{aligned} \qquad (1.6)$$

and

$$\begin{aligned} f(y)-f(y+\lambda v) &\leqslant \lambda(f(x)-f((x+v))) \\ &\leqslant \lambda|f((x+v)-f(x))| \end{aligned} \qquad (1.7)$$

This shows that

$$|f(y)-f(y+\lambda v)| \leqslant \lambda|f((x+v)-f(x))| \qquad (1.8)$$

Then

$$\begin{aligned} \|f(y)-f(y+\lambda v)\| &\leqslant \lambda\|f((x+v)-f(x))\| \\ &< \varepsilon \end{aligned}$$

So, f is continuous at y.

2. Order Lipschitzian Vector-Valued Functions

Lipchitzian and contractive real functions have important properties that we want to extend to infinite dimensional analysis. For this purpose, we adopt the following definition.

Definition 2.1 *A mapping f from a subset B of a normed tvs $(X, \|\|)$ to a Banach lattice E is order Lipschitz continuous on B if there exists $e \in E_+$ such that for every $y,z \in B$*

$$|f(y) - f(z)|\ 6\ \|y - z\|e$$

If moreover $\|e\| < 1$ then f is called an order contraction.

The following gives examples of order Lipschitz continuous mappings.

Theorem 2.1 Let *f: $C \to E$ be o-convex positive mapping from a convex subset C of a normed tvs $(X, |||\cdot|)$ to a Banach lattice E. If f is continuous at the interior point x of C, then f is order Lipschitz continuous on a neighborhood of x. That is, there exists $\delta > 0$ and $e \in E_+$, such that $B_\delta(x) \subset C$ and for $y,z \in B_\delta(x)$, we have*

$$|f(y) - f(z)| \leqslant ||y - z||e$$

Proof. Since f is continuous at x, it follows from Theorem1.3 that there exists $e \in E+$ and $\delta > 0$ satisfying $B_{2\delta}(x) \subset C$ and $f(y) \leqslant e$. So, $w,z \in B_2\delta(x)$ implies $0 \leqslant f(w) \leqslant e$ and $-e \leqslant -f(z) \leqslant 0$. By addition, we achieve $|f(w) - f(z)| \leqslant e$, for all w,z $B_{2\delta}$. Let $y,z \in B_{2\delta}(x)$ and $\alpha = \|y-z\|$. Then $w = y + \frac{\delta}{\alpha}(y+z)$ belongs to $B_{2\delta}$ and we have $y = \frac{\alpha}{\alpha+\delta}w + \frac{\delta}{\alpha+\delta}z$.

Therefore

$$f(y) \leqslant \frac{\alpha}{\alpha+\delta}f(w) + \frac{\delta}{\alpha+\delta}f(z)$$

Subtracting $f(z)$ from each side gives

$$\begin{aligned} f(y) - f(z) &\leqslant \frac{\alpha}{\alpha+\delta}[f(w) - f(z)] \\ &\leqslant \frac{\alpha}{\alpha+\delta}e \\ &\leqslant \alpha e \end{aligned}$$

Switching the roles of y and z allows us to conclude

$$|f(y) - f(z)| \leqslant ||y - z||e$$

A net $\{x_\alpha\}$ in a Riesz space E is order convergent to some $x \in E$, written $\{x\alpha\} \to^o x$, if there is a net $\{q_\alpha\}$ (with the same directed set) satisfying $\{Q_\alpha\} \downarrow 0$ and $|x\alpha - x|$ 6 q_α for each α. A function f: $E \to F$ between two Riesz spaces is order uniformly continuous if $\{y_\alpha - z_\alpha\} \to^o 0$ in E implies $\{f(y_\alpha) - f(z_\alpha)\} \to^o 0$ in F.

Proposition 2.1 *Let $f;X \to E$ be an order Lipschitz continuous mapping between banach latties. If the lattice norm on X is order continuous then f is order uniformly continuous.*

Proof. Obvious.

Now we will generalize, to convex order applications, one of the important themes of the analysis, namely the extreme points of a convex functions on a compact convex set. Let C be a convex subset of a vector space X. Recall that an extreme subset of C, is a nonempty subset F of C with the property that if x belongs to F it cannot be written as a convex combination of points of C outside F. A point x is an extreme point of C if the singleton $\{x\}$ is an extreme set.

Proposition 2.2 *Let $f : C \to E$ be o-convex mapping from a convex subset C of a normed tvs $(X,|||\cdot|)$ to a Banach lattice E. The set of maximizers of f is either an extreme set or is empty.*

Proof. Suppose f achieves a maximum on C; that is, f satisfies the identitie $\sup\{f(x) : x \in C\} = f(e)$ for some $e \in C$. Put $M = \{x \in C : f(x) = f(e)\}$. Suppose that $x = \alpha y + (1 - \alpha)z \in M, 0 < \alpha < 1$ and $y,z \in C$. If $y \,/\!\in M$ then $f(y) < f(e)$, so

$$\begin{aligned} f(e) = f(x) &= f(\alpha y + (1-\alpha)z) \\ &\leqslant \alpha f(y) + (1-\alpha)f(z) \\ &< \alpha f(e) + (1-\alpha)f(e) = f(e) \end{aligned}$$

a contradiction. Hence $y,z \in M$, so M is an extreme subset of C.

Recall that the order $\leqslant$ of a Banach lattice E is continuous if $\leqslant$ is a closed subset of $E \times E$. Let us say that $\leqslant$ is upper semicontinuous if $\{x \in E : y \leqslant x\}$ is closed for each y.

Theorem 2.2 Let $f : K \to E$ be a continuous vector-valued function from a compact space into a Banach lattice with a continuous order. Suppose that $f(K)$ satisfies the condition (C): $c \vee d$ belongs to $f(K)$ for all $c,d \in f(K)$, then f attains a minimum value, and the nonempty set of minimizers is compact. Similarly, a continuous vector-valued function on a compact set attains a maximum value, provided that $f(K)$ satisfies the condition (C'): $c \wedge d$ belongs to $f(K)$ for all $c,d \in f(K)$ and the nonempty set of maximizers is compact.

Proof. Let K be a compact of a normed vector space X and let $f : K \to E$ be a continuous mapping from K to a Banach lattice E. For each $c \in f(K)$, put $Fc = \{x \in K : f(x) > c\}$. It follows from the continuity of f and of the order that the nonempty set F_c is closed ($c = f(x)$ implis $x \in F_c$). Moreover, the family $F = \{Fc : c \in f(K)\}$ has the finite intersection property. In deed, let $Fc_1, Fc_2,...F_{cn}$ be a finite familly in F. Since $F(K)$ satisfies condition *(C)*

so, $c_0 = \bigvee_{i=1}^{i=n} c_i \in f(K)$. For all $x \in F_{c_0}$ and $1 \leqslant i \leqslant n$ we have

$$0 \leqslant f(x) - c_0 \leqslant f(x) - c_i$$

so, $x \in \bigcap_{1\leqslant i\leqslant n} F_{c_i}$ and $F_{c_0} \subset \bigcap_{1\leqslant i\leqslant n} F_{c_i}$. Since K is compact, [Aliprantis,

Theorem.2.31] implies that the set of minimizers $\bigcap_{c\in f(K)} F_c$ is compact and Nonempty.

We realize that, in its real context, the assumptions (C) and (C') in Theorem 2.2 are ensured from the fact that the order in R is total.

A complete lattice is a lattice in which every nonempty subset that is order bounded from above has a supremum. (Equivalently, if every nonempty subset that is bounded from below has an infimum).

Now consider a vector form of the Contraction Mapping Theorem.

Theorem 2.3 *Let B be a closed subset of a Banach Lattice E and* *let $f: B \to B$ be an order contraction mapping. Then f has a unique fixed point x. Moreover, for any choice $x0$ in X, the sequence defined recursively by $xn+1 = f(x_n)$, $n = 0,1,2,...$, converges to the fixed point and $||x_n - x|| < ||e||^n||x_0 - x||$ for each n.*

Proof. Let $e \in E_+$ such that $||e|| < 1$ and $|f(y)-f(z)| \leqslant ||y - z||e$, for all $y, z \in B$. If $f(x) = x$ and $f(y) = y$ then $|x-y| = |f(x)-f(x)|| \leqslant ||x-y||e$. Since the norm of E is lattice, we have $||x - y|| \leqslant ||x - y||||e||$ and hence $||x - y|| = 0$. Thus f can have at most one fixed point.

Now, if x_0 is chosen in B then the formula $x_{n+1} = f(xn)$, $n = 0,1,2,...$ defines inductively the sequence (x_n) which satisfies: $|xn+1 - xn| \leqslant ||xn - xn-1||e$, for every $n > 1$. The lattice property verified by the norm of E implies that $||xn+1 - xn|| \leqslant ||xn - xn-1||||e||$ and by induction, we see that for all $n > 1$, $||xn+1 - xn|| \leqslant ||x1 - x0||||e||n$. Hence, for $n > m$ the triangle inequality yields.

$$\begin{aligned} ||x_m - x_n|| &\leqslant \sum_{k=m+1}^{n} ||x_k - x_{k-1}|| \\ &\leqslant ||x_1 - x_0|| \sum_{k=m+1}^{n} ||e||^k \\ &\leqslant ||x_1 - x_0|| \frac{||e||^m}{1 - ||e||} \end{aligned}$$

This implies that (x_n) is a Cauchy sequence.

Since B is closed in the complete space E then, $(x_n) \to x \in B$. obviously, f is continuous, and:

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x),$$

so x is the fixed point of f.

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