Review Article

Current Research in Statistics & Mathematics

Measurable Functional Calculus and Spectral theory

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Submitted: 04 Jun 2022; **Accepted:** 20 Jun 2022; **Published:** 25 Jun 2022

Citations: *Mykola Ivanovich Yaremenko (2022). Measurable Functional Calculus and Spectral theory, Current Research in Statistics & Mathematics; vol 1(1): 01-07.*

Abstract

In this article, the spectral theory is considered, we study the spectral families and their correspondence to the operators on the reflexive Banach spaces; assume is a well-bounded operator on reflexive Lebesgue spaces then the operator A is a scalar type spectral operator. It is proven that if a weak spectral family $E(\lambda)$ is concentrated on [a,b] then there is a linear well-bounded operator $A \in L(X)$ on the reflexive Banach space X such that $\langle A(x), y^* \rangle = b\langle x, y^* \rangle - \int_{[a,b]} \langle x, E(\lambda) y^* \rangle d\lambda$ holds for all $x \in X$ and $y^* \in X^*$.

Key Words: Functional Calculus, Banach Space, Spectral Theorem, C*-Algebra, Measurable Space, Spectral Integral,Well-Bounded Operator.

Subject classification codes: 47B01, 46E10, 47B40, 46E15.

Introduction

This article is dedicated to the spectral theory of the operators that are defined on the subset of the reflexive Banachspace X. An important example of such operators is a class of well-bounded operators, which have spectral decomposition with special properties. Let us presume that the functional calculus defined on the Banach algebra of the absolutely continuous functions $AC([a,b])$ on a compactinterval $[a,b]$ then its operator is well-bounded. Assuming that the functional calculus of the well-bounded operator on L^P, $1<P<\infty$ space is contractive then this operator has a scalar-type spectral. The last statement is not true in the cases when the Banach spaces are not reflexive, for example, on L^∞ [2, 5, 9].

Let us considera simpler example of the theory in Banach space, the structure of the projection measure in the Hilbert space H. Let (Z, Σ, η) be a measurable Borel space and $\{H_Z\}_{z\in Z}$ be a η - measurable set of separable Hilbert spaces [11]. The projection-valued measure E on (Z,Σ) can be defined as a mapping from Σ to the set of self-adjoint orthogonal projections on H that satisfies $E(Z)=ID_H$,and the mapping from σ - algebra Σ into the field $\phi \mapsto (E(\phi)x, y)$ is a complex measure on Σ. In terms of the functional calculus this definition can be reformulated in the following form: let (Φ, H) be functional calculus on a measurable space (Z, Σ), the projection-valued measure is a mapping E: $\Sigma \to L(H)$, $E(\phi) = \Phi(\chi_B) \in L(H)$ for any $\phi \in \Sigma$ The main result of the theory for separable Hilbert spaces is the statement that for each projection-valued measure on the measurable space there is a unique measurable functional calculus that generates this projection-valued measure, and conversely, for each measurable functional calculus on a measurable space, there is a uniquely defined projection-valued measure [9].

In the present article, these results are developed and extended on the case of the reflexive Banach spaces. We show that presuming (Φ,X) is a functional calculus on the measurable space (Z, Σ) then there are a semi-finite measure space (Ω,F,μ) and operator $U:X \to L^P(\Omega,F,\mu)$, and an injective pointwise continuous *-homomorphism F: $M(Z,\Sigma) \to M(\Omega,F)$, such that $\Phi(f) = UM_{Ff} U^{-1}$, where M_{Ff} is the operator of the multiplication byfunction .An important result of the representation theory is the following statement that if the AC functional calculus of the operator is contractive then the operator can be represented as the integral with respect to a spectral measure.

The Spectral Decomposition for The Operator In Reflexive Banach Spaces

Some definitions and notations.The letter p denotes the scalar field usually real or complex numbers, the letters X,Y,Z denote reflexiveBanach spaces; $L(X)$ denotes the Banach algebra of all bounded linear operators on X , for any real compact interval $AC([a,b])$ denotes the Banach algebra of all absolutely continuous functions with its natural norm, and $BV([a,b])$ denotes the Banach algebra of all functions of bounded variation with its natural norm. It is easy to show thatif a function f belongs to $AC([a,b])$ thenthis function f necessarily belongs to $BV([a,b])$, however not reciprocally, there is such function $g \in BV([a,b])$, which $BV([a,b])$, in other words,the algebra $AC([a,b])$ is a proper subalgebra of the algebra $BV([a,b])$. Indeed, let $f \in AC([a,b])$ then for any $E<0$ there is $\delta>0$ such that for any sequence of disjointed intervals $\{(a_i,b_i)\}_{i=1,...,n}$, theproperty: that from $\sum_{i=1,...,n} b_i - a_i < \delta$ follows $\sum_{i=1,...,n} \|f(b_i) - f(a_i)\| < \varepsilon$ is satisfied. Let us divide the interval $[a,b]$ by points $a = \lambda_1 < \lambda_2 < ... < \lambda_n = b$ into parts in such a way that $\lambda_{i+1} - \lambda_i < \delta$ for $i=1,....,n-1$. Then for any division $\{(\sigma_{j+1},\sigma_j)\}_{j=1,\dots,m-1}$ of the interval $[\lambda_{i+1}-\lambda_i]$, on these parts, the sum $\sum_{j=1,\dots,n-1}\|f(\sigma_{j+1})-f(\sigma_j)\|$ is $\sum_{j=1,\dots,n-1}\|f(\sigma_{j+1})-f(\sigma_j)\|<\varepsilon$,so the ,variation of the function f on the interval $[\lambda_{i+1}-\lambda_i]$ is necessarily less than εn , thus the variation of the function f on the [a, b] interval is less than , so the function f∈ BV ([a,b]) .

Definition 1. ***Let*** *A: X→Y* ***be an operator defined on Banach spaces*** **X** ***then the operator*** A^**:* $Y^*\to X^*$ ***is called the adjoint operator to*** *A: X → Y* ***, namely,*** $(A^*(f))(x)=f(A(x))$ ***for all*** $f\in Y^*$ ***and all*** $x\in X$**.**
In particular, assuming X is a reflexive Banach space then if operator $A: X\to X$ then the adjoint operator is $A^*:X^*\to X$ if operator $A: X\to X^*$ then the adjoint operator is $A^*:X^*\to X$.

Definition 2. ***Let operator*** *A: X→ X* ***then the set*** *ρ (A)* ***of all complex numbers such that***

$$\rho(A)=\{\lambda\in\mathbb{C}:\ \lambda I-A\quad has\quad inverse\}$$

is called the resolvent set.
The complement σ(*A*) to the resolvent set is a spectrum of the operator *A: X→ X*.
The operator $R(\lambda, A)=(\lambda I-A)^{-1}$ is called a resolvent of the operator *A* .

Definition 3. ***The set*** { *E* (λ), λ ∈ ℝ } ***of projection operators that satisfies the following***

1. $E(\lambda)E(\mu)=E(\mu)E(\lambda)=E(\lambda)$ ***for*** $\lambda\le\mu$ ***; and*** $\sup_\lambda\|E(\lambda)\|<\infty$ ***;***
2. $E(\lambda)=strong-\lim_{\mu\to\lambda}E(\mu)$ ***;***
3. $strong-\lim_{\lambda\to-\infty}E(\lambda)=O$ ***and*** $strong-\lim_{\lambda\to\infty}E(\lambda)=I$ ***;***
4. $A=\int_{\mathbb{R}}\lambda\,dE(\lambda)=strong-\lim_{N\to\infty}\int_{[-N,N]}\lambda\,dE(\lambda)$

is called the spectral family of the operator ***A*** **.**

Condition 1 is a definition of the projection, which means the operator *E*(λ) is a projection onto the subspace *X* of created by all eigenvectors corresponding to all eigenvalues that are no larger than λ .
An operator *A* ***can be written as***

$$A=\int_{\sigma(A)}\lambda\,dE(\lambda),$$

where is a spectral family of *A* , all limits are understood as limits with respect to the natural topologies. This integral is anoperator-valued Riemann-Stieltjes integral in the topology of the operator norm.
Let us consider the integral $\int_{[a,b]}f(\lambda)\,dE(\lambda)$ as an operator-valued Riemann-Stieltjes integral.
We can build a partition P of the compact interval $[a,b]$ as $a=\lambda_0<\lambda_1<\dots\lambda_n=b$ and the direction of the partition $|P|=\max_{i=1,\dots,n}|\lambda_i-\lambda_{i-1}|$ then if for any chosen set $\{\xi_i\}_{1,\dots,n}$ of points $\xi_i\in[\lambda_{i-1},\lambda_i]$ there is a limit

$$\lim_{|P|\to0}\sum_{i=1,\dots,n}f(\xi_i)\big(E(\lambda_i)-E(\lambda_{i-1})\big),$$

and this limit is independent of the specifics of the partitions, this limit is called the Riemann-Stieltjes integral of the continuous function ,and can be written as

$$\int_{[a,b]}f(\lambda)\,dE(\lambda)=\lim_{|P|\to0}\sum_{i=1,\dots,n}f(\xi_i)\big(E(\lambda_i)-E(\lambda_{i-1})\big).$$

Theorem 1. ***For the existence of the integral***

$$I=\int_{[a,b]}f(\lambda)\,dE(\lambda),$$

it is necessary and sufficient that

$$\lim_{|P|\to0}\sum_{i=1,\dots,n}\left(\sup_{\xi_i\in[\lambda_{i-1},\lambda_i]}f(\xi_i)-\inf_{\xi_i\in[\lambda_{i-1},\lambda_i]}f(\xi_i)\right)\big(E(\lambda_i)-E(\lambda_{i-1})\big)=0.$$

Theproof of this theorem is rather standard: first is building the upper and lower Darboux-Stieltjes sums and finding their difference next showing that the conditions of the theorem are the necessary and sufficient conditions that the difference between the upper and lower Darboux-Stieltjes's sums converges to zero.

Theorem 2. ***If the function*** f ***is continuous and*** $\|E(\lambda)\|$ ***belongs to*** *B*V([a,b]) ***as a function of*** λ ***then the integral*** $I=\int_{[a,b]}f(\lambda)\,dE(\lambda)$ exists.
This theorem is the consequence of theorem 1.

Theorem 3. If the function $f\in AC$ ([a,b]) then the integral $I=\int_{[a,b]}f(\lambda)\,dE(\lambda)$ exists.
Proof. For any $f\in AC$ ([a,b]) the mapping

$$\Psi(f)=f(a)E(a)+\int_{[a,b]}f(\lambda)\,dE(\lambda)$$

is defined the homomorphism Ψ(*f*): *AC*([a,b]) →*L(X)* for which the following estimation

$$\|\Psi(f)\|\le\sup_{\lambda\in[a,b]}\|E(\lambda)\|\Big(|f(b)|+\operatorname*{var}_{[a,b]}f\Big)$$

holds for all $f\in AC$ ([a,b]) .

Lemma 1. ***Let*** $f\in AC$ ([a,b]) ***and let*** φ ***be a continuous function of the real argument*** *t* ***defined on*** [a,b] ***then***

$$Stieltjes\int_{[a,b]}\varphi(t)\,d\,f(t)=Lebesque\int_{[a,b]}\varphi(t)\,f'(t)\,dt.$$

Proof. The existence of both integrals is obvious.
By definition, the Stieltes integral is the limit of the following integral sums

$$\sum_{i=1,\dots,n}\varphi(\xi_i)\big(f(t_i)-f(t_{i-1})\big).$$

Since

$$f(t_i)-f(t_{i-1})=\int_{[t_{i-1},t_i]}f'(t)d$$

we have

$$\sum_{i=1,\dots,n}\varphi(\xi_i)\big(f(t_i)-f(t_{i-1})\big)-\int_{[a,b]}\varphi(t)\,f'(t)\,dt=$$
$$=\sum_{i=1,\dots,n}\int_{[t_{i-1},t_i]}\big(\varphi(\xi_i)-\varphi(t)\big)f'(t)dt$$

and

$$\left|\sum_{i=1,\dots,n}\varphi(\xi_i)\big(f(t_i)-f(t_{i-1})\big)-\int_{[a,b]}\varphi(t)\,f'(t)\,dt\right|\le$$

$$\le\sum_{i=1,\dots,n}\left(\sup_{\xi\in[t_{i-1},t_i]}\varphi(\xi)-\inf_{\xi\in[t_{i-1},t_i]}\varphi(\xi)\right)\int_{[t_{i-1},t_i]}|f'(t)|dt.$$

Next, we have that $\sup_{i=1,\dots,n}\left(\sup_{\xi\in[t_{i-1},t_i]}\varphi(\xi)-\inf_{\xi\in[t_{i-1},t_i]}\varphi(\xi)\right)$ converge to zero when the maximal longitude of the segments of the partitions converges to zero. The lemma has been proven.

Theorem 4. ***Let*** X ***be a reflexive Banach space and let the operator*** $A\in L(X)$ ***be well-bounded then there is a unique spectral family*** $E(.)$ ***in*** X ***such that***

$$A=a\,E(a)+\int_{[a,b]}\lambda\,dE(\lambda)$$

Remarks. The spectral family is concentrated on a compact interval.

Proof. Let us define a functional calculus $\Upsilon\ :AC([a,b])\to LB(X)$ We define a set $F(\lambda,\eta)$ of all real-valued absolutely continuous functions $f\in AC([a,b])$ such that

$$f=\begin{cases}1 \quad \text{on} \quad [a,\lambda]\\ \text{decreasing} \quad on \quad [\lambda,\lambda+\eta]\\ 0 \quad \text{on} \quad [\lambda+\eta,b]\end{cases}$$

for all $\lambda\in[a,b]$ and $0<\eta<(b-\lambda)$ .Next, we have $\|f\|_{Bound}\le 1$ for any $f\in F(\lambda,\eta)$. The class $K(\lambda,\eta)$ can be defined as a closure in the weak topology

$$K(\lambda,\eta)=weak\ cl\{\Upsilon(f):f\in F(\lambda,\eta)\}\subset LB(X^*)$$

For $\eta_1<\eta_2$ we obtain $K(\lambda,\eta_1)\subset K(\lambda,\eta_2)$ and it can be deduced that set $K(\lambda)=\bigcap_{\eta>0}K(\lambda,\eta)$ is a weakly compact uniformly bounded set.

The set is a subset of the reflexive Banach space defined by the formula

$$Z(\lambda)=\left\{x\in X:\Upsilon(f)x=0,\quad\text{for}\quad\text{all}\quad f\in\bigcup_{\eta>0}\big(1-F(\lambda,\eta)\big)\right\}$$

Let $y\in Z(\lambda)\in K(\lambda)=\bigcap_{\eta>0}K(\lambda,\eta)$ then there is a net $\{g_\alpha\}_{\alpha\in\Lambda}\subset K(\lambda,\eta)$ with the following property

$$\langle Ex,y^*\rangle=\lim_{\alpha\in\Lambda}\langle\Upsilon(g_\alpha)x,y^*\rangle=\lim_{\alpha\in\Lambda}\langle\big(1-\Upsilon(1-f_\alpha)\big)x,y^*\rangle$$

for all $x\in X$. Since $\langle Ex,y^*\rangle=\langle x,y^*\rangle$ we have $x\in$ Rang (E) thus set $Z(\lambda)$ is the range of each $K(\lambda)=\bigcap_{\eta>0}K(\lambda,\eta)$

For any $\theta>0$,there is $\eta_0>0$ such that $0\le f(t)\le\theta/2$ for all t $\in[\lambda,\lambda+\eta_0]$, so for $E\in K(\lambda,\eta_0)$ there is a net $\{g_\alpha\}_{\alpha\in\Lambda}\subset F(\lambda,\eta_0)$ with the property $weak-\lim_{\alpha\in\Lambda}\Upsilon(g_\alpha)=E$.

Now, we are going to apply the fourth condition of the definition

$$\int_{[a,b]}\left|(fg_\alpha)'\right|=\int_{[a,b]}\left|f'g_\alpha+f\,g_\alpha'\right|\le$$

$$\le\int_{[\lambda,\lambda+\eta_0]}\left|f'g_\alpha\right|+\int_{[\lambda,\lambda+\eta_0]}\left|+f\,g_\alpha'\right|\le$$

$$\le\frac{\theta}{2}+\frac{\theta}{2}=\theta,$$

so

$$\left|\langle\Upsilon(f)x,x^*\rangle\right|\le\left|\langle\Upsilon(f)x,y^*\rangle\right|=\left|\langle\Upsilon(f)Ex,y^*\rangle\right|=$$

$$=\left|\langle Ex,(\Upsilon(f))^*y^*\rangle\right|=\left|\lim_{\alpha\in\Lambda}\langle\Upsilon(g_\alpha)x,(\Upsilon(f))^*y^*\rangle\right|=$$

$$=\left|\lim_{\alpha\in\Lambda}\langle\Upsilon(f\,g_\alpha)x,y^*\rangle\right|\le\sup_{\alpha\in\Lambda}\left\|\Upsilon(f\,g_\alpha)\right\|\|x\|\|y^*\|$$

for all $y^*\in X^*$ so $E\in K(\lambda)=\bigcap_{\eta>0}K(\lambda,\eta)$ Thus, from the inequality $\left|\langle\Upsilon(f)x,y^*\rangle\right|\le\theta\|\Upsilon\|\|x\|\|y^*\|$ follows $\Upsilon(f)x=0$ so the range of E coincides with $Z(\lambda)$; the set E is a projection.

Let us establish that $K(\lambda,\eta)$ is a commutative multiplicative semigroup. Let $\widehat{K},\breve{K}\in K(\lambda,\eta)$, us have that there are nets $\{g_\alpha\}_{\alpha\in\Lambda},\{h_\beta\}_{\beta\in B}\in F(\lambda,\eta)$ such that

$$\widehat{K}=weak-\lim_{\alpha\in\Lambda}\Upsilon(g_\alpha)$$

and

$$\breve{K}=weak-\lim_{\beta\in B}\Upsilon(h_\beta)$$

For all $x\in X$, we have

$$\langle\widehat{K}\,\breve{K}x,y^*\rangle=\lim_{\alpha\in\Lambda}\langle\Upsilon(g_\alpha)\,\breve{K}x,y^*\rangle=$$

$$=\lim_{\alpha\in\Lambda}\langle\breve{K}x,(\Upsilon(g_\alpha))^*y^*\rangle=\lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\langle\Upsilon(h_\beta)x,(\Upsilon(g_\alpha))^*y^*\rangle\right\}=$$

$$=\lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\langle\Upsilon(g_\alpha h_\beta)x,y^*\rangle\right\}=\lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\langle\Upsilon(h_\beta)\Upsilon(g_\alpha)x,y^*\rangle\right\}=$$

$$=\lim_{\alpha\in\Lambda}\langle\breve{K}\Upsilon(g_\alpha)x,\ y^*\rangle=\lim_{\alpha\in\Lambda}\langle\Upsilon(g_\alpha)x,(\breve{K})^*y^*\rangle=\langle\widehat{K}\,x,(\breve{K})^*y^*\rangle=$$

$$=\langle\breve{K}\widehat{K}x,\ y^*\rangle,$$

so $\widehat{K}\,\breve{K}=\breve{K}\widehat{K}$, thus $E(\lambda)\in K(\lambda)=\bigcap_{\eta>0}K(\lambda,\eta)$, uniqueness is following from the properties of the projections. We define the set of the projection $\{E(\lambda)\}_{\lambda\in[a,b]}$ on X by presuming $E(\lambda)=O$ for $\lambda<a$ and $E(\lambda)=I$ for $\lambda>b$.

Now, let us establish the properties of $\{E(\lambda)\}_{\lambda\in[a,b]}$. Assuming that $a\le\lambda<\mu<b$, and assuming η is large enough, we are going to obtain that from $E(\lambda), E(\mu)\in K(\lambda,\eta)$ follows $E(\lambda), E(\mu)=E(\mu)$ $E(\lambda)=E(\lambda)$. If $\eta=\mu-\lambda$, then from $E(\lambda)\in K(\lambda,\eta)$ follows existence of the nets $\{g_\alpha\}_{\alpha\in\Lambda}\in F(\lambda,\eta)$ and $\{h_\beta\}_{\beta\alpha B}\in F(\lambda,\eta)$ with the properties $weak-\lim_{\alpha\in\Lambda}\Upsilon(g_\alpha)=E(\lambda)$ and $weak-\lim_{\beta\in B}\Upsilon(h_\beta)=E(\mu)$. Next, since $g_\alpha h_\beta=g_\alpha$ we have

$$\left\langle E(\lambda)\, E(\mu)x, y^*\right\rangle = \lim_{\alpha\in\Lambda}\left\langle \Upsilon(g_\alpha)\, E(\mu)x, y^*\right\rangle =$$

$$= \lim_{\alpha\in\Lambda}\left\langle E(\mu)x, \left(\Upsilon(g_\alpha)\right)^* y^*\right\rangle = \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\left\langle \Upsilon(h_\beta)x, \left(\Upsilon(g_\alpha)\right)^* y^*\right\rangle\right\} =$$

$$= \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\left\langle \Upsilon(g_\alpha)\Upsilon(h_\beta)x, y^*\right\rangle\right\} =$$

$$= \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\left\langle \Upsilon(g_\alpha h_\beta)x, y^*\right\rangle\right\} = \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\left\langle \Upsilon(g_\alpha)x, y^*\right\rangle\right\}$$

for all $x \in X$, $y^* \in X^*$. Thus, it has been obtained $\left\langle E(\lambda)\, E(\mu)x, y^*\right\rangle = \left\langle E(\lambda)x, y^*\right\rangle$ and so equality of projection

$$E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\lambda)$$

holds for all $a \le \lambda < \mu < b$.

Since $strong - \lim_{\mu\to\lambda+0} E(\mu) = E(\lambda+0)$ we have $E(\lambda+0) \in K(\lambda)$

For any pair $x \in X$, $y^* \in X^*$ and any $f \in AC([a,b])$, the morphism $f \mapsto \left\langle \Upsilon(f)x, y^*\right\rangle$ is anelement of the dual space $AC([a,b])$to and since $AC([a,b])$ is isometric to $L^1([a,b]) \oplus C$, from the duality argument, we have that there are $\gamma\left\langle x, y^*\right\rangle \in L^\infty([a,b])$ $\tilde{c}\left\langle x, y^*\right\rangle \in C$,which satisfy the following equality

$$\left\langle \Upsilon(f)x, y^*\right\rangle = \tilde{c}\left\langle x, y^*\right\rangle f(b) + \int_{[a,b]} f'(t)\gamma\left\langle x, y^*\right\rangle(t)\,dt$$

for all $f \in AC([a,b])$.

For any $\lambda \in [a,b]$, we assume $0 < \lambda+\eta < b$ then the function

$$g(\lambda,\eta)(t) = \begin{cases} 1 \quad \text{on} \quad [a,\lambda] \\ \text{ecreasing} \quad on \quad [\lambda, \lambda+\eta] \\ 0 \quad on \quad [\lambda+\eta, b] \end{cases}$$

belongs to $F(\lambda,\eta)$ and

$$\left\langle \Upsilon(g(\lambda,\eta))x, y^*\right\rangle = -\frac{1}{\eta}\int_{[\lambda,\lambda+\eta]} \gamma\left\langle x, y^*\right\rangle(t)\,dt$$

Thus, there is a weak limit $g(\lambda,\eta) \xrightarrow{weak-\eta\to 0+} E(\lambda)$

So, λ -almost everywhere, we obtain $\gamma\left\langle x, y^*\right\rangle(\lambda) = -\left\langle E(\lambda)x, y^*\right\rangle$, and for arbitrary $x \in X$, $y^* \in X^*$, the integral equality

$$\left\langle \Upsilon(f)x, y^*\right\rangle = \left\langle x, y^*\right\rangle f(b) - \int_{[a,b]} f'(\lambda)\left\langle E(\lambda)x, y^*\right\rangle d\lambda$$

holds for all $f \in AC([a,b])$.

Next, we have

$$\left\langle\left(\int_{[a,b]}^{\oplus} f\, dE\right)x, y^*\right\rangle =$$

$$= \lim_{\Lambda\in\Pi}\left\{\left\langle E(b)x, y^*\right\rangle f(b) - \left\langle \sum_{\Lambda}\left(f(\lambda_i) - f(\lambda_{i-1})\right)E(\lambda_i)x, y^*\right\rangle\right\} =$$

$$= \left\langle x, y^*\right\rangle f(b) - \lim_{\Lambda\in\Pi}\left\{\sum_{\Lambda}\left(f(\lambda_i) - f(\lambda_{i-1})\right)\left\langle E(\lambda_i)x, y^*\right\rangle\right\} =$$

$$= \left\langle x, y^*\right\rangle f(b) - \int_{[a,b]} f'(\lambda)\left\langle E(\lambda)x, y^*\right\rangle d\lambda = \left\langle \Upsilon(f)x, y^*\right\rangle.$$

Thus, by taking $f(\lambda) = \lambda$, we have

$$\left\langle Ax, y^*\right\rangle = b\left\langle x, y^*\right\rangle - \int_{[a,b]} \left\langle E(\lambda)x, y^*\right\rangle d\lambda$$

3. The characteristic of well-bounded operators in terms of the weak spectral family

Definition 4. ***The set*** $\{E(\lambda) \in L(X^*), \lambda \in \square\}$ ***of projection operators that satisfies the following conditions***

1. $E(.)$ ***is concentrated on a compact interval*** $[a,b]$;
2. $E(\lambda)\, E(\mu) = E(\mu)\, E(\lambda) = E(\lambda)$ ***for*** $\lambda \le \mu$; ***and*** $\sup_{\lambda}\|E(\lambda)\| < \infty$;
3. $E(\lambda) = O$ ***for all*** $\lambda < a$ ***and*** $E(\lambda) = I$ ***for all*** $b < \lambda$;
4. ***there is*** $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{[t,t+\varepsilon]}\left\langle x, E(\lambda)y^*\right\rangle d\lambda = \left\langle x, E(t)y^*\right\rangle$ ***for all*** $x \in X$,$y^* \in X$ ***and for all*** $t \in (a,b)$

is called a weak spectral family.

Theorem 5. Let $A \in L(X)$ be linear well-bounded operator then there is unique weak spectral family $\{E(\lambda) \in L(X^*), \lambda \in \square$ concentrated on $[a,b]$ such thatthe equality

$$\left\langle A(x), y^*\right\rangle = b\left\langle x, y^*\right\rangle - \int_{[a,b]} \left\langle x, E(\lambda)y^*\right\rangle d\lambda$$

holds for all $x \in X$,$y^* \in X$.

Proof. Let Φ denotes a functional calculus $\Phi : AC([a,b]) \to LB(X)$ then we define a functional calculus $\Upsilon : AC([a,b]) \to LB(X^*)$ by the formula $\Upsilon(f) = \left(\Phi(f)\right)^*$.

The is compact functional calculus in the weak topology of $AC[(a,b)]$ for the operator $A^* \in LB(X)$.

Let us define a set $F(\lambda,\eta)$ of all real-valued functions $f \in AC[(a,b)]$ such that

$$f = \begin{cases} 1 \quad \text{on} \quad [a,\lambda] \\ \text{decreasing} \quad on \quad [\lambda, \lambda+\eta] \\ 0 \quad \text{on} \quad [\lambda+\eta, b] \end{cases}$$

for all $\lambda \in [a,b]$ and $0<\eta<(b-\lambda)$ .The class $K(\lambda,\eta)$ can be defined as a closure in the weak topology

$$K(\lambda,\eta) = weak\ cl\left\{\Upsilon(f) : f \in F(\lambda,\eta)\right\} \subset LB(X^*)$$

From $\eta_1 < \eta_2$ follows $K(\lambda,\eta_1) \subset K(\lambda,\eta_2)$ and we can deduce that set $K(\lambda) = \bigcap_{\eta>0} K(\lambda,\eta)$ is a weakly compact uniformly bounded set.

We define the subset of the reflexive Banach space by the formula

$$Z(\lambda) = \left\{x^* \in X^* : \Upsilon(f)x^* = 0, \quad \text{for} \quad \text{all} \quad f \in \bigcup_{\eta>0}\left(1 - F(\lambda,\eta)\right)\right\}$$

Let $y^* \in Z(\lambda) \in K(\lambda) = \bigcap_{\eta>0} K(\lambda,\eta)$ then there is a net $\{g_\alpha\}_{\alpha\in\Lambda} \subset K(\lambda,\eta)$ with the following property

$$\left\langle x, Ey^*\right\rangle = \lim_{\alpha\in\Lambda}\left\langle x, \Upsilon(g_\alpha)y^*\right\rangle = \lim_{\alpha\in\Lambda}\left\langle x, \left(1 - \Upsilon(1 - f_\alpha)\right)y^*\right\rangle$$

for all $x \in X$. Since $\left\langle x, Ey^*\right\rangle = \left\langle x, y^*\right\rangle$ we have $y^* \in Rang(E)$.

For any $\theta > 0$,there is $\eta_0 > 0$ such that $0 \le f(t) \le \theta/2$ for all $t \in [\lambda, \lambda+\eta_0]$, so for $E \in K(\lambda,\eta_0)$ there is a net $\{g_\alpha\}_{\alpha\in\Lambda} \subset F(\lambda,\eta_0)$ with the property $weak - \lim_{\alpha\in\Lambda}\Upsilon(g_\alpha) = E$.

Now, we are going to apply the fourth condition of the definition

$$\int_{[a,b]}\left|\left(fg_\alpha\right)'\right| = \int_{[a,b]}\left|f'g_\alpha + f\,g_\alpha'\right| \le$$

$$\le \int_{[\lambda,\lambda+\eta_0]}\left|f'g_\alpha\right| + \int_{[\lambda,\lambda+\eta_0]}\left|+f\,g_\alpha'\right| \le$$

$$\le \frac{\theta}{2}+\frac{\theta}{2} = \theta,$$

so

$$\left|\left\langle x, \Upsilon(f)x^*\right\rangle\right| \le \left|\left\langle x, \Upsilon(f)y^*\right\rangle\right| = \left|\left\langle x, \Upsilon(f)Ey^*\right\rangle\right| =$$

$$= \left|\left\langle \Phi(f)x, Ey^*\right\rangle\right| = \left|\lim_{\alpha\in\Lambda}\left\langle \Phi(f)x, \Upsilon(g_\alpha)y^*\right\rangle\right| =$$

$$= \left|\lim_{\alpha\in\Lambda}\left\langle x, \Upsilon(f\,g_\alpha)y^*\right\rangle\right| \le \sup_{\alpha\in\Lambda}\left\|\Upsilon(f\,g_\alpha)\right\|\left\|x\right\|\left\|y^*\right\|$$

for all $y^* \in X^*$ so $E \in K(\lambda) = \bigcap_{\eta>0} K(\lambda,\eta)$. Thus, from the inequality $\left|\left\langle \Upsilon(f)x, y^*\right\rangle\right| \le \theta\|\Upsilon\|\|x\|\|y^*\|$ follows $\Upsilon(f)x_\lambda^* = 0$ so the range of coincides with $Z(\lambda)$; the set E is a projection.

Let us establish that $K(\lambda,\eta)$ is a commutative multiplicative semigroup. Let $\hat{K}, \breve{K} \in K(\lambda,\eta)$, us have that there are nets $\{g_\alpha\}_{\alpha\in\Lambda}$, $\{h_\beta\}_{\beta\in B} \in F(\lambda,\eta)$ such that

$$\hat{K} = weak - \lim_{\alpha\in\Lambda}\Upsilon(g_\alpha)$$

and

$$\breve{K} = weak - \lim_{\beta\in B}\Upsilon(h_\beta)$$

For all $x \in X$, we have

$$\left\langle x, \hat{K}\breve{K}y^*\right\rangle = \lim_{\alpha\in\Lambda}\left\langle x, \Upsilon(g_\alpha)\breve{K}y^*\right\rangle =$$

$$= \lim_{\alpha\in\Lambda}\left\langle \left(\Upsilon(g_\alpha)\right)^* x, \breve{K}y^*\right\rangle = \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\left\langle \left(\Upsilon(g_\alpha)\right)^* x, \Upsilon(h_\beta)y^*\right\rangle\right\} =$$

$$= \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\left\langle x, \Upsilon(g_\alpha h_\beta)y^*\right\rangle\right\} = \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\left\langle x, \Upsilon(h_\beta)\Upsilon(g_\alpha)y^*\right\rangle\right\} =$$

$$= \lim_{\alpha\in\Lambda}\left\langle x, \breve{K}\Upsilon(g_\alpha)y^*\right\rangle = \lim_{\alpha\in\Lambda}\left\langle \left(\breve{K}\right)^* x, \Upsilon(g_\alpha)y^*\right\rangle = \left\langle \left(\breve{K}\right)^* x, \hat{K}y^*\right\rangle =$$

$$= \left\langle x, \breve{K}\hat{K}y^*\right\rangle,$$

so $\hat{K}\breve{K} = \breve{K}\hat{K}$; thus $E(\lambda) \in K(\lambda) = \bigcap_{\eta>0} K(\lambda,\eta)$ uniqueness is following from the properties of the projections. We define the set of the projection $\{E(\lambda)\}_{\lambda\in[a,b]}$ on X by presuming $E(\lambda) = O$ for and $E(\lambda) = I$ for $\lambda > b$.

Now, let us establish the properties of $\{E(\lambda)\}_{\lambda\in[a,b]}$. Assuming that $a \le \lambda < \mu < b$, and assuming η is large enough, we are going to obtain that from $E(\lambda), E(\mu) \in K(\lambda,\eta)$ follows $E(\lambda), E(\mu) = E(\mu)$ $E(\lambda) = E(\lambda)$. If $\eta = \mu - \lambda$, then from $E(\lambda) \in K(\lambda,\eta)$ follows existence of the nets $\{g_\alpha\}_{\alpha\in\Lambda} \in F(\lambda,\eta)$ and $\{h_\beta\}_{\beta\in B}$ $F(\lambda,\eta)$ with the properties $weak - \lim_{\alpha\in\Lambda}\Upsilon(g_\alpha) = E(\lambda)$ and $weak - \lim_{\beta\in B}\Upsilon(h_\beta) = E(\mu)$. Next, since $g_\alpha h_\beta = g_\alpha$ we have

$$\left\langle x, E(\lambda)E(\mu)y^*\right\rangle = \lim_{\alpha\in\Lambda}\left\langle x, \Upsilon(g_\alpha)E(\mu)y^*\right\rangle =$$

$$= \lim_{\alpha\in\Lambda}\left\langle \Phi(g_\alpha)x, E(\mu)y^*\right\rangle = \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\left\langle \Phi(g_\alpha)x, \Upsilon(h_\beta)y^*\right\rangle\right\} =$$

$$= \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\left\langle \Phi(h_\beta)\Phi(g_\alpha)x, y^*\right\rangle\right\} = \lim_{\alpha\in\Lambda}\left\{\lim_{\beta\in B}\left\langle \Phi(g_\alpha)x, y^*\right\rangle\right\}$$

for all $x \in X^*$, $y^* \in X^*$. So, we have obtained $\left\langle x, E(\lambda)E(\mu)y^*\right\rangle = \left\langle x, E(\lambda)y^*\right\rangle$ and thus equality

$$E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\lambda)$$

holds for all $a \le \lambda < \mu < b$.

Since $strong - \lim_{\mu\to\lambda+0} E(\mu) = E(\lambda+0)$ we have $E(\lambda+0) \in K(\lambda)$.

For any pair $x \in X, y^* \in X^*$ and any $f \in AC([a,b])$, the morphism $f \mapsto \left\langle x, \Upsilon(f)y^*\right\rangle$ is an element of the dual space to $AC([a,b])$ and since $AC([a,b])$ is isometric to $L^1([a,b]) \oplus C$, from the duality argument, we have that there are $\gamma\left\langle x, y^*\right\rangle \in L^\infty([a,b])$ $\tilde{c}\left\langle x, y^*\right\rangle \in C$, which satisfy the following equality

$$\left\langle x, \Upsilon(f)y^*\right\rangle = \tilde{c}\left\langle x, y^*\right\rangle f(b) + \int_{[a,b]} f'(t)\gamma\left\langle x, y^*\right\rangle(t)\,dt$$

For any $\lambda \in [a,b]$, we assume $0 < \lambda + \eta < b$ then the function

$$g(\lambda,\eta)(t) = \begin{cases} 1 \quad \text{on} \quad [a,\lambda] \\ \text{ecreasing} \quad on \quad [\lambda, \lambda+\eta] \\ 0 \quad on \quad [\lambda+\eta, b] \end{cases}$$

belongs to $F(\lambda,\eta)$ and

$$\left\langle x, \Upsilon(g(\lambda,\eta))y^*\right\rangle = -\frac{1}{\eta}\int_{[\lambda,\lambda+\eta]} \gamma\left\langle x, y^*\right\rangle(t)\,dt$$

Thus, there is a weak limit $g(\lambda,\eta) \xrightarrow{weak-\eta\to 0+} E(\lambda)$.

So, λ -almost everywhere, we obtain $\gamma\langle x,y\rangle(\lambda) = -\langle x, E(\lambda)y^*\rangle$, and for arbitrary $x \in X, y^* \in X^*$, the integral equality

$$\left\langle x, \Upsilon(f)y^*\right\rangle = \left\langle x, y^*\right\rangle f(b) - \int_{[a,b]} f'(\lambda)\left\langle x, E(\lambda)y^*\right\rangle d\lambda$$

holds for all $f \in AC([a,b])$ .Thus, by taking $f(\lambda) = \lambda$, we have

$$\left\langle Ax, y^*\right\rangle = \left\langle x, A^*y^*\right\rangle = b\left\langle x, y^*\right\rangle - \int_{[a,b]}\left\langle x, E(\lambda)y^*\right\rangle d\lambda$$

Let function $\varphi \in L^1([a,b])$ then we can define $f(\varphi) = \int_{[\lambda,b]}\varphi(t)\,dt$ thus $f(\varphi) \in AC([a,b])$ and almost everywhere $f'(\varphi)(\lambda) = -\varphi(\lambda)$. For any fixed $x \in X$, the mapping $A(x)(\varphi) = \Phi(f(\varphi))(x)$ is continuous as the mapping $L^1([a,b]) \to X$. So, we have

$$\left\langle A(\varphi)x, y^*\right\rangle = \left\langle \Phi(f(\varphi))x, y^*\right\rangle = \int_{[a,b]}\varphi(\lambda)\left\langle x, E(\lambda)y^*\right\rangle d\lambda$$

and the mapping $A^*(x): X \to L^\infty([a,b])$ is such that

$$\left\langle \varphi, A^*(x)y^* \right\rangle = \int_{[a,b]} \varphi(\lambda)\left\langle x, E(\lambda)y^* \right\rangle d\lambda$$

Theorem 6. ***Let*** $\{E(\lambda) \in L(X^*), \lambda \in \square\}$ ***be a weak spectral family concentrated on*** $[a,b]$ ***then there is a linear well-bounded operator*** $A \in L(X)$ ***on the reflexive Banach space*** X ***such that***

$$\left\langle A(x), y^* \right\rangle = b\left\langle x, y^* \right\rangle - \int_{[a,b]} \left\langle x, E(\lambda)y^* \right\rangle d\lambda$$

holds for all $x \in X$, $y \in X^*$.

Proof. Assuming $\{E(\lambda) \in L(X^*), \lambda \in \square\}$ is a weak spectral family concentrated on $[a,b]$, the linear operator $A \in L(X)$ can be defined by the following formula

$$\left\langle A(x), y^* \right\rangle = b\left\langle x, y^* \right\rangle - \int_{[a,b]} \left\langle x, E(\lambda)y^* \right\rangle d\lambda$$

it is easy to see that this operator is linear and the only property of it that has to be established is well-boundedness.

By the induction and the Fubini theorem, we have

$$\left\langle (A(x))^n, y^* \right\rangle = b^n \left\langle x, y^* \right\rangle - \int_{[a,b]} n\lambda^{n-1} \left\langle x, E(\lambda)y^* \right\rangle d\lambda$$

thus

$$\left\| (A(x))^n \right\| \le b^n + n \sup_{\lambda \in [a,b]} \left\{ \left\| E(\lambda) \right\| \right\} \int_{[a,b]} \lambda^{n-1} d\lambda$$

andthe operator A is well-bounded.

4. Absolutely continuous functional calculus on Lebesgue spaces

Theorem 7. Let ***be a well-bounded linear operator on*** Lebesgue spaces $L^p(\Omega,\Sigma,\mu)$, $p \in (1,\infty)$. ***Then the operator*** A ***is a scalar type spectral operator.***

Proof. The spectral family $\{E(\lambda)\}$ of the operator A is concentrated on the interval $[a,b] \subset \square$.

Let us assume that $u \in L^p(\Omega,\Sigma,\mu)$, $P \in (1,\infty)$ and $v \in L^q(\Omega,\Sigma,\mu)$, where $1/p + 1/q = 1$. We have to show that the variation of the function $\langle E(\lambda)u,v\rangle$ is boundedas the function of λ. Assume that $a = \lambda_0 < \lambda_1 < \ldots < \lambda_n = b$ is a partition of the interval $[a,b]$. For arbitrary elements $u \in L^p(\Omega,\Sigma,\mu)$, $P \in (1,\infty)$ and $v \in L^q(\Omega,\Sigma,\mu)$, the variation of the function $\langle E(\lambda)u,v\rangle$ equals

$$\operatorname*{var}_{[a,b]}\left\langle E(\lambda)u, v \right\rangle = \sum_{i=1,\ldots,n} \left| \left\langle E(\lambda_i)u, v \right\rangle - \left\langle E(\lambda_{i-1})u, v \right\rangle \right| =$$
$$= \sum_{i=1,\ldots,n} \left| \left\langle \left(E(\lambda_i) - E(\lambda_{i-1})\right)u, v \right\rangle \right| \le$$
$$\le \left\| \sum_{i=1,\ldots,n} \left(E(\lambda_i) - E(\lambda_{i-1})\right) \right\| \|u\| \|v\|.$$

Let m be an integer such that $\lambda_{m-1} < c < \lambda$, so we have

$$\left\| \sum_{i=1,\ldots,n} \left(E(\lambda_i) - E(\lambda_{i-1})\right) \right\| \le \left\| \sum_{i=1,\ldots,m-1} \left(E(\lambda_i) - E(\lambda_{i-1})\right) \right\| +$$
$$+ \left\| \left(E(\lambda_m) - E(\lambda_{m-1})\right) \right\| + \left\| \sum_{i=m+1,\ldots,n} \left(E(\lambda_i) - E(\lambda_{i-1})\right) \right\|$$

thus for $\lambda < c$ we have $\|E(\lambda)\| \le 1$, and for $\lambda \ge c$ we have $\|I - E(\lambda)\| \le 1$. So $\|E(\lambda_m)\| \le 2$ and $\|E(\lambda_{m-1})\| \le 1$. Since $\{E(\lambda_i)\}_{i=1,\ldots,m-1}$ and $\{I - E(\lambda_{n-i})\}_{i=m-1,\ldots,n}$ arethe increasing sequences of contractive projections, we have

$$\left\| \sum_{i=1,\ldots,m-1} \left(E(\lambda_i) - E(\lambda_{i-1})\right) \right\| \le 2(q-1)$$

and

$$\left\| \sum_{i=m-1,\ldots,n} \left(E(\lambda_i) - E(\lambda_{i-1})\right) \right\| \le 2(q-1)$$

In the final conclusion, we obtain

$$\left\| \sum_{i=1,\ldots,n} \left(E(\lambda_i) - E(\lambda_{i-1})\right) \right\| \le 4(q-1) + 3$$

thus the variation of $\langle E(\lambda)u,v\rangle$ can not exceed the value $(4(q-1)+3)\|u\|\,\|v\|$. The theorem is proven.

Definition 5. A solitary operator is a bounded linear surjective operator $U: X \to X$ on a Banach space that for all $x \in X$ and $y \in X^*$ satisfies the following equality $\left\langle Ux, U^* y \right\rangle = \left\langle x, y \right\rangle$,
where $U^*: X^* \to X^*$.

Theorem 8. ***Assuming*** (Φ, X) ***is a functional calculus on the measurable space*** (Z,Σ). ***Then there are a semi-finite measure space*** (Ω,F,μ) ***and solitary operator*** $U: X \to L^P(\Omega,\mathrm{F},\mu)$ ***and an injective pointwise continuous*** $*$ ***-homomorphism*** $F: M(Z,\Sigma) \to M(\Omega,\mathrm{F})$, ***such that*** $\Phi(F) = UM_{Ff}U^{-1}$, ***where*** M_{Ff} is ***the operator of the multiplication by*** f.

Proof. For every set $A \in \Sigma$, we define measure $\mu_x(A) = \langle \Phi(X_A)x, x^* \rangle$ as a function of $x \in X$, so $\left\langle \Phi(f)x, x^* \right\rangle = \left\langle \Phi(f) \right\rangle_{\mu_x}$

for every bounded f. Now, for every bounded f, we define the space $B_x = \left[\{\Phi(f)x, f \in M_b(Z,\Sigma)\}\right]$, thus there is a solitary operator $W_x : L^p(Z,\Sigma,\mu_x) \to B_x$ as an extension of mappings $M_b(Z,\Sigma) \to B_x$ and $f \to \Phi(f)x$.
Let $\{x_i\}$ and $\{x_i^*\}$ be two sets of unit vectors in X and X^* spaces, respectively, with properties

$$\left\langle x_k, x_k^* \right\rangle = \|x_k\| \|x_k^*\|_* = 1 \quad \forall k \in N$$

and

$$\left\langle x_i, x_k^* \right\rangle = 0$$

for every $i \neq k$.
For every k, we can define the set $Z_k = Z \times \{k\}$ as an exemplar of Z then the set Ω can be represented as the disjoint union $\bigcup_k Z_k$. Let
Let us define an additive set function μ by the following formula

$$\mu(A) = \sum_k \mu_{x_k}(A \cap Z_k) \quad \forall A \in \mathsf{F}$$

The additive set function μ is the measure on the maximal sigma-algebra F on Ω, which includes all measurable mapping $Z_k = Z \times \{k\}$ into Ω.

The operator W_{xk} is correctly defined on $L^P(Z_k,\Sigma,\mu_{xk})$ and $W_{xk} : L^P(Z_k,\Sigma,\mu_{xk}) \to B_{xk}$ so we define the operator $U: X \to L^P(\Omega,\mathrm{F},\mu)$ by the condition $U^{-1} = W_{xk}$ on .

$$L^p(Z_k,\Sigma,\mu_{x_k}) \subseteq L^p(\Omega,\mathsf{F},\mu)$$

Then the $*$-homomorphism $F: M(Z,\Sigma) \to M(\Omega,\mathrm{F})$, we introduce by the formula $(F\,f)(x,k) = f(x), \quad x \in X.$

For all $f \in (Z,\Sigma)$ we define the multiplication operator calculus as $M_{Ff} = U\Phi(f)U^{-1}$, so the theorem has been proven.

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