ISSN: 2834-7706

Research Article

Journal of Mathematical Techniques and Computational Mathematics

On the H^m-Regularity for Second Order Elliptic Equations

Subham De*

Department of Mathematics, Indian Institute of Technology, India

***Corresponding Author**
Subham De, Department of Mathematics, Indian Institute of Technology, India
Website: www.sites.google.com/view/subhamde

Submitted: 2024, Feb 28; **Accepted:** 2024, Mar 21; **Published:** 2024, Mar 26

Citation: De, S. (2024). On the H^m-Regularity for Second Order Elliptic Equations. *J Math Techniques Comput Math, 3*(3), 01-16.

Abstract
Our primary objective in this paper is to discuss the H^m-Regularity for second order elliptic equations over Sobolev Spaces. We here consider the cases $m = 1$ and $m \geq 2$ separately. We revisit some elementary concepts in Functional Analysis and Abstract Harmonic Analysis before providing a proper definiton to the notion of weak solution of a Dirichlet Problem. While towards the later stages, we shall classify different types of regularity conditions, the main focus lies upon deducing appropriate ellipticity conditions in support of commenting about the existence and uniqueness of the weak solutions to a given problem. Appropriate references are provided in the bibliography section to facilitate further reading for ardent readers and researchers in this field.

Key Words and Phrases: Fourier Transform, Sobolev Space, Schwartz Space, Hilbert Space, Weak Derivative, Trace Theorem, Integration by Parts, Outward Normal, Bounded Domain, Boundary Value Problem, Dirichlet Problem, Weak Solution, Uniqueness, Existence, Interior Regularity, Boundary Regularity, Higher Regularity, Elliptic Equation, Elliptic Operator.

2020 MSC: Primary 35A01, 35A02, 35A15, 35J20, 35D30.
Secondary 35B65, 35A20, 35J25, 35J35, 58J05.

1. Introduction

1.1 Notations

We shall use the following notations throughout the article. They are as follows:

- $\mathbb{R}^n_+ \; := \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_n > 0\}.$

- A priori given integers, $\alpha_i \geq 0 \; \forall \; i = 1, 2, ..., n$, we denote,

$$\begin{aligned} \alpha \; &:= (\alpha_1, \alpha_2, ..., \alpha_n) \\ \alpha' \; &:= (\alpha_1, \alpha_2, ..., \alpha_{n-1}, 0). \end{aligned}$$

- For a multi-index α, we define,

$$\begin{aligned} |\alpha| \; &:= \alpha_1 + \alpha_2 + ... + \alpha_n \\ D^\alpha &= \tfrac{1}{i^{|\alpha|}} \cdot \tfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} .... \partial x_n^{\alpha_n}} \end{aligned}$$

Where, $i = \sqrt{-1}$.

- For every $1 \leq k \leq n$, and $j \in \mathbb{N}$,

$$D^j_k \; := \tfrac{1}{i^j} \cdot \tfrac{\partial^j}{\partial x^j_k}$$

It is assumed for simplicity that, for $j = 1$, we just write D_k.

- $C_0^\infty(\overline{\mathbb{R}_+^n})$:= Set of all C^∞ functions upto the boundary of $\mathbb{R}^n_+$ and having *compact support* in $\overline{\mathbb{R}_+^n}$.

1.2 Important results in Fourier Analysis

We recall the definition of *Fourier transform* and shall discuss about some of its important properties pertinent to our topic. For further details, one can refer to [2].

Definition 1.2.1. (Fourier Transform) For any $u \in L^1(\mathbb{R}^n)$, $v \in L^1(\mathbb{R}^n_+)$, we define the fourier transforms of u and v as,

$$\hat{u}(\xi) := \int_{\mathbb{R}^n} e^{-ix.\xi} u(x)dx \ , \qquad \xi \in \mathbb{R}^n$$

and,

$$\hat{v}(\xi', x_n) := \int_{\mathbb{R}^{n-1}} e^{-ix'.\xi'} u(x', x_n)dx' \ , \qquad \xi' \in \mathbb{R}^{n-1} \ , \ x = (x', x_n)$$

Furthermore, for $u \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, we have the Parseval Relation,

$$\int_{\mathbb{R}^n} |u(x)|^2 dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 dx \tag{1.1}$$

And,

$$\int_{\mathbb{R}^n} |D^\alpha u(x)|^2 dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi^\alpha|^2 |\hat{u}(\xi)|^2 dx \tag{1.2}$$

The relations (1.1) and (1.2) yields,

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} |v(x', x_n)|^2 dx' dx_n = \frac{1}{(2\pi)^{n-1}} \int_0^\infty \int_{\mathbb{R}^{n-1}} |\hat{v}(\xi', x_n)|^2 d\xi' dx_n$$

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} |D^{\alpha'} v(x', x_n)|^2 dx' dx_n = \frac{1}{(2\pi)^{n-1}} \int_0^\infty \int_{\mathbb{R}^{n-1}} |\xi^{\alpha'}|^2 |\hat{v}(\xi', x_n)|^2 d\xi' dx_n$$

A priori given a *Schwartz Space S* of rapidly decreasing functions in $\mathbb{R}^n$ with its dual as S', we can deduce the following,

$$\hat{u}(\phi) = u(\hat{\phi}) \ , \quad u \in \mathcal{S}' \ , \ \phi \in \mathcal{S}$$

Remark 1.2.1. An important observation is that, $\hat{u} \in S'$ as well for every $u \in S'$, by properties of *Fourier Transform* on Schwartz Spaces.

Using the above results, we provide a formal definition of the Sobolev Space as follows :

Definition 1.2.2. (Sobolev Space) For every $s \in \mathbb{R}$, we define the Sobolev Space $H^s(\mathbb{R}^n)$ as,

$$H^s(\mathbb{R}^n) := \left\{ u \in \mathcal{S}' \ : \ \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty \right\} \tag{1.3}$$

It can be verified that, $H^s(\mathbb{R}^n)$ is indeed a **Hilbert Space** with respect to the following *inner product* defined on it,

$$< u, v > \ := \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi$$

Remark 1.2.2. For $s \in \mathbb{N} \cup \{0\}$, we can interpret the above as following,

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) \ : \ D^\alpha u \in L^2(\mathbb{R}^n) \ , \text{for } |\alpha| \leq s \right\}$$

Where, D^α_u denotes the weak (distributional) derivative of u with an equivalent norm defined on $H^s(\mathbb{R}^n)$ as,

$$||u||_s^2 = \sum_{|\alpha|\leq s} \int_{\mathbb{R}^n} |D^\alpha u|^2 \tag{1.4}$$

Important to note that, the above definition has a valid extension to $\mathbb{R}^n_+$ as,

$$H^s(\mathbb{R}^n_+) := \{u \in L^2(\mathbb{R}^n_+) \ : \ D^\alpha u \in L^2(\mathbb{R}^n_+) \ , \text{for } |\alpha| \leq s\}$$

which in turn forms a Hilbert Space with respect to the inner product,

$$< u, v >:= \sum_{|\alpha|\leq s} \int_{\mathbb{R}^n_+} D^\alpha u . D^\alpha v$$

For future reference and for simplicity, since we shall be working in the case when $s \in \mathbb{N} \cup 0$, we take, $s = m$, and for notational purposes, we shall consider the Sobolev Space $H^m(\mathbb{R}^n_+)$ in our future deductions.

1.3 Equivalent Norm on $H^m(\mathbb{R}^n_+)$

A priori from the definition of norm as mentioned in (1.4) and using the property that, ∃ constants $c_1, c_2 > 0$ such that,

$$c_1(1+|\xi'|^2)^{m-j} \leq \sum_{|\alpha'|\leq m-j} |\xi^{\alpha'}|^2 \leq c_2(1+|\xi'|^2)^{m-j} \ , \forall j \in 0, 1, 2, ...m$$

For every $u \in H^m(\mathbb{R}^n_+)$, it helps us provide the definition of an equivalent norm on $H^m(\mathbb{R}^n_+)$, denoted by $|.|m$, and defined as,

Definition 1.3.1.

$$|u|^2_m := \sum_{j=0}^{m} \int_0^\infty \int_{\mathbb{R}^{n-1}} (1+|\xi'|^2)^{m-j} |D^j_n \hat{u}(\xi', x_n)|^2 d\xi' dx_n \tag{1.5}$$

Definition 1.3.2. For every, $0 \leq j \leq m - 1$, we define, $\gamma_j : C_0^\infty(\overline{\mathbb{R}_+}^n) \longrightarrow C_0^\infty(\overline{\mathbb{R}_+}^n)$ as,

$$(\gamma_j u)(x') := (D^j_n u)(x', 0) \tag{1.6}$$

1.4 Trace Theorem

Theorem 1.4.1. *The map γ_j (as defined in (1.6)) has in fact an extension from $H^m(\mathbb{R}^n_+)$ onto $H^{m-j-1/2}(\mathbb{R}^{n-1})$ as an operator which is both bounded and linear.*

Proof. Using (1.5), we can interpret the following for every $u \in C_0^\infty(\overline{\mathbb{R}_+}^n)$,

$$|D^j_n \hat{u_n}(\xi', 0)|^2 \leq \frac{1}{A} \int_0^\infty |D^{j+1}_n \hat{u_n}(\xi', x_n)|^2 dt + A \int_0^\infty |D^j_n \hat{u_n}(\xi', x_n)|^2 dt$$

Where, we take, $A = (1 + |\xi'|^2)^{1/2}$ for the purpose of this proof (in fact any value of A would work in general).

Therefore, multiplying both sides by $(1 + |\xi'|^2)^{m-j-1/2}$, we obtain,

$$\int_{\mathbb{R}^{n-1}} (1+|\xi'|^2)^{m-j-1/2} |D^j_n \hat{u_n}(\xi', 0)|^2 d\xi' \leq |u|^2_m$$

$$\text{i.e.,} \qquad ||\gamma_j u||_{m-j-1/2} \leq ||u||_m$$

Important to note that, such an extension of γ_j is unique, since, $C_0^\infty(\overline{\mathbb{R}_+}^n)$ is *dense* in $H^m(\mathbb{R}^n_+)$.

In order to establish *surjectivity*, assume any $\rho_j \in H^{m-j-1/2}(\mathbb{R}^{n-1})$ for j = 0...m−1. Define,

$\chi \in C_0^\infty(\mathbb{R})$ such that,

$$\chi = \begin{cases} 1, & \text{in } B(0,1/2) \\ 0, & \text{outside } B(0,1). \end{cases}$$

Now, we define,

$$\hat{u}(\xi', x_n) = \chi\left((1+|\xi'|^2)^{1/2}x_n\right)\sum_{j=0}^{m-1}\hat{\rho_j}(\xi')\frac{(ix_n)^j}{j!}$$

Where, $\hat{\rho_j}(\xi') = D_n^j\hat{u}(\xi', 0)$.

We claim that, $u \in H^m(\mathbb{R}^n_+)$.

A priori using Leibniz Rule, we can justify that, ∃ constants C_{kl} satisfying,

$$D_n^k\hat{u}(\xi', x_n) = \sum_{l=0}^{k}C_{kl}D_n^{k-l}\left\{\chi\left((1+|\xi'|^2)^{1/2}x_n\right)\right\}\left\{\sum_{j=l}^{m}\hat{\rho_j}(\xi')\frac{j(j-1)...(j-l)}{j!}(ix_n)^{j-l}\right\}.$$

Again, on *Supp*(χ), we have, $|x_n| \leq \frac{1}{(1+|\xi'|^2)^{1/2}}$. Thus, we get,

$$\left|x_n^{j-l}D_n^{k-l}\left(\chi\left((1+|\xi'|^2)^{1/2}x_n\right)\right)\right| \leq (1+|\xi'|^2)^{(k-j)/2}$$

Hence, ∃ a constant $c > 0$ such that,

$$\int_{\mathbb{R}^{n-1}}\int_0^\infty\left|D_n^k\hat{u}(\xi', x_n)\right|^2(1+|\xi'|^2)^{m-k}dx_nd\xi' \leq c\int_{\mathbb{R}^{n-1}}|\hat{\rho_j}(\xi')|^2(1+|\xi'|^2)^{m-j-1/2}d\xi'$$

$$\leq c||\rho_j||^2_{m-j-1/2}$$

It helps us conclude that,

$$|u|_m \leq c\sup_{0\leq j\leq m-1}||\rho_j||_{m-j-1/2}$$

Hence our proof is complete.

1.5 Integration by Parts

Theorem 1.5.1. *Assume any u, v* $\in H^1(\mathbb{R}^n_+)$. *Then,*

$$\int_{\mathbb{R}^n_+}(D_nu)v\ dx = -\int_{\mathbb{R}^n_+}uD_nv\ dx - i\int_{\mathbb{R}^{n-1}}\gamma_0(u)\gamma_0(v)dx' \tag{1.7}$$

Proof. Choose any two sequences u_k, $v_k \in C_0^\infty(\overline{\mathbb{R}_+}^n)$ satisfying, $|u_k - v_k| \to 0$ as $k \to \infty$.
Therefore, we obtain,

$$\int_{\mathbb{R}^n_+}(D_nu_k)v_k\ dx = i\int_{\mathbb{R}^{n-1}}\left\{\int_0^\infty\frac{\partial u_k}{\partial x_n}v_kdx_n\right\}dx'$$

$$= -\int_{\mathbb{R}^n_+}u_kD_nv_k\ dx - i\int_{\mathbb{R}^{n-1}}u_k(x',0)v_k(x',0)dx'$$

A priori from the fact that, $u_k \to u$, $v_k \to v$, $D_n u_k \to D_n u$, $D_n v_k \to D_n v$, $u_k(x', 0) \to \gamma_0(u)$, $v_k(x', 0) \to \gamma_0(v)$ as $k \to \infty$ in L^2. Hence, we obtain our desired result.

As a corollary to the above theorem, we can indeed deduce the following.

Corollary 1.5.2. *Suppose,* $H_0^1(\mathbb{R}_+^n) := \{u \in H^1(\mathbb{R}_+^n) \ : \ \gamma_0(u) = 0\}$. *is dense in* $H_0^1(\mathbb{R}_+^n)$.

2 Calculus on the Boundary of Sobolev Spaces

2.1 Domain with Smooth Boundary

Suppose, for $1 \leq i \leq n$ & $r_1, r_2 > 0$, and every $x = (x_1, x_2, ..., x_n)$, we denote,

$$x' := (x_1, x_2, .., x_{i-1}, x_{i+1}, ...x_n) \in \mathbb{R}^{n-1}$$

We define the following sets in $\mathbb{R}^{n-1}$ as,

$$B_i(x', r_1) := \{y \in \mathbb{R}^{n-1} \ : \ |x' - y| < r_1\}$$

$$V_i(x, r_1, r_2) = \{(y_1, y_2, ..., y_n) \in \mathbb{R}^n \ : \ |x' - y'| < r_1 \ , \ |x_i - y_i| < r_2\}$$

Using the above notations, we explain the concept of a domain with smooth boundary as follows.

Definition 2.1.1. $\Omega \subset \mathbb{R}^n$ is defined to be a domain with smooth boundary if for every $x_0 \in \partial\Omega$, $\exists\, r_1, r_2 > 0$, $i \in 1, 2, ..., n$ and a smooth function, $\psi_i : B_i(x', r_1) \to \mathbb{R}$ satisfying,

$$V_i(x_0, r_1, r_2) \cap \Omega = \{x \in V_i(x_0, r_1, r_2) \ : \ \psi_i(x') < x_i\} \tag{2.1}$$

or,

$$V_i(x_0, r_1, r_2) \cap \Omega = \{x \in V_i(x_0, r_1, r_2) \ : \ \psi_i(x') > x_i\} \tag{2.2}$$

and,

$$V_i(x_0, r_1, r_2) \cap \partial\Omega = \{x \ : \ \psi_i(x') = x_i\} = \{(x_1, ..., x_{i-1}, \psi_i(x'), x_{i+1}, ..., x_n) \ : \ x' \in B_i\} \tag{2.3}$$

Example 2.1.1. Consider $\Omega = \mathbb{R}_+^n$. Then, $\psi_n(x') = 0$, where, $x' = (x_1, x_2, ..., x_{n-1})$. Furthermore, we can deduce that, $r_1 = r_2 = \infty$.

Definition 2.1.2. (Outward Normal) Suppose, $x_0 \in \partial\Omega$ and, $r_1, r_2 > 0$, such that, (2.1) holds true.

For every $x \in V_i(x_0, r_1, r_2) \cap \partial\Omega$, we define,

$$\nu(x) = \frac{\nabla(-x_i + \psi_i(x'))}{|\nabla(-x_i + \psi_i(x'))|} = \frac{\left(\frac{\partial\psi_i}{\partial x_1}, ..., \frac{\partial\psi_i}{\partial x_{i-1}}, -1, \frac{\partial\psi_i}{\partial x_{i+1}}, ..., \frac{\partial\psi_i}{\partial x_n}\right)}{\sqrt{1 + |\nabla\psi_i|^2}} \tag{2.4}$$

Subsequently,

$$|\nabla\psi_i|^2 = \left|\frac{\partial\psi_i}{\partial x_1}\right|^2 + \left|\frac{\partial\psi_i}{\partial x_2}\right|^2 + ... + \left|\frac{\partial\psi_i}{\partial x_{i-1}}\right|^2 + \left|\frac{\partial\psi_i}{\partial x_{i+1}}\right|^2 + ... + \left|\frac{\partial\psi_i}{\partial x_n}\right|^2 \tag{2.5}$$

v is denoted as the unit outward normal to $\partial\Omega$ at the point x.

Definition 2.1.3. (Measure on $\partial\Omega$) We define the measure $d\sigma$ on $V_i(x_0, r_1, r_2) \cap \partial\Omega$ as follows,

$$d\sigma = \sqrt{1 + |\nabla\psi_i|^2} dx_1 dx_2 ... dx_{i-1} dx_{i+1} ... dx_n \tag{2.6}$$

We can use the Integration by Parts (Theorem (1.5.1)) to infer about the relation of v and $d\sigma$ as defined above with the smooth boundary of a domain Ω.

Lemma 2.1.2. For any $\Omega \subset \mathbb{R}n$, the unit outward normal, ν and the measure $d\sigma$ as defined earlier are in fact independent of the description of $\partial\Omega$.

Theorem 2.1.3. *In any bounded domain $\Omega \subset \mathbb{R}$ with smooth boundary, we choose any $u, v \in C^1(\Omega) \cap C^0(\bar{\Omega})$. Then,*

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v dx = -\int_{\Omega} u \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} uv\nu_i d\sigma \tag{2.7}$$

Where, $\nu(x) = (\nu_1(x), \nu_2(x), ..., \nu_n(x))$ is the unit outward normal to $\partial\Omega$ and $d\sigma$ is the measure on $\partial\Omega$.

Proof. We shall introduce two important results which shall be used to prove this theorem.

Lemma 2.1.4. *Assume K to be the support of u as defined in the statement of the theorem, such that, K is compact in Ω. Furthermore, let, $\phi \in C^1_0(\Omega)$ be such that, $\phi = 1$ on K. Also, let $v_1 = \phi v$. Hence, $u, v_1 \in C^1_0(\mathbb{R}^n)$ and $Supp(u), Supp(v_1) \in \Omega$, and*

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v dx = -\int_{\Omega} u \frac{\partial v}{\partial x_i} dx$$

Lemma 2.1.5. *Let $x_0 \in \partial\Omega$, and, $r_1, r_2 > 0$, $1 \le i \le n$. A priori using the notations ψ_i, V_i as defined before,let us take, $\eta_1 < r_1$, $\eta_2 < r_2$ and, $u, v \in C^1(\Omega) \cap C^0(\bar{\Omega})$ satisfying, $u \subset V(x_0, \eta_1, \eta_2) \subset V(x_0, r_1, r_2)$. Then,*

$$\int_{\Omega} \frac{\partial u}{\partial x_j} v dx = -\int_{\Omega} u \frac{\partial v}{\partial x_j} dx + \int_{\partial\Omega} uv\nu_j d\sigma$$

Now we an indeed proceed to the proof of the main theorem.

Clearly, $\bar{\Omega}$ is *compact*, hence, it has a finite cover, say, $\bar{\Omega} \subset V^0 \cup_{\alpha=1}^{l} V^\alpha$, where, $\bar{V}^0 \subset \Omega$ is compact and, $V^\alpha = V_i^\alpha(x^\alpha, r_1^\alpha, r_2^\alpha), x^\alpha \in \partial\Omega$.

Suppose, $\{\chi_i\}_{i=0}^{l}$ be a *partition of unity* subordinate to this cover. Therefore, $u = \sum_{\alpha=0}^{l} u\chi_\alpha = \sum_{\alpha=0}^{l} u_\alpha$, where, we define, $u_\alpha = u\chi_\alpha$. As a result, $Supp(u_0) \subset V^0 \subset \Omega$ and, $Supp(u_\alpha) \subset V^\alpha$.

From the results mentioned in Lemmas (2.1.4) and (2.1.5), we obtain,

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v dx = \sum_{\alpha=0}^{l} \int_{\Omega} \frac{\partial u^\alpha}{\partial x_i} v dx$$

$$= \sum_{\alpha=0}^{l} \left(-\int_{\Omega} u^\alpha \frac{\partial v}{\partial x_i} dx \right) + \sum_{\alpha=0}^{l} \left(\int_{\partial\Omega} u^\alpha v\nu_i d\sigma \right)$$

$$= \int_{\Omega} \left(\sum_{\alpha=0}^{l} u^\alpha \right) \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} \left(\sum_{\alpha=0}^{l} u^\alpha \right) v\nu_i d\sigma$$

$$= -\int_{\Omega} u \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} uv\nu_i d\sigma$$

And, the proof is done.

2.2 Boundary of Sobolev Spaces

Suppose, Ω be a bounded domain with a smooth boundary. Let, $\partial\Omega \subset \cup_{\alpha=1}^{l} V^\alpha$, where, $V^\alpha := V_i^\alpha(x^\alpha, r_1^\alpha, r_2^\alpha)$ and assume $\exists\ \psi_i^\alpha : B(x^{\alpha'}, r_1^\alpha) \longrightarrow \mathbb{R}$ describing $\partial\Omega$ as in (2.1.1). Furthermore, let $\{\chi_\alpha\}$ be a C^∞-partition of unity subordinate to this cover. $u : \partial\Omega \to \mathbb{R}$ be a function. A priori under the assumptions as discussed in the previous section, we can infer that, $u = \sum u_\alpha$, where $u_\alpha = \chi_\alpha u$ has *support* in $V_i^\alpha(x^\alpha, r_1^\alpha, r_2^\alpha)$. Hence, we define another function $\tilde{u}_\alpha$ on $\mathbb{R}^{n-1}$ by,

$$\tilde{u}_\alpha(x) = u_\alpha(x_1, ..., x_{i-1}, \psi_i^\alpha(x'), x_{i+1}, ..., x_n)$$

Then, we give the follwing definition.

Definition 2.2.1. $u \in H^s(\partial\Omega)$ iff $\tilde{u}_\alpha \in H^s(\mathbb{R}^{n-1})$ and,

$$||u||_s = \sum_{\alpha=1}^{l} ||\tilde{u}_\alpha||_s$$

***Remark* 2.2.1.** Important to observe that, the norm as defined above depends upon the partition of unity and the cover $\{V^\alpha\}$ and any such two norms obtained by different covers and partitions of unity are in fact equivalent.

Using the definition of $H^s(\partial\Omega)$ and the Trace Theorem (Theorem (1.4.1)) in $\mathbb{R}^n_+$, the following theorem follows.

Theorem 2.2.2. *For a bounded domain Ω with smooth boundary, we define $\gamma_j : C^\infty(\bar{\Omega}) \to C^\infty(\partial\Omega)$ as,*

$$\gamma_0 u = u|_{\partial\Omega} \ \ if,\ j = 0$$
$$\gamma_j u = \frac{\partial^j u}{\partial^j \nu}|_{\partial\Omega} \ \ if,\ j > 0$$

Where, $\frac{\partial}{\partial\nu} = \sum_{l=1}^{n} \nu_l \frac{\partial}{\partial x_l}$, *$\nu(x)$ being the unit outward normal to $\partial\Omega$ at x.*

Let, $m \in \mathbb{N}$ and $0 \leq j \leq m - 1$, therefore, γ_j extends to a continuous surjective map from $H^m(\Omega)$ to $H^{m-j-1/2}(\partial\Omega)$. Moreover, if

$$H_0^m(\Omega) = \{u \in H^m(\Omega) \ : \ \gamma_j u = 0 \ ,\ 0 \leq j \leq m-1\}$$

*Then, $C_0^\infty(\Omega)$ is **dense** in $H_0^m(\Omega)$.*

Corollary 2.2.3. *Suppose, $u, v \in H^1(\Omega)$, then,*

$$\int_\Omega \frac{\partial u}{\partial x_i} v dx = -\int_\Omega u \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)\nu_i d\sigma$$

i.e., if, $u \in H_0^1(\Omega)$, then,

$$\int_\Omega \frac{\partial u}{\partial x_i} v dx = -\int_\Omega u \frac{\partial v}{\partial x_i} dx$$

Remark 2.2.4. The above formula in Corollary (2.2.3) holds true for C^1-functions and thus by approximation process, the same holds for $H^1(\Omega)$.

3 Boundary Value Problems

3.1 Introducing Boundary Conditions on Solutions

Let, $\Omega \subset \mathbb{R}^n$ be open, and $a_{ij} \in C^1(\bar{\Omega})$, $c \in C^0(\bar{\Omega})$. Furthermore, suppose, $a_{ij} = a_{ji}$, and they satisfy the uniform ellipticity condition,

$$m|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq M|\xi|^2 \ ,\ \forall\ x \in \bar{\Omega}\ ,\ \xi \in \mathbb{R}^n \tag{3.1}$$

for some $m, M > 0$. Let, $f \in C^0(\bar{\Omega})$, $g \in C^1(\partial\Omega)$, and $u \in C^2(\bar{\Omega})$ be satisfying the following,

$$\mathcal{L}u + c(x)u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u = f \tag{3.2}$$

with the following two kinds of boundary conditions imposed on $\partial\Omega$ as follows,

$$(1) \quad u = g \quad \text{on } \partial\Omega \text{ (\textbf{Dirichlet Boundary Condition})} \tag{3.3}$$

$$(2) \quad \frac{\partial u}{\partial \nu_{\mathcal{L}}} = \sum a_{ij}\nu_i \frac{\partial u}{\partial x_j} = g \quad \text{on } \partial\Omega \text{ (\textbf{Neumann Boundary Condition})} \tag{3.4}$$

Therefore, multiplying (3.2) by $v \in H^2(\bar{\Omega})$ and integrating by parts, it yields,

$$\int_\Omega fv = -\sum_{i,j=1}^{n} \int_\Omega \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) v + \int_\Omega c(x)v$$

$$= \sum_{i,j=1}^{n} \int_\Omega a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \int_\Omega c(x)uv - \int_{\partial\Omega} \frac{\partial u}{\partial \nu_{\mathcal{L}}} v d\sigma$$

Interested readers may ask certain questions like, whether a solution to (3.2) really exists for given f, g and under the so called boundary conditions as descibed in (3.3) and (3.4). Furthermore, if a solution does in fact exists, one might also raise a question as to whether it is unique!

Remark 3.1.1. Existence of solutions to (3.2) in $C^2(\bar{\Omega})$ is not true in general. Hence, we relax the condition to a certain extent and look for solution in a much bigger space, i.e., the Sobolev Space. There, we consider only the *Dirichlet Problem* and the other boundary conditions can bemodified in a similar manner.

3.2 Weak Formulation of a Dirichlet Problem

A priori from the fact that, the operator $\mathcal{L}$ maps $H^2(\Omega)$ into $L^2(\Omega)$ and γ_0 maps $H^2(\Omega)$ into $H^{3/2}(\partial\Omega)$. It allows us to reformulate the problem in the following manner:

Given $f_1 \in L^2(\Omega)$, $g \in H^{3/2}(\partial\Omega)$, we intend to find a $u_1 \in H^2(\Omega)$ satisfying,

$$\begin{cases} \mathcal{L}u_1 + c(x)u_1 = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u_1}{\partial x_j}\right) + c(x)u_1 = f_1 & \text{in } \Omega \\ \gamma_0 u_1 = g & \text{on } \partial\Omega. \end{cases}$$

As $\gamma_0 : H^2(\Omega) \longrightarrow H^{3/2}(\partial\Omega)$ is surjective, thus, $\exists\, u_0 \in H^2(\Omega)$ with $\gamma_0 u_0 = g$. Suppose, $u = u_1 - u_0$.

Hence, u satisfies,

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + cu = f_1 - \mathcal{L}u_0 - cu_0 & \text{in } \Omega \\ \gamma_0 u = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, putting $f = f_1 - \mathcal{L}u_0 - cu_0$, the problem further reduces to obtaining one such $u \in H^2(\Omega)$ such that,

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + cu = f & \text{in } \Omega \\ \gamma_0 u = 0 & \text{on } \partial\Omega. \end{cases}$$

It suffices to look for a solution to the problem,

$$\begin{cases} u \in H^2(\Omega) \cap H_0^1(\Omega) & , \\ -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + cu = f & \text{in } \Omega. \end{cases} \qquad .....(i)$$

Multiplying (i) by $v \in H_0^1(\Omega)$ and applying Integration by Parts, we obtain,

$$\int_{\Omega} \left(a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + cuv \right) dx = \int_{\Omega} fv \qquad (3.5)$$

Important to note that, u is a solution of (*i*) iff (3.5) is satisfied for every $v \in H_0^1(\Omega)$.

3.3 Weak Solution of a Dirichlet Problem

Using the above concept, we can introduce the notion of weak solution for a boundary value problem as follows.

Definition 3.3.1. (Weak Solution) Suppose, $u \in H_0^1(\Omega)$. We define u to be a weak solution of the problem,

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + cu = f & \text{in } \Omega \\ \gamma_0 u = 0 & \text{on } \partial\Omega. \end{cases}$$

iff $\forall\ v \in H_0^1(\Omega)$, (3.5) holds true.

Remark 3.3.1. In case for weak solution, the assumption that, $u \in H^2(\Omega)$ can be relaxed. Further, using the above definition, we can in fact search for answers to some critical questions related to the existence and uniqueness of a weak solution. Even, one can comment on some regularity properties of the weak solution in $H^2(\Omega)$.

We shall introduce the following results in order to discuss these concepts in the next section.

Lemma 3.3.2. *(Poincare's Lemma) ∃ a constant c > 0 which satisfies the following relation for every* $u \in H_0^1(\Omega)$,

$$\int_{\Omega} |u|^2 dx \le c \int_{\Omega} |\nabla u|^2 dx \qquad (3.6)$$

Proof. A priori from the fact that, Ω is *bounded*, $\exists\ M > 0$ such that,

$$\Omega \subset \{x\ :\ -M \le x_i \le M\ ,\ i = 1, 2, ..., n\}$$

Suppose, $u \in C_0^\infty(\Omega)$, thus, for $x \in \Omega$,

$$u(x_1, x_2, ..., x_n) = \int_{-M}^{x_1} \frac{\partial u}{\partial t}(t, x_2, ..., x_n) dt$$

Hence,

$$|u|^2 \le (x_1 + M) \int_{-M}^{x_1} \left|\frac{\partial u}{\partial t}\right|^2 dt \le 2M \left|\frac{\partial u}{\partial t}(t, x_2, ..., x_n)\right|^2 dt$$

We thus conclude,

$$\int_{\Omega} |u(x)|^2 dx \le (2M)^2 \int_{\Omega} |\nabla u|^2 dx$$

And the proof is complete.

As a corollary to the above result, we can infer the following.

Corollary 3.3.3. *Given any u, v ∈* $H_0^1(\Omega)$ *and,* $c(x) \ge 0$, $c \in L^\infty(\Omega)$*, suppose,*

$$a(u, v) = \int_{\Omega} \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + cuv \right) dx$$

Then, a satisfies the following conditions:
(i) (Symmetry) a(u, v) = a(v, u).
(ii) (Continuity) $\exists\ c_1 > 0$ *such that,* $|a(u, v)| \leq c1||u||1||v||1$.
(iii) (Coercive) $\exists\ c_2 > 0$ *such that,* $a(u, u) \geq c_2||u||^2_1$, *where,* $||.||1$ *is the norm defined on* $H^1_0(\Omega)$.

Lemma 3.3.4. ***(Lax-Milgram Lemma)*** *Suppose H be a Hilbert Space and* $a : H \times H \to H$ *be symmetric, continuous bilinear map. Furthermore, let us assume that,* $\exists\ c > 0$ *such that,*

$$a(u, u) \geq c||u||^2 \quad \forall u \in H.$$

Then, for every $l \in H^*$, $\exists!\ u_0 \in H$ *satisfying,*

$$a(u_0, v) = l(v)$$

$\forall$ v $\in$ H.

Proof. H being an inner product space with respect to the bilinear form a(., .), which also happens to be both continuous and coercive (Corollary (3.3.3)), we can conclude that, the two norms, $||.||$ and $||.||_1$ are indeed equivalent.

Therefore, *H* with the inner product a is complete and, subsequently is a *Hilbert Space*.

Applying Riesz Representation Theorem, for every $l \in H^*$, $\exists\ u_0 \in H$ such that,

$$a(u_0, v) = l(v) \qquad \forall v \in H \text{ holds true.}$$

This eventually estblishes the statement of the lemma.

3.4 Existence and Uniqueness of Weak Solution

Theorem 3.4.1. *Consider any* a_{ij} , $c \in L^\infty(\Omega)$, *such that,* $\exists\ m,M > 0$ *satisfying,*
(a) $c \geq 0$.
(b) $m|\xi|^2 \leq \Sigma\, a_{ij}(x)\xi_i\xi_j \leq M|\xi|^2$, $\forall\ x \in \Omega,\ \xi \in \mathbb{R}^n$.

Furthermore, if $f \in L^2(\Omega)$, *then* $\exists!$ *weak solution* $u \in H^1_0(\Omega)$ *of the problem,*

$$\begin{cases} -\sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + cu = f & in\ \Omega \\ \gamma_0 u = 0 & on\ \partial\Omega. \end{cases}$$

Proof. Let us choose, $v \in H^1_0(\Omega)$ and,

$$a(u, u) = \int\limits_\Omega \left(\sum a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + cu\right)$$

$$l(v) = \int\limits_\Omega fv$$

Hence, we have,

$$|l(v)| \leq |f|_{L^2(\Omega)}|v|_{L^2(\Omega)} \leq |f|_{L^2(\Omega)}||v||_1$$

4 Hm-Regularity for Second Order Elliptic Equations

In this section, we shall discuss the existence of the weak solution, $u \in H^1_0(\Omega)$ in the space $H^m(\Omega)$. For our simplicity, we shall consider a special case, $m = 2$ for our analysis.

4.1 Interior Regularity

Given $h \in \mathbb{R}$, $e_i := (0, 0, ..., 0, 1, 0, ..., 0)$, for $1 \leq i \leq n$. Moreover, for every $u \in \mathbb{R}^n$, we define,

$$(\tau_h^i u)(x) := \frac{u(x+he_i)-u(x)}{h} \tag{4.1}$$

Lemma 4.1.1. (*I*) *For every* $u \in H^1(\mathbb{R}^n)$*, and* $1 \leq i \leq n$*,*

$$||\tau_h^i u||_{L^2} \leq ||\nabla u||_{L^2}$$

(*II*) *For every* $u \in H^1(\mathbb{R}_+^n)$*, and* $1 \leq i \leq n-1$*,*

$$||\tau_h^i u||_{L^2} \leq ||\nabla u||_{L^2}$$

Theorem 4.1.2. ***(Interior Regularity)*** *A priori given* $a_{ij} \in C^1(\mathbb{R}^n)$*,* $u \in H^1(\mathbb{R}^n)$ *such that, Supp(u) is indeed compact in* $\mathbb{R}^n$*. Then,* $f \in L^2(\mathbb{R}^n)$*, and moreover,* $\exists\ m > 0$ *satisfying,*

1.

$$m|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j\ , \qquad \forall x \in \bar{\Omega}$$

Where, Ω *is an open set such that, Supp(u)* $\subset \Omega$*.*

2. $\forall\ v \in H^1(\mathbb{R}^n)$,

$$\sum_{i,j}\int_{\mathbb{R}^n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} = \int_{\mathbb{R}^n} fv$$

Then, $u \in H^2(\mathbb{R}^n)$.

Proof. To prove the above Theorem, one can in fact proceed through the following steps systematically, using the results and concepts which we shall briefly discuss below.

Lemma 4.1.3. $\exists\ c, c_0 > 0$ *satisfying,*

$$\left|\left|\tau_h^i \frac{\partial u}{\partial x_j}\right|\right|_{L^2} \leq c\ , \qquad \forall \quad 0 < |h| < c_0\ , \quad 1 \leq i,j \leq n$$

From the above Lemma (4.1.3), we can observe that, $\left\{\tau_h^i \frac{\partial u}{\partial x_j}\right\}_{0<|h|<c_0}$ is indeed bounded in L^2. Thus, for any subsequence, $\tau_{h_k}^i \frac{\partial u}{\partial x_j}$ converging weakly to v_j^i in $L^2(\mathbb{R}^n)$, and for every $\phi \in C_0^\infty(\mathbb{R}^n)$, we shall have,

$$\int_{\mathbb{R}^n} \tau_{h_k}^i \frac{\partial u}{\partial x_j}\phi dx = \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_j}\left\{\frac{\phi(x-h_ke_i)-\phi(x)}{h_k}\right\} dx$$

As $k \to \infty$, we apply dominated convergence theorem to deduce,

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = v_j^i \in L^2(\mathbb{R}^n)$$

Hence, we conclude that, $u \in H^2(\mathbb{R}^n)$, and the proof is complete.

We can indeed generalize the above result to comment on the H^m-Interior Regularity for a Dirichlet problem.

Corollary 4.1.4. *Assume that,* $a_{ij} \in C^{m+1}(\mathbb{R}^n)$*,* $f \in H^m(\mathbb{R}^n)$*,* $u \in H^1(\mathbb{R}^n)$ *,* $\forall\ 1 \leq i, j \leq n$ *satisfy the same hypothesis as described in Theorem (4.1.2). Then,* $u \in H^{m+2}(\mathbb{R}^n)$*.*

In addition, one can further deduce the following.

Corollary 4.1.5. *If a_{ij}, $f \in C^\infty(\mathbb{R}^n)$, $\forall$ $1 \le i, j \le n$, then, $u \in C^\infty(\mathbb{R}^n)$.*

4.2 Boundary Regularity

Theorem 4.2.1. *Suppose, $a_{ij} \in C^1(\overline{\mathbb{R}_+}^n)$, $u \in H_0^1(\overline{\mathbb{R}_+}^n)$ satisfying, Supp(u) is compact in $\overline{\mathbb{R}_+}^n$. Moreover, let $f \in L^2(\mathbb{R}^n)$ and $\exists$ $m > 0$ such that,*

1.

$$m|\xi^2|^2 \le \sum a_{ij}(x)\xi_i\xi_j \ , \qquad \forall x \in \bar{\Omega}$$

Where, Ω is an open set in $\overline{\mathbb{R}_+}^n$ such that, Supp$(u) \subset \Omega$.

2. $\forall\ v \in H^1(\mathbb{R}^n)$,

$$\sum_{i,j} \int_{\mathbb{R}_+^n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} = \int_{\mathbb{R}_+^n} fv$$

Then, $u \in H^2(\overline{\mathbb{R}_+}^n)$.

Proof. Suppose, $0 < |h| < \epsilon_0$ satisfies that, for $i \le k \le n-1$, the support of $x \mapsto u(x + he_k)$ is in fact contained in $\bar{\Omega}$ [Since, $k \le n-1$].

A priori using concepts similar tothe case for interior regularity, it follows that,

$$||\tau_h^k \nabla u||_{L^2(\mathbb{R}_+^n)} \le c \ , \quad 0 < |h| < \epsilon_0 \ , \quad k \le n-1.$$

Thus, for a subsequence $h_k \to 0$, we obtain,

(i) $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2$, $\forall$ $1 \le i, j \le n-1$.

(ii) $\frac{\partial^2 u}{\partial x_n \partial x_i} \in L^2$, $\forall$ $1 \le i, j \le n-1$.

Therefore, it only suffices to prove that, $\frac{\partial^2 u}{\partial x_n^2} \in L^2$. Let, $\phi \in C_0^1(\overline{\mathbb{R}_+}^n)$. Hence,

$$\int_{\mathbb{R}_+^n} a_{nn} \frac{\partial u}{\partial x_n} \frac{\partial \phi}{\partial x_n}$$

$$= -\int_{\mathbb{R}_+^n} \left\{ \sum_{i,j \le n-1} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_j} \right\} - \sum_{j \le n-1} \left\{ \int_{\mathbb{R}_+^n} a_{nj} \frac{\partial u}{\partial x_j} \frac{\partial \phi}{\partial x_n} + \int_{\mathbb{R}_+^n} a_{jn} \frac{\partial u}{\partial x_n} \frac{\partial \phi}{\partial x_j} \right\} + \int_{\mathbb{R}_+^n} f\phi$$

$$= \int_{\mathbb{R}_+^n} F\phi$$

Where,

$$F = \left\{ \sum_{i,j \le n-1} \frac{\partial a_{ij}}{\partial x_j} \frac{\partial u}{\partial x_i} + a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \sum_{j \le n-1} \frac{\partial a_{nj}}{\partial x_n} \frac{\partial u}{\partial x_j} + \sum_{j \le n-1} \frac{\partial a_{jn}}{\partial x_j} \frac{\partial u}{\partial x_n} + 2a_{jn} \frac{\partial^2 u}{\partial x_j \partial x_n} + f \right\}$$

Therefore,

$$\frac{\partial}{\partial x_n}\left(a_{nn}\frac{\partial u}{\partial x_n}\right) = F \in L^2$$

i.e.,

$$-a_{nn}\frac{\partial^2 u}{\partial x_n^2} = F + \frac{\partial a_{nn}}{\partial x_n}\frac{\partial u}{\partial x_n} \in L^2$$

Applying the concept that, $a_{nn} > 0$ in $\bar{\Omega}$ and therefore, $\frac{\partial^2 u}{\partial x_n^2} \in L^2$, and the proof.

As a corollary, one can establish the following.

Corollary 4.2.2. *(**Higher Regularity**) Assuming $a_{ij} \in C^{m+1}$, $f \in H^m(\mathbb{R}^n_+)$, $u \in H^1_0(\mathbb{R}^n_+)$ satisfying similar conditions as*
above. Then, $u \in H^{m+2}(\mathbb{R}^n_+)$.

Corollary 4.2.3. *Let, $a_{ij} \in C^\infty, f \in C^\infty(\mathbb{R}^n_+)$, then, $u \in C^\infty$.*

4.3 A Special Diffeomorphism on Second Order Elliptic Operators

Theorem 4.3.1. *Given Ω_1 and Ω_2 being bounded open sets, and $F : \Omega_1 \to \Omega_2$ be a **diffeomorphism**. We further define, $J(F)$ $:= \left(\frac{\partial F_k}{\partial y_i}\right)_{i,k}$, such that, $J(F)$ and $J^{-1}(F)$ exist $\forall\ y \in \bar{\Omega}_1$ and **phism**. We further define, $J(F) := \left(\frac{\partial F_k}{\partial y_i}\right)_{i,k}$, such that, $J(F)$*

and $J^{-1}(F)$ exist $\forall\ y \in \bar{\Omega}_1$ and is continuous there. Then F transforms any second order elliptic operator $\mathcal{L}$ on Ω_1 to a second order elliptic operator on Ω_2.

Proof. A priori given any two bounded domains ω_1 and Ω_2, and a diffeomorphism defined as,

$$F := (F_1, F_2, ..., F_n)\ :\ \bar{\Omega}_1 \to \bar{\Omega}_2$$

We an in fact observe that, F is C^∞, one-one and onto and the inverse is indeed differentiable.

Furthermore, for any function u on Ω_2, we define $\tilde{u} : \Omega_1 \to \mathbb{R}$ by,

$$\tilde{u}(y) := u(F(y))\ ,\ y \in \Omega_1$$

Consequently,

$$\frac{\partial \tilde{u}}{\partial y_i} = \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}(F(y))\frac{\partial F_k}{\partial y_i}(y)$$

and,

$$\frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j} = \sum_{k,l=1}^{n} \frac{\partial^2 u}{\partial x_l \partial x_k}(F(y))\frac{\partial F_l}{\partial y_i}\frac{\partial F_k}{\partial y_j} + \sum_{k=1}^{n}\frac{\partial u}{\partial x_k}(F(y))\frac{\partial^2 F_k}{\partial y_i \partial y_j}.$$

Let,

$$\mathcal{L} := \sum \tilde{a}_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j} + \sum \tilde{b}_i(y)\frac{\partial}{\partial y_i} + \tilde{c}(y)$$

be any elliptic differential operator on Ω_1. Hence, in Ω_2, we can in fact deduce that,

$$\mathcal{L} := \sum a_{il}(y)\frac{\partial^2}{\partial x_k \partial x_l} + \sum b_k(y)\frac{\partial}{\partial x_k} + c(x). \tag{4.2}$$

Where,

$$a_{kl}(x) = \sum_{i,j} \tilde{a}_{ij}(F^{-1}(x))\frac{\partial F_k}{\partial y_i}(F^{-1}(x))\frac{\partial F_l}{\partial y_j}(F^{-1}(x))$$

and,

$$b_k(x) = \sum_i \tilde{b}_i(F^{-1}(x))\frac{\partial F_k}{\partial y_i}(F^{-1}(x)) + \sum_{i,j} \tilde{a}_{ij}(F^{-1}(x))\frac{\partial^2 F_k}{\partial y_i \partial y_j}(F^{-1}(x))$$

and,

$$c(x) = \tilde{c}(F^{-1}(x))$$

One can indeed verify the neccessary conditions for $\mathcal{L}$ to be an *elliptic differential operator* on Ω_2.

Suppose, for $r_1, r_2 > 0, x' := (x_1, x_2, ..., x_{n-1})$, we define, $x_0 := (x'_0, x^0_n) \in \mathbb{R}^n$. Furthermore, $\psi : \overline{B(x'_0, r_1)} \to \mathbb{R}$ be a *smooth map* such that, $x^0_n = \psi(x'_0)$. Let,

$$\Omega_1 := \left\{x \mid x' \in B(x'_0, r_1)\ ,\ \psi(x') < x_n < r_2\right\}$$

and,

$$\Omega_2 := \left\{y \mid y' \in B(x'_0, r_1)\ ,\ 0 < y_n < r_2 - \psi(x')\right\}$$

We define,

$$F := (F_1, F_2, ..., F_n) : \bar{\Omega}_1 \to \bar{\Omega}_2$$

by,

$$\begin{cases} F_i(x) = x_i & 1 \leq i \leq n-1 \\ F_n(x) = x_n - \psi(x'). \end{cases}$$

Putting $y = F(x)$, we can deduce,

$$B(x'_0, r_1) = B(y'_0, r_1)\ , \qquad 0 < y_n < r_2 - \psi(y')$$

and,

$$JF(x) = \left(\frac{\partial F}{\partial x_i}\right) = \left\{ \begin{matrix} I_{(n-1)\times(n-1)} & 0 \\ -\nabla\psi & 1 \end{matrix} \right\}.$$

Moreover, the diffeomorphism F maps $\{x : x_n = \psi(x')\}$ to $\{y : y_n = 0\}$.

Using the above concept, we can indeed conclude the following result.

Theorem 4.3.2. *Given a bounded domain Ω with smooth boundary, suppose we have the following second order elliptic operator,*

$$\mathcal{L} = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c \tag{4.3}$$

where, the coefficients satisfy, $a_{ij} \in C^1(\bar{\Omega}), b_i, c \in L^\infty(\Omega)$ *and the ellipticity condition given by,*

$$m|\xi|^2 \leq \sum a_{ij}\xi_i\xi_j \leq M|\xi|^2 \;, \quad \forall\; \xi \in \mathbb{R}^n. \tag{4.4}$$

for some $m, M > 0$. *Let,* $f \in L^2(\Omega)$ *and,* $u \in H_0^1(\Omega)$ *be a* ***weak solution*** *of the problem,*

$$\mathcal{L}u = f \qquad in\ \Omega \tag{4.5}$$

Then, $u \in H^2(\Omega)$.

Proof. Suppose, $\{V^\alpha\}$ be an *open cover* satisfying,

(*i*) $\overline{V^0} \subset \Omega$.

(*ii*) $V^\alpha = V_i(x^\alpha, r_1^\alpha, r_2^\alpha)$, and, $\psi_i^\alpha : \overline{B(x^\alpha, r_1^\alpha)} \to \mathbb{R}$ be a smooth function such that,

$$\Omega \cap V^\alpha = \left\{x \in V_i^\alpha \mid x' \in B(x^\alpha, r_1^\alpha), \psi^\alpha(x') < x_i < r_2^\alpha\right\},$$

$$\partial\Omega \cap V^\alpha = \left\{x \in V_i^\alpha \mid \psi^\alpha(x') = x_i\right\}$$

Let, $F_\alpha : \overline{\Omega \cap V^\alpha} \to \overline{\mathbb{R}_+}^n - \{y : y_i > 0\}$ be defined as,

and,

$$W^\alpha = F_\alpha(\overline{\Omega \cap V^\alpha}) = \left\{y \mid y_i' \in B(x^\alpha, r_1^\alpha), 0 < y_i < r_2^\alpha - \psi(y')\right\}.$$

Consider a C^∞ partition of unity as $\{\chi^\alpha\}_{\alpha=0}^l$ subordinate to the assumed covering such that, $u^\alpha = \chi^\alpha u$.

We can observe that, $u = \sum u^\alpha$ and, $Supp(u^\alpha) \subset V^\alpha$, where,

$$\frac{\partial}{\partial x_i}(u^\alpha) = \frac{\partial}{\partial x_i}(\chi^\alpha u) = \chi^\alpha \frac{\partial u}{\partial x_i} + u\frac{\partial \chi^\alpha}{\partial x_i}$$

As a result, we can obtain $b_i^\alpha, c^\alpha \in L^\infty(\Omega)$, $Supp(b_i^\alpha), c^\alpha \subset \overline{V^\alpha}$ such that,

$$\mathcal{L}u^\alpha = \chi^\alpha \mathcal{L}(u) + \sum b_i^\alpha \frac{\partial u}{\partial x_i} + c^\alpha u$$

$$= \chi^\alpha + \sum b_i^\alpha \frac{\partial u}{\partial x_i} + c^\alpha u = f^\alpha \in L^2(V^\alpha)$$

Clearly, we have, $Supp(f^\alpha) \subset Supp(\chi^\alpha) = K^\alpha$ is *compact*. Moreover, $Supp(u^\alpha) \subset Supp(\chi^\alpha) = K^\alpha$.

For $\alpha = 0$, we have, $Supp(u^0) \subset K^0 \subset \Omega$ and, $\mathcal{L}u^0 = f^0$ in Ω. Thus, $\mathcal{L}u^0 = f^0$ in $\mathbb{R}^n$.
A priori from the fact that, $\mathcal{L}$ is indeed *elliptic* in Ω, hence, by **interior regularity** property,

we conclude, $u^0 \in H^2(\Omega)$.

In case when, $\alpha \neq 0$, we define, $F^\alpha : (\overline{\Omega \cap V^\alpha}) \to \overline{W^\alpha}$ such that,

$$\tilde{u}^\alpha(y) = u^\alpha((F^\alpha)^{-1}(y)).$$

Therefore, $\tilde{u}^\alpha \in H_0^1(W^\alpha)$ implying, $Supp(\tilde{u}^\alpha)$ is in fact compact in $\overline{\mathbb{R}_+}^n$. Consequently, $\tilde{u}^\alpha$ is a *weak solution* of a second order uniformly elliptic operator $\mathcal{L}^\alpha$, in other words,

$$\begin{cases} \mathcal{L}^\alpha \tilde{u}^\alpha = \tilde{f}^\alpha & \text{in } \mathbb{R}_+^n \\ \tilde{u}^\alpha \in H_0^1(\mathbb{R}_+^n). \end{cases}$$

Hence, $\tilde{u}^\alpha \in H^2(W^\alpha) \implies u^\alpha \in H^2(\Omega \cap V^\alpha) \implies u := \sum u^\alpha \in H^2(\Omega)$. The result is thus established.

As a corollary, we can further comment on the H^m-regularity for the case $m \geq 2$ as follows.

Corollary 4.3.3. *Suppose,* $f \in H^m(\Omega)$, $g \in H^{m+\frac{3}{2}}(\partial\Omega)$, *and* $u \in H^1(\Omega)$ *satisfies,*

$$\begin{cases} \mathcal{L}u = f & \textit{in } \Omega \\ \gamma_0 u = g & \textit{on } \partial\Omega \end{cases}$$

Then, $u \in H^{m+2}(\Omega)$.

Proof. A priori applying the fact that, $\gamma_0 : H^{m+2}(\Omega) \to H^{m+\frac{3}{2}}(\partial\Omega)$ is surjective, and hence, $\exists$ $u_1 \in H^{m+2}$ with, $\gamma_0(u_1) = g$. Let, $v = u - u_1$. then, $v \in H_0^1(\Omega)$. Moreover, $\mathcal{L}v = f - \mathcal{L}u_1 \in H^m(\Omega)$, and, $\gamma_0 v = 0$ onto $\partial\Omega$.

It follows that, u^α indeed satisfies the *Interior Regalarity* and *Boundary Regularity* conditions as described in Theorems (4.1.2) and (4.2.1) respectively. Therefore, we conclude that, $u^\alpha \in H^{m+2}(\Omega)$, $\forall \;\; \alpha$, implying that, $u \in H^{m+2}(\Omega)$. And the proof is thus complete. □

Statements and Declarations

Conflicts of Interest Statement

I as the sole author of this article certify that I have no affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affi liations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

Data Availability Statement

I as the sole author of this aarticle confirm that the data supporting the findings of this study are available within the article [and/or] its supplementary materials.

References

1. Bahuguna, D., Raghavendra, V., & Kumar, B. R. (Eds.). (2002). Topics in sobolev spaces and applications. Alpha Science Int'l Ltd.
2. Stein, E. M., & Shakarchi, R. (2011). Fourier analysis: an introduction (Vol. 1). Princeton University Press.
3. Hormander, L. (1969). Linear partial differential operators. Springer.
4. L. C. Evans. (1994). Partial Differential Equations, Vol. 1 & Vol. 2, Berkeley Mathematics Lecture Notes.
5. V. G. Mazya. (1985). Sobolev Spaces, Springer-Verlag, Springer Series in Soviet Mathematics.
6. Sobolev, S. L. (1936). On some estimates relating to families of functions having derivatives that are square integrable. In Dokl. Akad. Nauk SSSR (Vol. 1, No. 10, pp. 267-270).
7. Sobolev, S. L. (1938). On a theorem of functional analysis. Mat. Sbornik, 4, 471-497.
8. M. Struwe. (1990). Variational Methods : Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer-Verlag.
9. W. P. Ziemer. (1980). Weakly Differentiable Functions, Sobolev Spaces and Functions of Bounded Variation, Springer-Verlag.