ISSN: 2996 - 2498

Research Article

Environmental Science and Climate Research

Validation of Advanced Kinematic Model of Earth-Moon System by Observed Length of Day Curve

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Submitted: 2023, Dec 19; **Accepted:** 2024, Jan 10 ;**Published:** 2024, Mar 23

Citation: Sharma, B. K. (2024). Validation of Advanced Kinematic Model of Earth-Moon system by observed Length of Day Curve. *Env Sci Climate Res, 2*(1), 01-21.

Abstract
NASA's press release of 'Moon having receded by 1 m from Earth in a quarter of century' on the Silver Jubilee Anniversary (20 July 1994) of Man's landing on Moon led to the development of the Kinematic Model(KM) of evolving Earth-Moon System. Best fit KM parameters are adopted for the analysis of evolving E-M system, The present length of day of Earth is assumed to be 24 hours, orbital period of Moon divided by spin period of Earth=LOM/LOD=27.322 and the age of Moon=4.467Gy. Using this best fit model parameters the velocity of recession of Moon is derived as 2.3cm/y as compared to 3.7cm/y by Lunar Laser Ranging experiment.KM is used to plot lengthening of day(LOD) curve from pre-cambrian period (2.5Gy ago) to the present time. This theoretical plot is superimposed on the observed lengthening of day curve. The observed plot is generated from the observed length of day in different geologic epochs by John West Wells, Charles P.Sonnet & Chan and Kaula & Harris by the study of coral fossils, ancient tidalites and marine creatures respectively. The superposition gives more than 25% mismatch in remote past around 2.5Gya. This KM did not consider the obliquity angle 23.44° of Earth and the inclination angle of 5.14° of Lunar Orbital Plane with respect to Ecliptic plane. The classical KM assumed monotonic spiral expansion of Moon from Earth as shown in Figure 1. A recent paper "Tidal evolution of Moon in high angular momentum, high obliquity Earth" has revolutionized the Earth-Moon modeling and paved the way for near-match between observated and theoretical LOD curve. In the new scenario Moon's tidal evolution is anything but sedate and monotonic. It experienced a bumpy launch through gravitational sling shot at the first geo-synchronous orbit of 15,000Km. Moon was gravitationally catapulted and launched on an expanding spiral path but this expansion was turbulent, chaotic and got stalled at times due to huge lunar obliquity tides during Laplace Plane transition and due to Cassini state transition as shown in Figure 2. Only after Moon settles down in Cassini state II that it resumes its monotonic and sedate spiral expansion as envisaged in the classical picture but this spiral expansion was at accelerated pace so as to reach 3,84,499Km orbital radius in 4.5Gy. It transits in fits , in chaotic and turbulent phase, for 3.3Gy whereas it spends1.2Gy in bounded state therefore this new model is referred to as 'Fits and Bound' model of E-M system. On this time scale the modern time recession rate comes to be 3.7cm/y. If this accelerated time frame is adopted then we get near-exact match between theory and observed LOD curve with worst mis-match being -4%. In this accelerated time frame, all E-M system parameters are obtained and it is self consistent model with velocity of recession = 3.7cm/y, LOM/LOD = 27.322 and near match between theory and observed LOD curve and spending 3.3Gy in turbulent and stalled evolutionary phase and spending 1.2Gy in the present accelerated phase. This gives the final vindication to the advanced KM.

Keywords: Monotonic Spiral Expansion, Stalled Tidal Evolution, Laplace-Plane Transition, Cassini State Transition, Observed Lod Curve, Theoretical Lod Curve.

1. Introduction

In 1994 at the Silver Jubilee Anniversary of Man's landing on Moon, NASA issued a Press Release stating that Moon had receded by 1 meter in a quarter of a century from 20th July 1969 to 20th July 1994. This enabled me to first calculate the length of day curve and compare it with observed LOD curve [1]. This analysis was revised and presented as "Lengthening of Day curve could be experiencing chaotic fluctuations with implications for Earth-Quake Prediction" at the World Space Congrss -2002 , Houston,Texas,USA. [2]. In the classical Earth-Moon Giant impact scenario, through a glancing angle impact of newly-formed Earth by a Mars-sized planetesimal a circum-terrestrial disc of impact generated material is created which is a mix of the impactor material(Mars-sized planetesimal) and the target material.(newly formed Earth). Beyond Roche's Limit = $3R_E$ = 18,000Km which is greater than first geo-synchronous orbit aG1 = 15,000Km, full size Moon accretes and is catapulted by gravitational sling shot on a monotonically expanding spiral orbit as shown in Figure 1 [3].

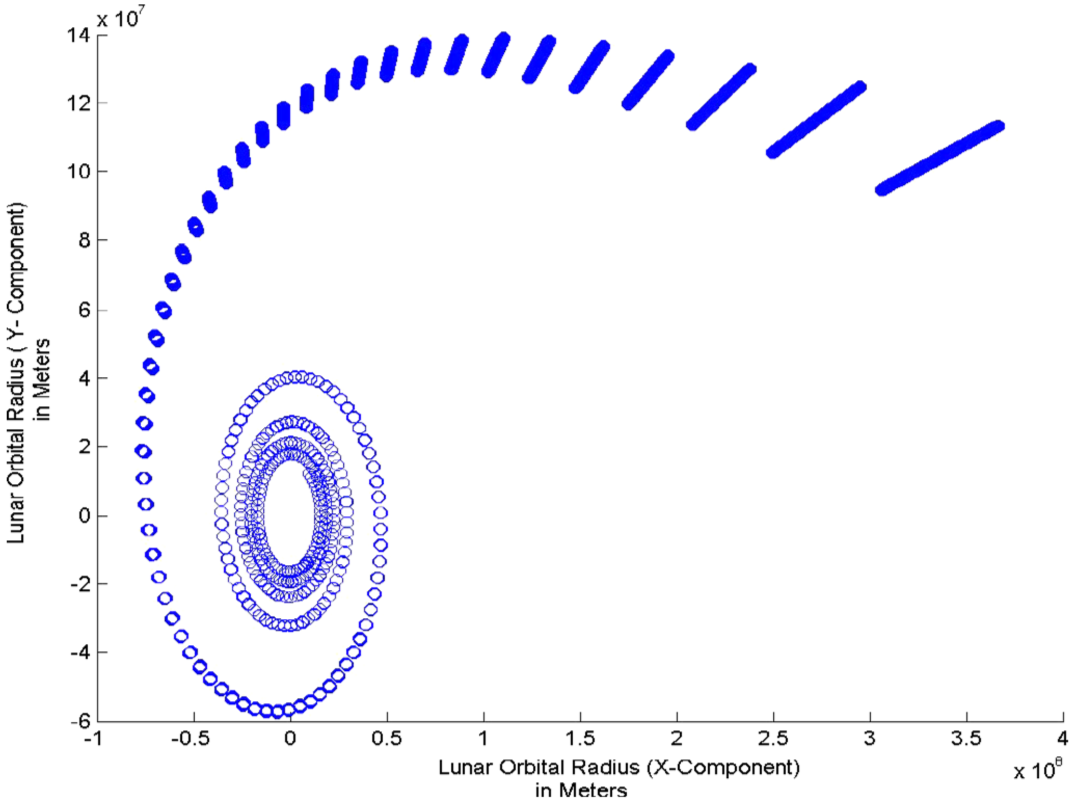

Figure 1: Lunar Orbital Radius outward expanding spiral trajectory obtained from the simulation for the age of Moon (i.e. from the time of Giant Impact to the present times covering a time span of 4.53Gyrs).

1.2. Deduction of LOM/LOD = ω/Ω equation [4].

$$J_T = C\omega + \left(m^* a^2_{present} + I\right)\Omega = [C + (m^* a^2_{G1} + I)]\Omega_{aG1} = [C + (m^* a^2_{G2} + I)]\Omega_{aG2} \qquad 4$$

In (4):
C = Moment of Inertia of the Primary around its spin axis.
I= Moment of Inertia of the Secondary around its spin axis.
And m*= reduced mass of the secondary = m/(1+m/M) where m = the mass of the secondary and
M= mass of the primary.
From Kepler's Third Law:

$$\Omega_{aG1} = \frac{B}{a_{G1}^{\frac{3}{2}}} \ (first\ triple\ synchrony\ state\ where\ \Omega = \omega)$$

$$and\ \ \Omega_{aG2}(second\ triple\ synchrony\ state\ where\ also\ \Omega = \omega) = \frac{B}{a_{G2}^{\frac{3}{2}}}$$

$$where\ B = \sqrt{G(M+m)} \qquad 5$$

Substituting (5) in (4) we get:

$$J_T = C\omega + \left(m^* a_{present}^2 + I\right)\Omega = [C + (m^* a_{G1}^2 + I)]\frac{B}{a_{G1}^{3/2}}$$

$$= [C + (m^* a_{G2}^2 + I)]\frac{B}{a_{G2}^{3/2}} \qquad 6$$

The roots of (6) give inner geo-synchronous orbit(aG1) and outer geo-synchronous orbit(aG2).

Using Globe-Spin parameters of E-M system given in Appendix A, we get the following kinematic parameters:

Reduced mass of Moon=m/(1+m/M)=m*=7.25284×10^{22}Kg.

B=√(G(M+m))=2.00811×10^{7}m$^{3/2}$/s, G=6.67×10^{-11} m^3/(kg-s^2),

C = moment of inertia of Earth around its spin axis=0.33086MR^2= 8.0209×10^{37}Kg-m^2,

I=moment of inertia of Moon around its spin axis=0.394mRmoon2= 8.72791×10^{34}Kg-m^2,

Define θ_1= I/C = 0.00108815, θ_2= m*/C=9.04243×10^{-16} 1/m^2,

JT= total angular momentum of E-M system= 3.43584×10^{34}Kg-m^2/s,

Spin Period of Earth = 1 Solar Day =24 hours.

Sidereal Spin Period of Moon = 27.322 Solar Day,

Sidereal Orbital period of E-M system =27.3217 Solar Day,

Moon is in synchronous orbit i.e. it is tidally locked and shows the same face to Earth. Generally this synchronism property is true for all compact binaries. Because of this synchrony of our Moon and because of the circular path, our Moon experiences no stretching and squeezing and hence no tidal heating and hence no volcanic effect. 'Io', a moon of Jupiter, is similarly placed as our Moon is with respect to Earth but still Io is the most volcanically active natural satellite because of tidal dissipation. In synchronous orbit there should have been no volcanic activity but its eccentric orbit makes it the most volcanically active natural satellite in our solar system. Io is in 2:1 resonance with Europa hence it has eccentric orbit and hence it is evolving and gradually circularizing. As its eccentricity is reduced in the process of circularization, the volcanic activity is on the wane.

Solving (6) we obtain the two geo-synchronous orbits:

$$a_{G1} = 1.46402 \times 10^7 m, \qquad a_{G2} = 5.5247 \times 10^8 \qquad 8$$

According to, tidal flexing does not allow the solid particles to coalesce within Roche's limit represented by the symbol a_R.

$$a_R = 2.456\left(\frac{\rho_E}{\rho_M}\right)^{1/3} R_E \sim 2.9R_E = 18496.606\ Km\ where\ \rho_E = \frac{5.5gm}{cc}\ and\ \rho_M = \frac{3.34gm}{cc} \qquad 9$$

and Roche Zone is defined within the range:

$$0.8\ to\ 1.35a_R\ or\ 2.32\ to\ 3.915R_E\ i.e.within\ 14{,}797Km\ to\ 24{,}970Km$$

This implies that impact generated debris will be prevented from accretion within 1.48×10^7m and those in 1.48×10^7m to 2.5×10^7m range also known as transitional zone will experience limited accretion growth whereas those lying beyond this zone will be unaffected by tidal forces. It is a happy coincidence that the Roche zone lies just beyond the inner Geo-Synchronous orbit of the Earth-Moon System. This implies that if accretional criteria is satisfied along with the impact velocity condition that is the rebound velocity should be smaller than the mutual surface escape velocity then merged body formation of Moon starts within the Roche zone. The accreted Moon gradually migrates outward sweeping the remnant debris [5].

Rewriting (6) we obtain:

$$\frac{J_T}{C\Omega} = \left[\frac{\omega}{\Omega} + \left(\left(\frac{m^*}{C}\right)a_{present}^2 + \frac{I}{C}\right)\right] = \left[\frac{\omega}{\Omega} + \theta_2 a_{present}^2 + \theta_1\right] \qquad 10$$

Substituting Ω= B/a$^{3/2}$ in (10) we obtain:

$$\left(\frac{J_T}{CB}\right)a^{\frac{3}{2}} = \left[\frac{\omega}{\Omega} + \theta_2 a_{present}^2 + \theta_1\right] \quad 11$$

Rearranging the terms in (11) we get:

$$\frac{\omega}{\Omega} = \frac{LOM}{LOD} = \left(\frac{J_T}{CB}\right)a^{\frac{3}{2}} - (\theta_2 a^2 + \theta_1) = Aa^{\frac{3}{2}} - Fa^2 \quad 12$$

$$where\ A = \frac{J_T}{BC}\ and\ F = \left(\theta_2 + \frac{\theta_1}{a^2}\right)$$

Substituting the numerical values of the parameters we get:
$A=2.1331\times10^{-11}(1/m^{3/2})$ and $F=9.0425\times10^{-16}(1/m^2)$.
Substituting these values in (10) we get LOM/LOD =27.1479.
The actual value of LOM/LOD is 27.322. This error is due to uncertainty in Globe-Spin parameters. A and F are adjusted to obtain the exact value of 27.322.
The best fit values of LOM/LOD constants are:
$A=2.13853\times10^{-11}(1/m^{3/2})$ and $F=9.05842\times10^{-16}(1/m^2)$.
The best fit parameters give the following geo-synchronous orbits:

$a_{G1}=1.46402\times10^7$m, $a_{G2}=5.5247\times10^8$, $a_2=2.40649\times10^7$m 13

Calculating LOM/LOD (12) using the best fit parameters of 'A' and 'F' we get : LOM/LOD = 27.322 which is the present era Sidereal Lunar Month/Solar Day observed values.

1.3. Tidal Torque Formulation [4].

For the calculation of the spiral trajectory we need the radial velocity of recession in case of super-synchronous configuration and velocity of approach in case of sub-synchronous configuration. The time integration of the reciprocal of radial velocity gives the non-Keplerian Transit time from its inception to the present orbit. This transit time should be equal to the age of the secondary or the natural satellites. The starting point of this time integral will be the tidal torque.
The Tidal Torque of Satellite on the Planet and of Planet on the Satellite = Rate of change of angular momentum hence

$$Tidal\ Torque = T = \frac{dJ_{orb}}{dt} \quad 14$$

But Orbital Angular Momentum:

$$J_{orb} = m^* a^2 \times \frac{B}{a^{3/2}} = m^* B\sqrt{a} \quad 15$$

Time Derivative of (15) is:

$$T = \frac{dJ_{orb}}{dt} = \frac{m^* B}{2\sqrt{a}} \times \frac{da}{dt} \quad 16$$

In super-synchronous orbit, the radius vector joining the satellite and the center of the planet is lagging planetary tidal bulge hence the satellite is retarding the planetary spin and the tidal torque is BRAKING TORQUE.
In sub-synchronous orbit, the radius vector joining the satellite and the center of the planet is leading planetary tidal bulge hence the satellite is spinning up the planet and the tidal torque is ACCELERATING TORQUE.
These two kinds of Torques are illustrated in Figure 1 and Figure 2.

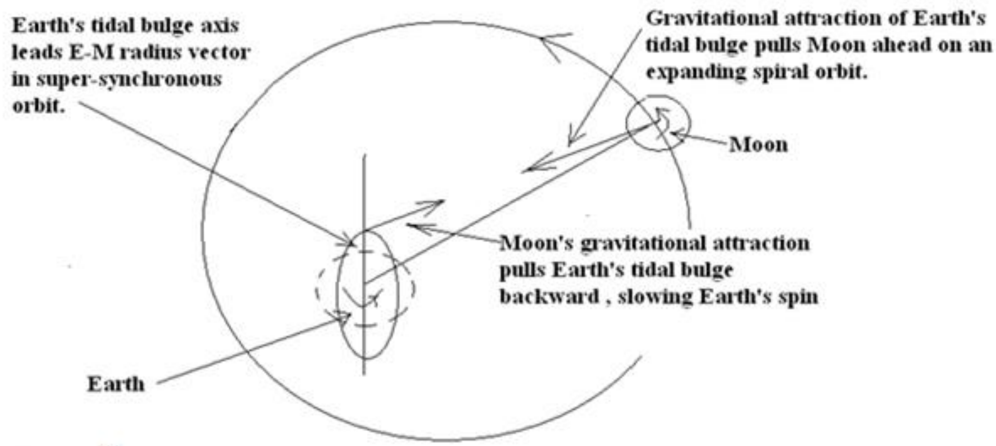

Figure 7. In Earth-Moon System, Moon is in super-synchronous orbit. The off-setting of the line of bulge in Earth with respect to E-M radius vector creates a tidal drag and de-spinning of Earth leading to secular lengthening of day. The de-spinning of Earth leads to increased angular momentum of Moon . During the conservative phase of the evolutionary phase of E-M System, by gravitational sling shot impulsive torque Moon is launched on an expanding spiral path around Earth. After the conservative phase, Earth coasts on its own towards the outer Clarke's orbit where it terminates its non-keplerian journey.

Figure 2: In super-synchronous orbit, the radius vector joining the satellite and the center of the planet is lagging planetary tidal bulge hence the satellite is retarding the planetary spin and the tidal torque is BRAKING TORQUE. This is shown in context of Earth and Moon.

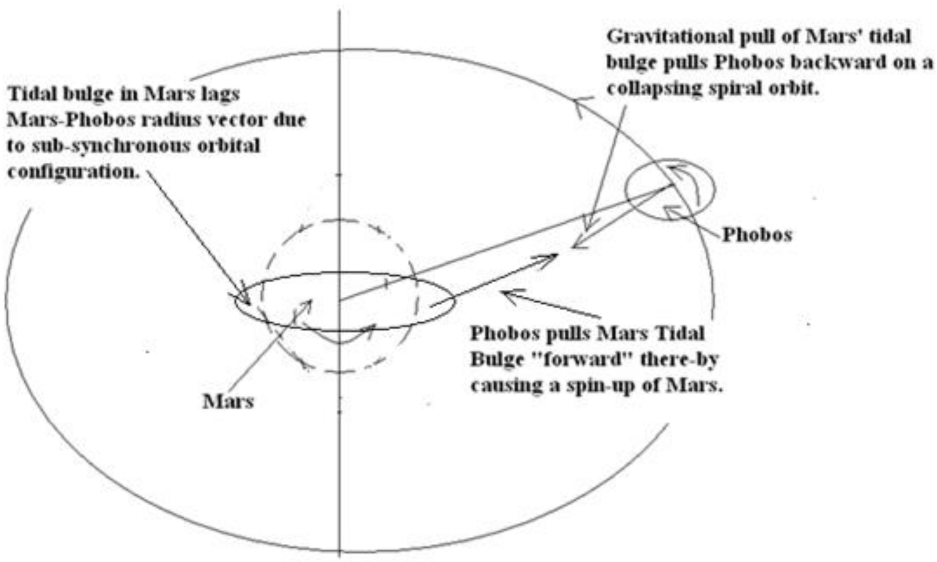

Figure 8 In Mars-Phobos System, Phobos is in sub-synchronous orbit and is speeding up Mars spin and losing angular momentum to conserve total angular momentum. In the process it is launched on a gravitational runaway collapsing spiral orbit. The offset of the tidal bulge axis of Mars with respect to Mars-Phobos radius vector creates a tidal acceleration of the spinning Mars.

Figure 3: In sub-synchronous orbit, the radius vector joining the satellite and the center of the planet is leading planetary tidal bulge hence the satellite is spinning up the planet and the tidal torque is ACCELERATING TORQUE. This is shown in context of Mars-Phobos System.

I have assumed the empirical form of the Tidal Torque as follows:

$$T = \frac{K}{a^Q}\left[\frac{\omega}{\Omega} - 1\right] \qquad 17$$

(17) implies that at Inner Clarke's Orbit and at Outer Clarke's Orbit, tidal torque is zero and (16) implies that radial velocity is zero and there is no spiral-in or spiral-out.

At Triple Synchrony, Satellite-Planet Radius Vector is aligned with planetary tidal bulge and the system is in equilibrium. But there are two roots of ω/Ω=1: Inner Clarke's Orbit and Outer Clarke's Orbit. As already shown in Total Energy Profile (S5.1. in supplememtary information, Inner Clarke's Orbit a_{G1} is unstable equilibrium state and Outer Clarke's Orbit a_{G2} is stable equilibrium state [5]. In any Binary System, secondary is conceived at a_{G1}. This is the CONJECTURE assumed in Kinematic Model. From this point of inception Secondary may either tumble short of a_{G1} or tumble long of a_{G1} . If it tumbles short, satellite gets trapped in Death Spiral and it is doomed to its destruction. If it tumbles long, satellite gets launched on an expanding spiral orbit due to gravitational sling shot impulsive torque which quickly decays due to the growing differential of ω and **Ω** and the resulting tidal heating. After the impulsive torque has decayed, the satellite coasts on it own toward final lock-in at a_{G2}.

Equating the magnitudes of the torque in (16) and (17) we get:

$$\frac{m^*B}{2\sqrt{a}} \times \frac{da}{dt} = \frac{K}{a^Q}\left[\frac{\omega}{\Omega} - 1\right] \qquad 18$$

Rearranging the terms in (18) we get:

$$V(a) = Velocity\ of\ recession = \frac{2K}{m^*B} \times \frac{1}{a^Q}\left[Aa^2 - Fa^{2.5} - \sqrt{a}\right] m/s \qquad 19$$

The Velocity in (19) is given in m/s but we want to work in m/y therefore (19) R.H.S is multiplied by 31.5569088×10^6s/(solar year).

$$V(a) = \frac{2K}{m^*B} \times \frac{1}{a^Q}\left[Aa^2 - Fa^{2.5} - \sqrt{a}\right] \times 31.5569088 \times 10^6 m/y \qquad 20$$

In (20) 'a' refers to the semi-major axis of the evolving Satellite. There are two unknowns: exponent 'Q' and structure constant 'K'. Therefore two unequivocal boundary conditions are required for the complete determination of the Velocity of Recession.

First boundary condition is at a = a2 which is a Gravitational Resonance Point where ω/Ω = 2,

i.e. $(Aa^{3/2} – Fa^2) = 2$ has a root at a2 [6].

In E-M case, $a_2 = 2.40649\times10^7$ m.

At a2 the velocity of recession maxima occurs. i.e. V(a2) = Vmax .

Therefore at a = a_2, $(\delta V(a)/\delta a)(\delta a/\delta t)|_{a2} = 0$.

On carrying out the partial derivative of V(a) with respect to 'a' we get the following:

$$At\ a_2\ ,\quad (2 - Q)A \times a^{1.5} - (2.5 - Q)F \times a^2 - (0.5 - Q) = 0 \qquad 21$$

Solving (21) at 2:1 Mean Motion Resonance orbit 'a_2' we obtain :

$$Q = 3.23771 \qquad 22$$

Now structure constant (K) has to be determined. This will be done by trial error so as to get the right age of Moon i.e. 4.467Gy. [The birth of the Solar System is the time when the condensation of the first solid took place from the Solar Nebula. This is taken as 4.567Gya. The last giant impact on Earth formed the Moon and initiated the final phase of core formation by melting the mantle of the Earth. The date of this last impact decides the birth date of Moon which was completed in a few hundred years by the accretion of the impact generated debris. Claim an age of 30My after the birth of Solar System [7-9]. Claim an younger Moon formed after 50 to 100My after the first solid condensed [10-12]. The concentration of highly siderophile elements (HSEs) in Earth's mantle constrains the mass of chondritic material added to Earth during Late Accretion [13, 14]. Using HSE abundance measurements, determine a Moon-formation age of 95± 32 Myr after the condensation [15-17]. This method is invariant of the geo-chemistry chronometer adopted by earlier researchers. So it will be realistic to take the age of Moon as 4.467Gya.]

Rewriting (20) and substituting the best fit values of the exponent and constants A and F we obtain the structure constant 'K'.

$$V(a) = \frac{2K}{m^*B} \times \frac{1}{a^Q}\left[A \times a^2 - F \times a^{2.5} - \sqrt{a}\right] \times 31.5569088 \times 10^6 m/y \qquad 23$$

We will assume the age of Moon 4.467Gy as already mentioned. The Transit Time from aG1 to the present 'a' is given as follows:

$$Transit\ Time = \int_{a_{G1}}^{a} \frac{1}{V(a)} da \qquad 24$$

Assuming a tentative value for Vmax and inserting it in (23) at a = a_2 we deduce the value of 'K'. Using this 'K' in (23) and inserting this trial expression in (24) we carry out the time integral to get the transit time from a_{G1} to present 'a' which should be the age of Moon. Several iterations are carried by adjusting V_{max} . By this iteration method we obtain the best fit structure constant as:

$$K = 6.48548 \times 10^{42} Newton - m^Q \qquad 25$$

So the best fit velocity of recession formalism is:

$$V(a) = \frac{2 \times 6.48548 \times 10^{42}}{m^*B} \times \frac{1}{a^{3.23771}}\left[A \times a^2 - F \times a^{2.5} - \sqrt{a}\right] \times 31.5569088 \times \frac{10^6 m}{y} \qquad 26$$

Transit Time expression gives 4.46209Gy using (24). This is the age of Moon as concluded above.

Present recession velocity of Moon according to classical theory is =2.325cm/y 27

1.3. The formalism of lengthening of day(LOD) curve , the observed LOD curve and its match.

Data set #	Time B.P.(years)	Orbital radii(m)[1]	Length of day(hrs)[2]	LOD[3](hrs)
1	Present	3.844×10^8	24	24
2	65Ma	3.8287×10^8	23.627	23.6
3	135Ma	3.81213×10^8	23.25	NA
4	136Ma	3.8118×10^8	23.2515	23.2
5	180Ma	3.80129×10^8	23.0074	23
6	230Ma	3.7891×10^8	22.7683	22.7
7	280Ma	3.7768×10^8	22.4764	22.4
8	345Ma	3.76055×10^8	22.136	22.1
9	380Ma	3.7517×10^8	21.9	NA
10	405Ma	3.74535×10^8	21.8055	21.7
11	500Ma	3.7208×10^8	21.276	21.3
12	600Ma	3.6943×10^8	20.674	20.7
13	715Ma	3.663×10^8	NA	20.1
14	850Ma	3.6251×10^8	NA	19.5
15	900Ma	3.61075×10^8	18.9	19.2
16	1200Ma	3.5205×10^8	NA	17.7
17	2000Ma	3.235×10^8	NA	14.2
18	2500Ma	3.012×10^8	NA	12.3
19	3000Ma	2.735×10^8	NA	10.5

20	3560Ma	2.3143×10^8	NA	8.7
21	4500Ma	NA	NA	6.1

Table 1: Observed Length of Day in different geological epochs.

1. Orbital radii according to the classical Model of E-M system
2. Length of Day according to John West Wells, and Charles [18-20].
3. Length of Day according to [23].

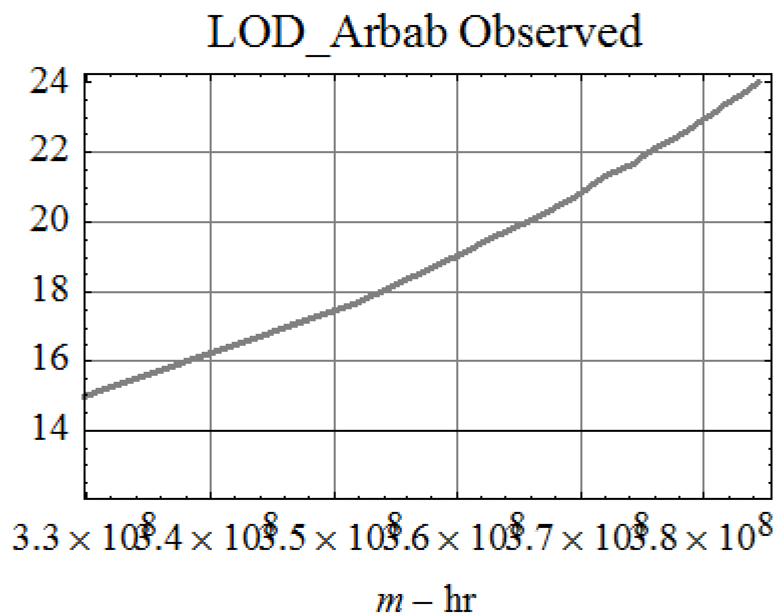

Figure 4: Observed Lengthening of Day curve (over a span of time 2.5Gy). X-axis is the lunar orbital radius in meter.Y-axis is Length of Earth Day(hours) in different geologic epochs corresponding to the lunar orbital radii.

1.4.1. Theoretical Formalism of Length of Earth Day in the life span of Moon from today to 2000Ma.

From Kepler's Third Law:

$$T_{orb} = \frac{2\pi \times a^{3/2}}{B} \qquad 27$$

From (12):

$$\frac{T_{orb}}{T_E} = \left(A \times a^{\frac{3}{2}} - F \times a^2\right) \qquad 28$$

Substituting (27) in (28) we obtain LOD:

$$T_E = \frac{2\pi \times a^{3/2}}{B\left(A \times a^{\frac{3}{2}} - F \times a^2\right)} \qquad 29$$

Substituting the best fit parameters in (29) the theoretical lengthening of day curve is obtained in Figure 4.

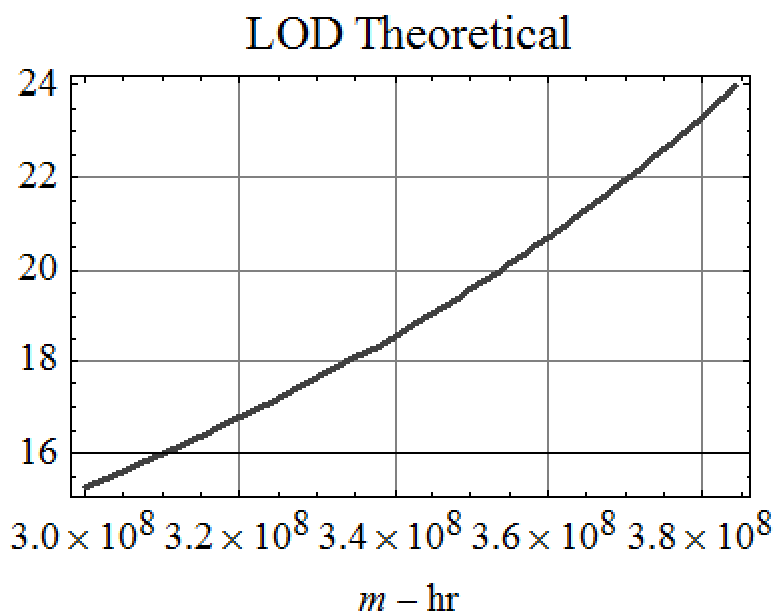

Figure 5: Theoretical Lengthening of Day curve (3.012×10^{8}m to the present) . X-axis is the lunar orbital radius in meter.Y-axis is Length of Earth Day in different geologic epochs corresponding to the lunar orbital radii.

Superposition of the theoretical curve and observed curve gives Figure 8.

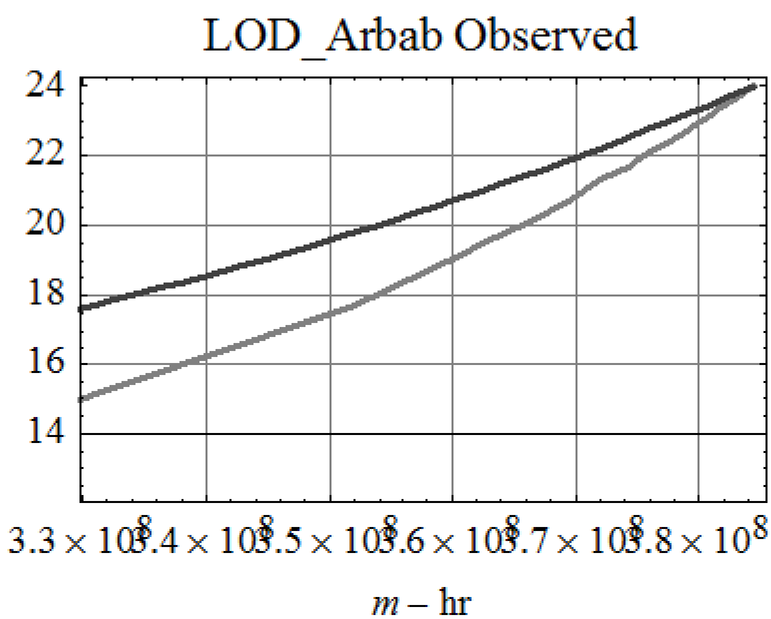

Figure 6: Superposition of Theoretical and Observed Lengthening of Day curve (over 2.5Gy). X-axis is the lunar orbital radius in meter.Y-axis is Length of Earth Day in different geologic epochs corresponding to the lunar orbital radii.

As we see in Figure 5 there is continuous mismatch over a 2.5Gy life span of Moon since the present and this mismatch is 24.7496% ~ 25% the worst at the remotest point.

2. Validation of Observed LOD for accelerated MOON(accelerated from 3.274×10^{8}m to the present orbit but Age of Moon =4.467Gy)

Matija Cuk have proposed a radically different model where Moon tidally evolves in fits and bound. Fits are due to stalling of Moon tidal evolution due to strong lunar obliquity tides created in Laplace Plane transition. Bound is due to accelerated transit time of 1.2Gy in spiraling out from 3.274×10^8m to the present lunar orbit of 3.844×10^8m as compared to 1.9Gy for the classical Moon for an identical orbital radius expansion. Application of Advanced Kinematic Model to fits and bound model of Moon at one stroke removes the tension between Lunar Laser Ranging measurement of 3.7cm/y and theoretically predicted Lunar recession of 2.3cm/y assuming 4.467Gy for the classical Moon on one hand and gives a perfect match between observed LOD curve and theoretically predicted LOD curve for last 900My and near perfect match over last 1.2Gy with a mismatch of -1.3% [24-26].

2.1. Development of Advanced Kinematic Model of Earth-Moon system

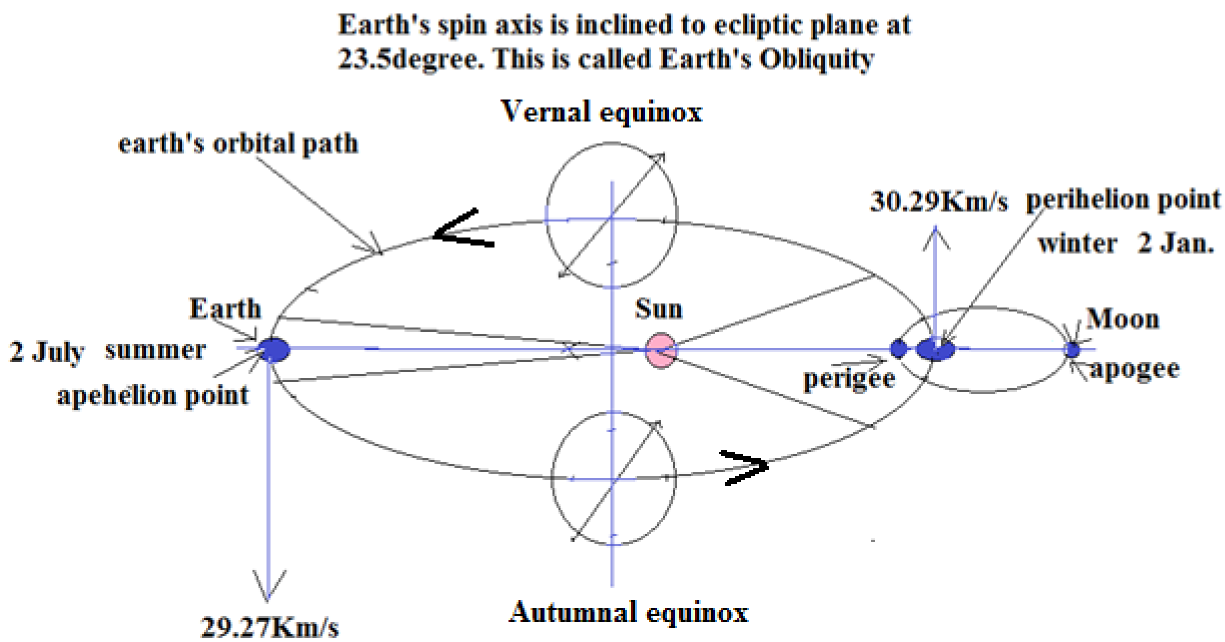

Figure 7: The orbital path of Earth around sun

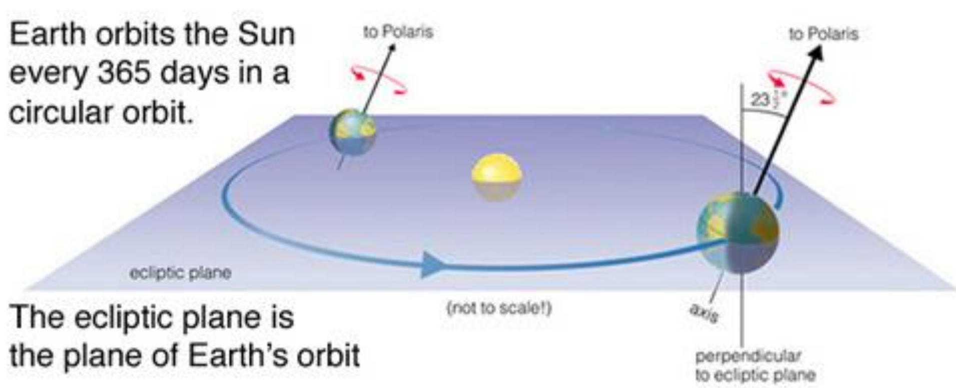

Figure 8: Earth’s orbit around the Sun.

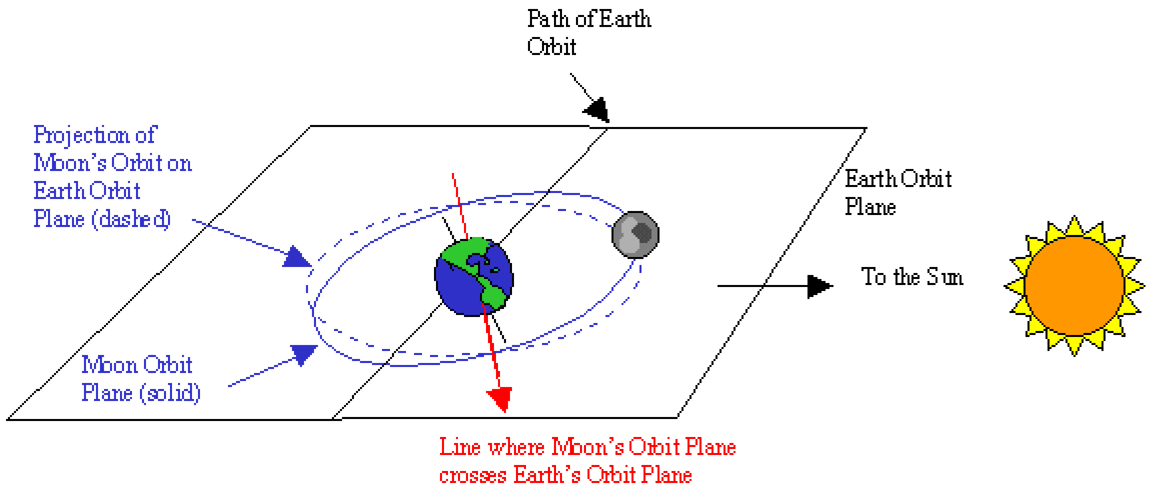

Figure 9: Illustration of ascending and descending nodes.

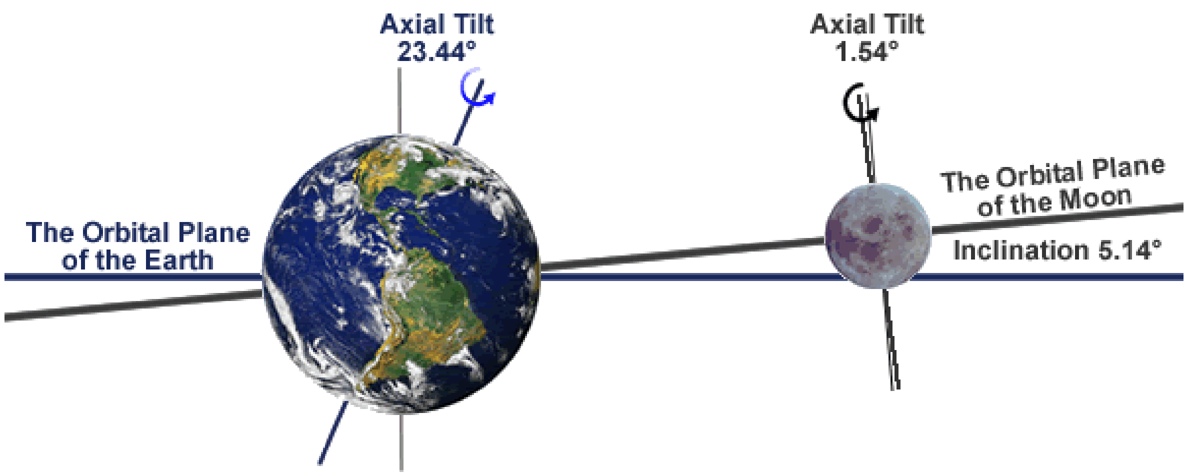

Figure 10: Axial tilt of Moon's spin axis to its orbit's normal is 1.54°.

Therefore Moon's axial tilt to Ecliptic normal = Axial tilt of Moon's spin to Moon's orbit's normal + inclination angle of Moon's orbital plane w.r.t. ecliptic = 1.54° + 5.14° = 6.68°

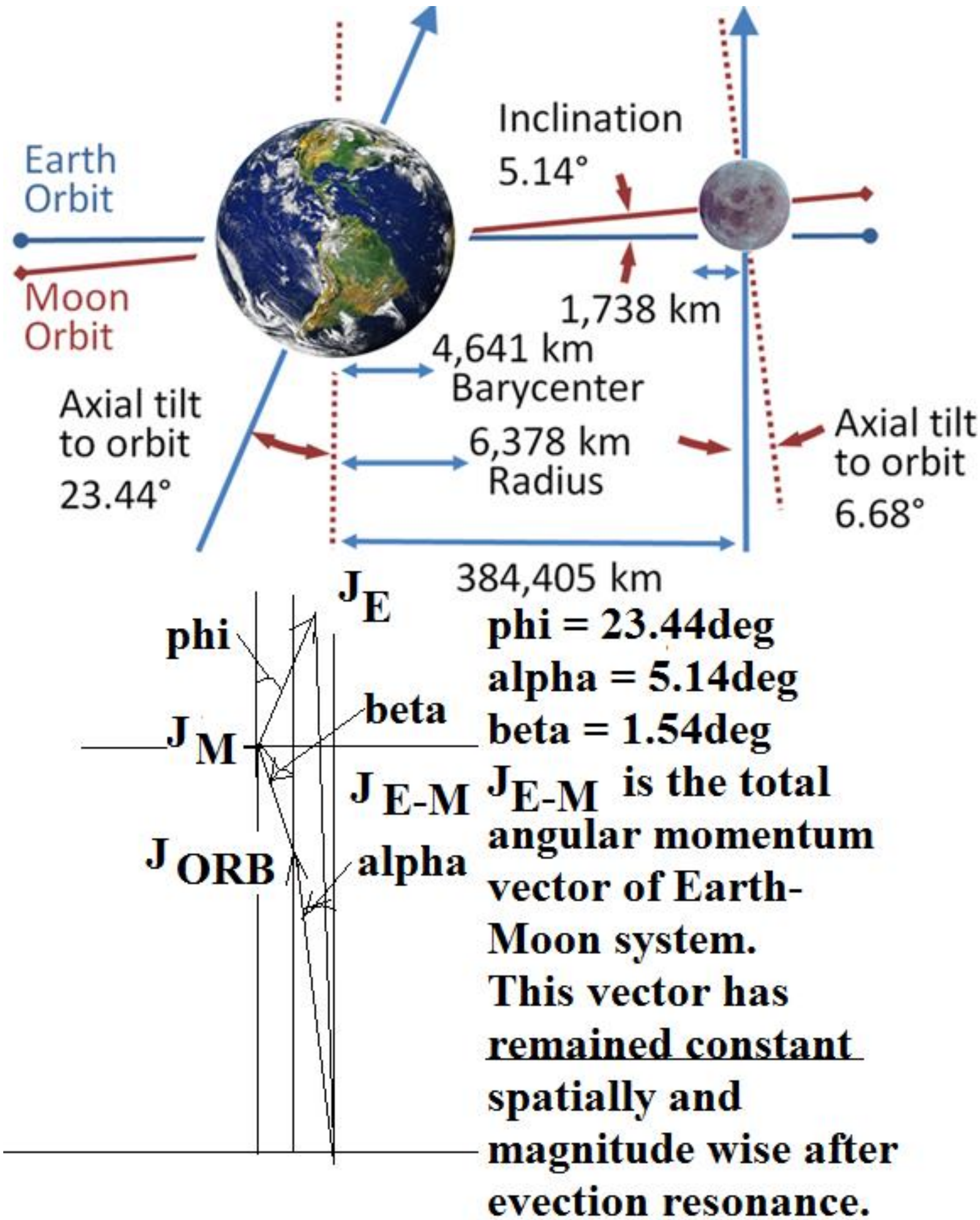

Figure 11: Resultant Angular Momentum Vector of Earth-Moon System.

Advanced Kinematic Model of Earth-Moon system was introduced in in the paper by the Author (Sharma 2023)
Rewriting its basic equation containing LOM/LOD we get:
After detailed analysis as shown in S6 of supplementary materials of Sharma 2023, we get:

$$(N)^2 \times a^3 = X^2 + (F)^2 \times (a^2)^2 + G^2 + 2(F \times a^2)(G)\sqrt{1-D^2} - 2 \times X \times \sqrt{(F \times a^2)^2 + (G)^2 + 2(F \times a^2)(G)\sqrt{1-D^2}} \times \{Z\} \qquad 20$$

Where the different symbols are defined as follows:

$$\frac{J_4}{C \times B} = N$$

$$here\ C = 0.3308 \times M_{Earth} \times R_{Earth}^2 = present\ day\ spin\ moment\ of\ inertia\ of\ E$$

$$B = \sqrt{G(M+m)} = 2.00873 \times 10^7 \frac{m^{3/2}}{s},$$

$$J_4 = total\ vector\ sum\ of\ the\ andular\ momentums\ of\ E - M\ system$$

$$\frac{F^*}{C} = F\ and\ \frac{I}{C} = G\ and\ X = \frac{LOM}{LOD}$$

$$here\ I = 0.394 \times m \times R_{Moon}^2 = spin\ moment\ of\ Inertia\ of\ Moon \qquad 22$$

Sin[β] = D and Cos[β] =√(1-D^2),

Sin[α] = A and Cos[α] =√(1-A^2),

Sin[Φ] = B and Cos[Φ] =√(1-B^2),

Let Z= - Cos[α] Cos[Φ]+ Sin[α] Sin[Φ]=A.B-√(1-A^2) √(1-B^2) 23

The empirical relation describing the evolution of Moon's orbital plane inclination with respect to the ecliptic is (Appendix A, Sharma 2023):.

$$Inclination\ angle\ \alpha = \frac{1.18751 \times 10^{25}}{a^3} - \frac{7.1812 \times 10^{16}}{a^2} + \frac{1.44103 \times 10^8}{a} - 8.250567342 \times 10^{-3} \qquad 31$$

The empirical relation describing the evolution of Moon's obliquity angle (β) is given as below(Appendix A,)

$$Moon's\ Obliquity\ angle\ \beta = 3.36402 - 1.37638 \times 10^{-8} a + 1.32216 \times 10^{-17} a^2 \qquad 32$$

The empirical relation describing the evolution of Moon's orbit eccentricity is (Appendix A, Sharma 2023):.

$$e = 0.210252 + 8.38285 \times 10^{-10} a - 3.23212 \times 10^{-18} a^2 \qquad 33$$

In Sharma 2023:

$$\frac{LOM}{LOD} = \frac{\omega}{\Omega} = -12.0501 + 2.6677 \times 10^{-7} \times a - 4.27538 \times 10^{-16} \times a^2 \qquad 16$$

$$\varphi = -0.732299 + 2.97166 \times 10^{-9} \times a \qquad 17$$

(31), (32) and (33) and (17) will be used to solve the quadratic equation given in (20). Two roots of (20) are obtained out of which positive root is retained and will be used for analysis purpose.
We have altogether 5 spatial function (31), (32), (33), (16) and (17) describing the evolution of inclination angle (α), Moon's obliquity (β), eccentricity(e) of lunar orbit, LOM/LOD and Earth's obliquity (Φ) respectively through different geologic epochs. Table 2 gives the evolution of these parameters through past geologic epochs.

a (×RE)	a (×10^8m)	ω/Ω	α radians	β radians	e	Φ(rad)	Sin[Φ]
30	1.9113	23.3752	0.480685 (27.4°)	1.21635 (69.69°)	0.2524	unstable	-0.464076
35	2.22985	26.1194	0.26478 (15.17°)	0.952317 (54.56°)	0.236	unstable	-0.216896
40	2.5484	28.1147	0.168969 (9.68°)	0.71512 (40.97°)	0.214	unstable	-0.0195376
45	2.86695	29.2938	0.124631 (7.1408°)	0.504756 (28.92°)	0.1849	0.113792 (6.51°)	0.113547
50	3.1855	29.5965	0.103801 (5.04736°)	0.321225 (18.4°)	0.1493	0.22 (12.6°) 0227	0.218451
55	3.50405	28.9877	0.0941394 (5.39379°)	0.164527 (9.4267°)	0.10714	0.314929 (18°)	0.309749
60	3.8226	27.4	0.0898729 (5.149°)	0.03466 (1.986°)	0.0584	0.398676 (22.84°)	0.388198
60.336	3.844	27.32	0.08971 (5.14°)	0.0268 (1.54°)	0.0549	0.409105 (23.44°)	0.397788

Table 2: Evolutionary history of ω/Ω (LOM/lOD), m α (Inclination angle), β (lunar obliquity), e (eccentricity) and Φ (terrestrial obliquity).

2.2. The formalism of the velocity of recession of Moon.

The radial velocity of lunar recession is the same as in the classical case:

$$V(a) = \frac{2K}{m^*B} \times \frac{1}{a^Q}\left[A \times a^2 - F \times a^{2.5} - \sqrt{a}\right] \times 31.5569088 \times 10^6 m/y \qquad 23$$

This is rewritten as:

$$V(a) = \frac{2K}{m^*B} \times \frac{\sqrt{a}}{a^Q}[X - 1] \times 31.5569088 \times 10^6 m/y \qquad 24$$

In (24) 'a' refers to the semi-major axis of the evolving Satellite. There are two unknowns: exponent 'Q' and structure constant 'K'. Therefore two unequivocal boundary conditions are required for the complete determination of the Velocity of Recession.

Equation (20) gives the expression of the permissible X in advanced Kinematic Model. That permissible X is substituted in (24) for analysis purpose.

By classical E-M model Q is calculated to be Q = 3.22684.

K = 5.5×10^{42}Newton-mQ ,Transit Time (from 3.012×108m to 3.844×10^8m) =2.38Gy.

This gives present epoch velocity of recession of Moon as =2.4cm/y. A.

K = 8.33269×1042Newton-mQ,Transit Time (from 3.012×10^8m to 3.844×10^8m) = 1.57732 Gy.

This gives present epoch velocity of recession of Moon as = 3.7cm/y. B

So for our calculations we will retain the structure constant in (B). This helps achieve correspondence with LLR result = 3.7cm/y. Now this can be justified.

From 3RE to 45RE Moon does not have a smooth transit. Infact it is bumpy. It is chaotic, gets stuck in resonances and comes out of the resonances and gets stalled and resumes its tidal evolution. In fact Moon takes 3.267Gy to spirally expand from 3RE to 45RE in fits and stalled manner. From 45R_E to 60.336R_E, Moon smoothly coasts in 1.2Gy. This accelerated spiral expansion results in present day velocity of recession of 3.7cm/y.

Since Adv.KM is well defined from 45R_E (Cassini State2) to 60.336R_E so data set within this range only is considered,

In Figure 9, tidal evolution of Moon's orbital radius is given based on the new study of Matija Cuk [26]

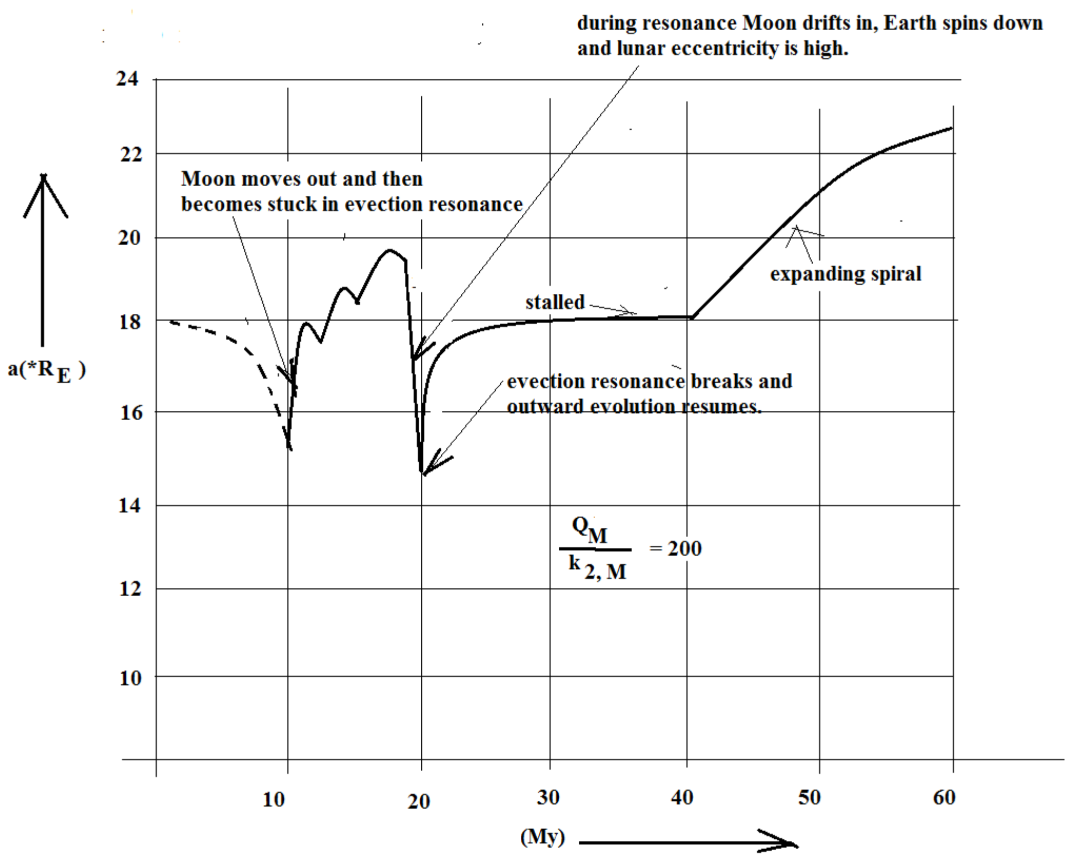

Figure 12: Evolution of semi-major axis of Moon for first 60My when Laplace plane transition is encountered at 20My and a = $17R_E$.

Validation of Observed LOD for accelerated MOON(accelerated from 3.274×10^8m to the present orbit but Age=4.467Gy) with time lost in the formation of the complete Moon from moonlets as well as time lost in stalled tidal evolution at Laplace Plane transition. Normal monotonically spiral expansion occurs only after Cassini State Transition) using 15 Data points given [18-22].

Data set #	Time B.P.(years)[1]	Orbital radii ($\times10^8$m)[3]	LOD(h)
1	Present	3.844	24
2	45Ma	3.8272	23.566(Ref.1)
3	65Ma	3.8197	23.627(Ref.2)
4	135Ma	3.7928	23.25(Ref.2)
5	136Ma	3.7924	23.2(Ref.5)
6	180Ma	3.7752	23(Ref.2)
7	230Ma	3.7553	22.7684(Ref.2)
8	280Ma	3.735	22.4765(Ref.2)
9	300Ma	3.7268	22.3(Ref.4)
10	345Ma	3.708	22.136(Ref.2)
11	380Ma	3.6933	21.9(Ref.2)
12	405Ma	3.6825	21.8(Ref.2)
13	500Ma	3.641	21.27(Ref.2)
14	600Ma	3.5954	20.674(Ref.2) 20.7(Ref.4)
15	715Ma	3.5405	20.1(Ref.5)

16	850Ma	3.4725	19.5(Ref.5)
17	900Ma	3.446	18.9(Ref.3) 19.2(Ref.4) (Ref.5)
18	1200Ma	3.274	17.7(Ref.5)
19	2000Ma	2.6(Not Permisible)	14.2(Ref.5)
	2450Ma (3.28×10^{8}m)	1.96(Not Permisible)	
20	2500Ma		12.3(Ref.5)
21	3000Ma		
22	3560Ma		
23	4500Ma		

Table 2: Tabulation of LOD in past geologic epochs for accelerated Moon.(Structure constant K= 8.333269 N-mQ, Q= 3.22684, present velocity of recession = 3.7cm/y)

[1]Based on annual bands in coral fossils.
[3]Orbital radii based on accelerated Moon.
Reference 1:Kaula & Harris (1975)
Reference 2:John West Wells (1965,1966)
Reference 3:Charles P. Sonnett & Chan(1998)
Reference 4:Leschiuta & Tavella (2001)
Reference 5:A.J.Arbab (2009)
In classical case, monotonic spiral expansion from 3.274×10^{8}m to the present Moon takes 1.9Gy but in accelerated Moon it takes only 1.2Gy.
ListPlot of LOD for accelerated Moon for the time span of 900My.

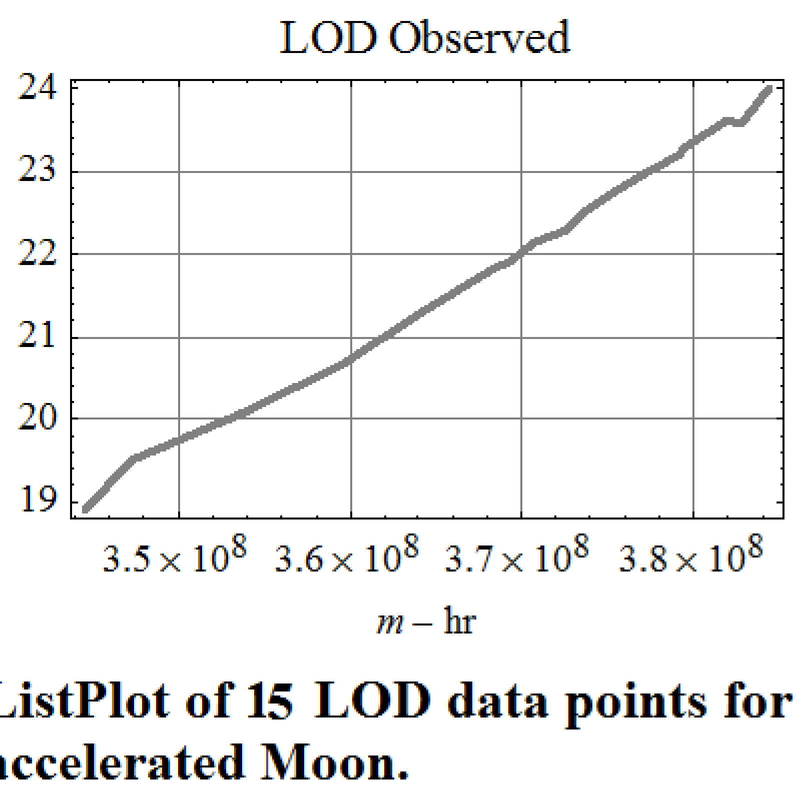

Figure 13: Observed LOD curve for fits and bound Moon covering a time span of 900My.

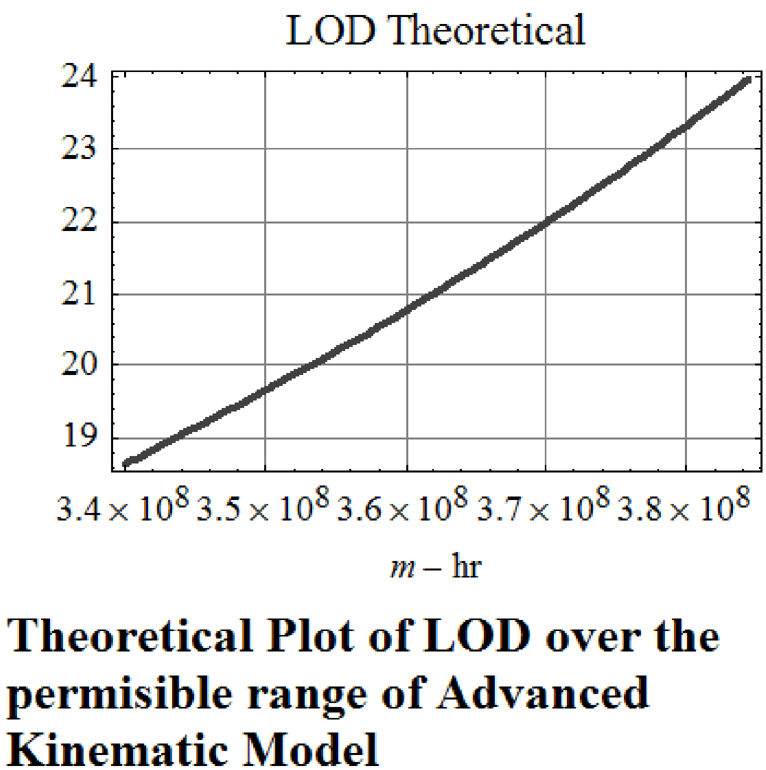

Figure 14: Theoretical Plot for LOD curve for accelerated Moon over 900My from the present.

Superposition of the observed and theoretical Plot for the accelerated Moon over 900My.

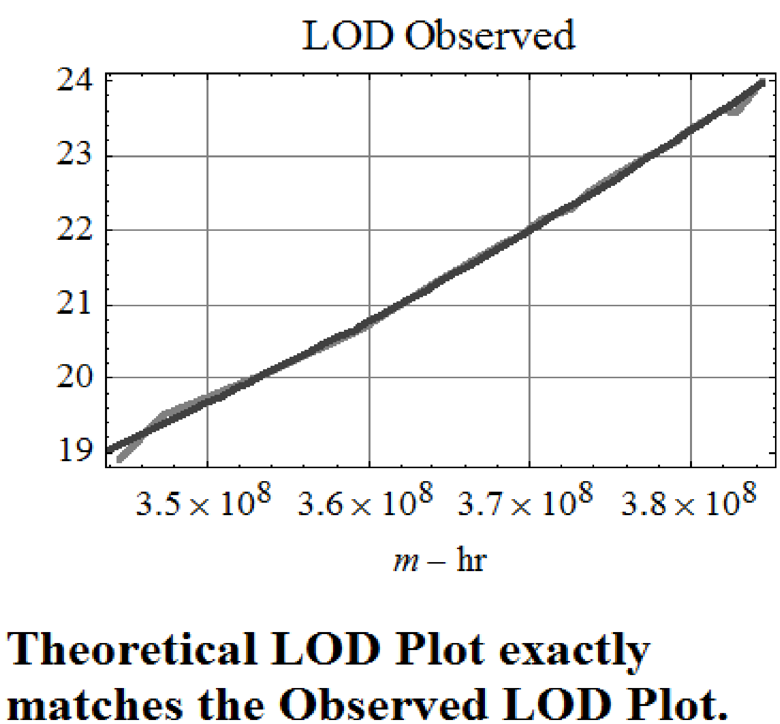

Figure 15: Exact match between Theoretical and Observed LOD Plot over last 900My.

Validation of Observed LOD for accelerated MOON (from 3.274×10^{8}m to the present orbit) using 17 Data points given by [20, 23].

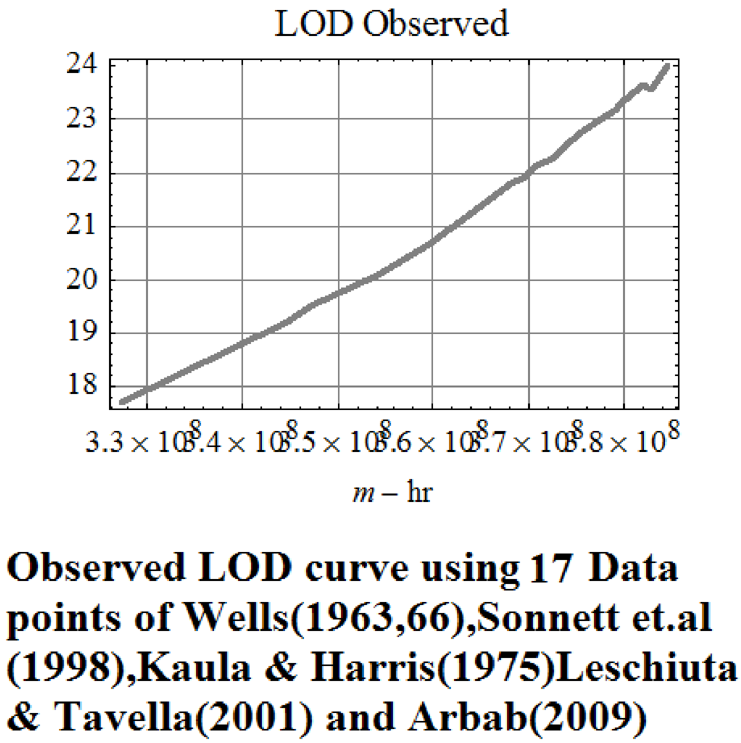

Figure 16: Observed LOD curve for acelerated Moon using 17 data points covering a time span of 1.2Gy.

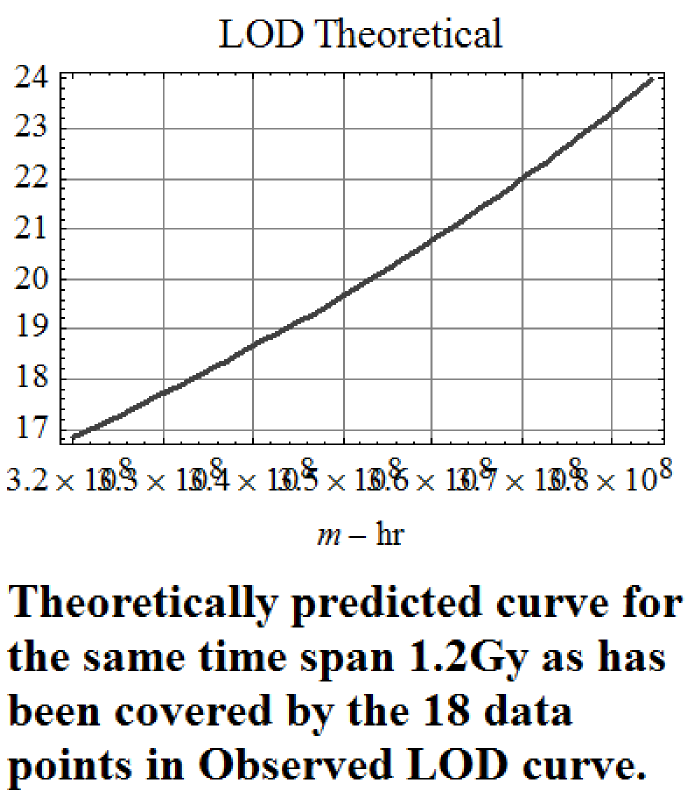

Figure 17: Theoretical Plot for LOD curve for accelerted Moon using 17 Data points..

The observed and theoretical curves are superposed in Figure 6.

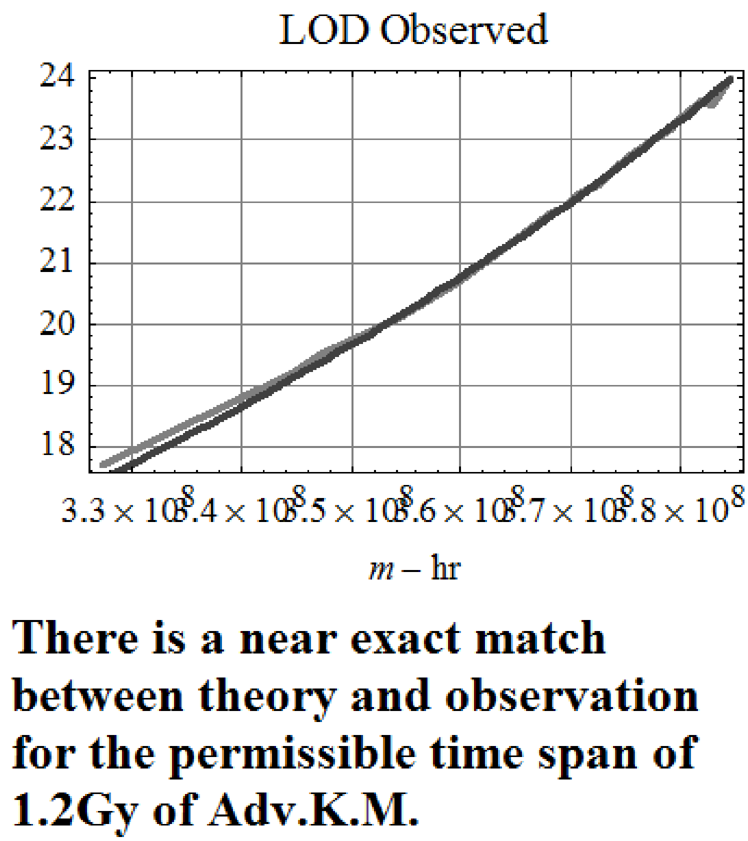

Figure 18: Near exact match between Theoretical and Observed LOD Plot over the permissible time span 1.2Gy of the Advanced kinematic Model of E-M system.

There is exact match over the entire life span except at 1.2Gy. The worst mismatch in the remote past at (1.2Gya) is -4%.

For classical Moon , from 3.274×108m to the present orbit the transit time is 1.9Gy.

For fits and bound Moon in its final phase , from 3.274×108m to the present orbit the transit time is 1.2Gy.

The observed curve of LOD is vastly different in classical Moon and fits and bound Moon because the accelerated phase has a scaled up time-scale resulting in Moon's recession velocity of 3.7cm/y as compared to that in monotonically expanding spirally evolving Moon where the velocity of recession is 2.3cm/y. Theoretical curve is the same in the two cases, accelerated and monotonically expanding orbit, because Structure Constant does not come anywhere in the picture while calculating theoretical LOD curve. This means if the FIT is exact in accelerated case then it is inevitable to be poor in classical case.

3. Discussion

The Advanced Kinematic Model has been developed by including Earth's obliquity (Φ), Moon's orbital plane inclination with respect to ecliptic(α) as well as lunar obliquity(β) with respect to the lunar orbital normal. The Laplace Plane transition and Cassini State transition occurring at a =(17 to 19)RE and at 33RE respectively have been kept out of the permissible range of Advanced KM. Advanced KM covers the range of Moon's tidal evolution from 45RE to 60.335RE (the present orbital radius). Because of instability and unpredictability of Laplace Plane transition and Cassini State transition, the range from 3RE to 45RE has been kept out of range for for Advanced KM. In classical Model of E-M system to satisfy the Age of Moon a lunar recession rate of 2.3cm/y was being adopted which was completely distorting the time scale of tidal evolution of Moon for last 1 Gy. If lunar recession rate of 3.7cm/y was being adopted then too short a Moon's age of 2.7Gy was being obtained which was completely contrary to the observed facts. This conundrum got resolved only after the publication of Cuk et. al. land mark paper on Moon's tidal evolution in high obliquity, high angular momentum Earth. By adopting the Lunar Laser Ranging data of lunar recession as the model present lunar recession the model in one stroke became self consistent in all respects namely we obtained present Earth day of 24h, present LOM/LOD= 27.322, present lunar recession rate of 3.7cm/y and most of all we got a perfect match between observed LOD curve and theoretical LOD curve.

4. Conclusion

Using Advanced Kinematic Model to obtain a perfect match between observed LOD curve and theoretical LOD curve has been a crowning achievement as well as the ultimate vindication of Advanced KM. This paper has laid to rest all the nagging doubts which have been there for the empirical nature of the tidal torque developed in Kinematic Model. This paper has proved with 95% confidence level that Advanced KM is a valid model and well tested model which can be used with accuracy and reliably for analyzing two body tidally interacting systems. The application range of Advanced KM can be planet-satellite, planet hosting star and planet, star binary, Neutron star binary, neutron star-black hole binary or black hole binary. In subsequent papers the validity of advanced KM will be proved in this wide range of binary pairs.

Acknowledgement

This continued research has been made possible under Emeritus Fellowship Scheme sponsored by University Grants Commission, India. The Grant number is EMERITUS/2012-13-GEN-855/. I acknowledge the infrastructural help given to me by the Director, IIT, Patna, in completing this paper. I acknowledge the cooperation of the staff members of IIT, Patna, who have helped me by material and moral support. I also acknowledge the moral support given by the Director, NIT, Patna, and my colleagues of NIT, Patna, in continuing my research in superannuation period. I also acknowledge the role of my D.Sc. supervisor Prof.Bhola Ishwar who mentored me through this new field of Astronomy and Astrophysics. Last of all I acknowledge the inspiration provided by George Gamow and his book 'The story of Moon', Carl Sagan "COSMOS" and books by Issac Asimov which laid the groundwork of this paper while I was 18 years old just introduced to General Mechanics in 2-year Physics course in my first 2 years of 5-year B.Tech. integrated course at IIT,Karagpur, in period 1962-67. This paper has been almost 60 years in making.

References

1. Sharma, B. K. (1995). Theoretical Formulation of Earth-Moon System revisited. In Proceedings of Indian Science Congress 82nd Session (p. 17).
2. Sharma, B. K. & Ishwar, B. "Lengthening of Day curve could be experiencing chaotic fluctuations with implications for Earth-Quake Prediction", World Space Congress-2002, 10th-19th October 2002, Houston, Texas, USA, Abstract 03078, Pg 3.
3. Sharma, B. K., Ishwar, B., & Rangesh, N. (2009). Simulation software for the spiral trajectory of our Moon. *Advances in space research, 4*3(3), 460-466.
4. Sharma, B. K. (2011). The Architectural Design Rules of Solar Systems Based on the New Perspective. *Earth, Moon, and Planets, 108,* 15-37.
5. Canup, R. M., & Esposito, L. W. (1996). Accretion of the Moon from an impact-generated disk. *Icarus, 119*(2), 427-446.
6. Rubicam. D. P. "Tidal Friction & the Early History of Moon"s Orbit", Journal of Geophysical Research, Vol. 80, No. 11, April 19, 1975, pp. 1537-1548.
7. Yin, Q., Jacobsen, S. B., Yamashita, K., Blichert-Toft, J., Télouk, P., & Albarede, F. (2002). A short timescale for terrestrial planet formation from Hf–W chronometry of meteorites. *Nature, 418*(6901), 949-952.
8. Jacobsen, S. B. (2005). The Hf-W isotopic system and the origin of the Earth and Moon. *Annu. Rev. Earth Planet. Sci., 33,* 531-570.
9. Taylor, D. J., McKeegan, K. D., & Harrison, T. M. (2009). Lu–Hf zircon evidence for rapid lunar differentiation. *Earth and Planetary Science Letters, 279*(3-4), 157-164.
10. Touboul, M., Kleine, T., Bourdon, B., Palme, H., & Wieler, R. (2007). Late formation and prolonged differentiation of the Moon inferred from W isotopes in lunar metals. *Nature, 450*(7173), 1206-1209.
11. Allègre, C. J., Manhès, G., & Göpel, C. (2008). The major differentiation of the Earth at□ 4.45 Ga. *Earth and Planetary Science Letters, 267*(1-2), 386-398.
12. Halliday, A.N.& Wood, B.J. 'Treatise on Geophysics' Vol 9 (ed. Schubert, G.) 13-50 (Elsevier, Amsterdam, 2007),
13. Chyba, C. F. (1991). Terrestrial mantle siderophiles and the lunar impact record. *Icarus, 92*(2), 217-233.
14. Bottke, W. F., Walker, R. J., Day, J. M., Nesvorny, D., & Elkins-Tanton, L. (2010). Stochastic late accretion to Earth, the Moon, and Mars. *science, 330*(6010), 1527-1530.
15. Becker, H., Horan, M. F., Walker, R. J., Gao, S., Lorand, J. P., & Rudnick, R. L. (2006). Highly siderophile element composition of the Earth's primitive upper mantle: constraints from new data on peridotite massifs and xenoliths. *Geochimica et Cosmochimica Acta, 70*(17), 4528-4550.
16. Walker, R. J. (2009). Highly siderophile elements in the Earth, Moon and Mars: update and implications for planetary accretion and differentiation. *Geochemistry, 69*(2), 101-125.
17. Jacobson, S. A., Morbidelli, A., Raymond, S. N., O'Brien, D. P., Walsh, K. J., & Rubie, D. C. (2014). Highly siderophile elements in Earth's mantle as a clock for the Moon-forming impact. *Nature, 508*(7494), 84-87.
18. Arbab, A. I. (2009). The length of the day: a cosmological perspective. Progress in Physics, 1(1), 8-11.
19. Ćuk, M., Hamilton, D. P., Lock, S. J., & Stewart, S. T. (2016). Tidal evolution of the Moon from a high-obliquity, high-angular-momentum Earth. *Nature, 539*(7629), 402-406.
20. Dickey, J. O., Bender, P. L., Faller, J. E., Newhall, X. X., Ricklefs, R. L., Ries, J. G., ... & Yoder, C. F. (1994). Lunar laser ranging: A continuing legacy of the Apollo program. *Science, 265*(5171), 482-490.
21. Ida, S., Canup, R. M., & Stewart, G. R. (1997). Lunar accretion from an impact-generated disk. *Nature, 389*(6649), 353-357.
22. Kaula, W. M., & Harris, A. W. (1975). Dynamics of lunar origin and orbital evolution. *Reviews of Geophysics, 13*(2), 363-371.
23. Leschiutta, S., & Tavella, P. (2001, January). Reckoning time, longitude and the history of the earth's rotation, using

the moon. In *Earth-Moon Relationships: Proceedings of the Conference held in Padova, Italy at the Accademia Galileiana di Scienze Lettere ed Arti, November 8–10, 2000* (pp. 225-236). Dordrecht: Springer Netherlands.

24. Sonett, C. P., & Chan, M. A. (1998). Neoproterozoic Earth-Moon dynamics: Rework of the 900 Ma Big Cottonwood Canyon tidal laminae. *Geophysical Research Letters, 25*(4), 539-542.
25. Wells, J. W. (1963). Coral growth and geochronometry. *Nature, 197,* 948-950.
26. Wells, J. W. (1966). Paleontological evidence of the rate of the Earth's rotation. In *The Earth-Moon System: Proceedings of an international conference, January 20–21, 1964, sponsored by the Institute for Space Studies of the Goddard Space Flight Center, NASA* (pp. 70-81). Boston, MA: Springer US.