ISSN: 2834-7706

Research Article

Journal of Mathematical Techniques and Computational Mathematics

Kervaire Conjecture on Weight of Group via Fundamental Group of Ribbon Sphere-Link

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Submitted: 2024, Mar 20; **Accepted:** 2024, Apr 10; **Published:** 2024, Apr 22

Citation: Kawauchi, A. (2024). Kervaire Conjecture on Weight of Group via Fundamental Group of Ribbon Sphere-Link. *J Math Techniques Comput Math, 3*(4), 1-3.

Abstract
Kervaire conjecture that the weight of the free product of every non-trivial group and the infinite cyclic group is not one is affirmatively confirmed by confirming affirmatively Conjecture Z on the knot exterior introduced by Gonzàlez Acuña and Ramírez as a conjecture equivalent to Kervaire conjecture.

Key Words: Weight, Kervaire Conjecture, Conjecture Z, Whitehead Aspherical Conjecture, Ribbon Sphere-Link.

1. Introduction

A *weight system* of a group G is a system of elements w_i, $(i = 1, 2, \dots, n)$ of G such that the normal closure $N(w_1, w_2, \dots, w_n)$ of w_i, $(i = 1, 2, \dots, n)$ in G (=: the smallest normal subgroup generated by w_i, $(i = 1, 2, \dots, n)$ in G) is equal to G. The *weight* of a group G is the least cardinal number $w(G)$ of a weight system of G. By convention, $w(G) = 0$ if and only if G is the trivial group. The rank of G is the least cardinal number $r(G)$ of generators of G. The difference $r(G) - w(G)$ is non- negative and in general taken sufficiently large. For example, let $G = \pi_1(S^3 \setminus k, x)$ be the fundamental group of a polygonal knot k in S^3. Then $G \cong \mathbf{Z}$ and $r(G) = 1$ for the trivial knot k, $r(G) = 2$ for the trefoil knot $k = 3_1$, and $r(G) = n$ for the $n - 1(\geq 2)$-fold connected sum $k = \#_{n-1}3_1$ of the trefoil knot 3_1. On the other hand, $w(G) = 1$ for every knot k, because $G/N(m(k)) = \{1\}$ for a meridian element $m(k)$ of k. Let $G * \boldsymbol{Z}$ denote the free product of a group G and the infinite cyclic group $\boldsymbol{Z}$. Kervaire's conjecture on the weight of a group is the following conjecture (see Kervaire, Magnus-Karrass-Solitar) [1,2].

Kervaire Conjecture

$w(G * \mathbf{Z}) > 1$ for every non-trivial group G.

Some partial affirmative confirmations of this conjecture are known. For example, the following result of Klyachko is used in this paper [3].

Theorem (Klyachko)

$w(G * \mathbf{Z}) > 1$ for every non-trivial torsion-free group G.

A *knot exterior* is a compact 3-manifold $E = \mathrm{cl}(S^3 \setminus N(k))$ for a tubular neighborhood $N(k)$ of a polygonal knot k in the 3-sphere S^3. Let F be a compact connected orientable non-separating proper surface of E where the boundary ∂F of F may be disconnected. Let $E(F) = \mathrm{cl}(E \setminus F \times I)$ be the compact piecewise-linear 3-manifold for a normal line bundle $F \times I$ of F in $E(F)$ where $I = [-1, 1]$. Let $E(F)^+$ be the 3-complex obtained from $E(F)$ by adding the cone $\mathrm{Cone}(v, F \times \partial I)$ over the base $F \times \partial I$ with a vertex v disjoint from E, where $\partial I = \{1, -1\}$. The 3-complex $E(F)^+$ is also considered to be obtained from E by shrinking the normal line bundle $F \times I$ into the vertex v. The result of Conjecture $\boldsymbol{Z}$ due to Gonzàlez Acunã and Ramírez in is stated as follows [4].

Theorem (Gonzàlez Acunã-Ramírez)

Kervaire's conjecture is equivalent to the following conjecture:

Conjecture *Z*. The fundamental group $\pi_1(E(F)^+, v)$ is isomorphic to $\boldsymbol{Z}$ for every knot exterior E and every compact connected orientable non-separating proper surface F in E.

There are knot theoretical investigations of this surface F and some partial confirmations [4-6]. In this paper, Kervaire conjecture is confirmed affirmatively by confirming Conjecture $\boldsymbol{Z}$ affirmatively.

Theorem 1

Conjecture $\boldsymbol{Z}$ is true.

Gonzàlez Acunã-Ramírez theorem and Theorem 1 imply:

Corollary 2

Kervaire conjectureis true.

An outline of the proof of Theorem 1 is explained as follows.

Outline of the Proof of Theorem 1

Let $E(F)^{++} = E(F) \cup \mathrm{Cone}(v_+, F \times 1) \cup \mathrm{Cone}(v_-, F \times (-1))$ be a 3-complex for distinct vertexes v_+ and v_- disjoint from E. Then the 3-complex $E(F)^+$ is homotopy equivalent to the bouquet $E(F)^{++} \vee S^1$. Hence the fundamental group $\pi_1(E(F)^+, v)$ is isomorphic to the free product $\pi_1(E(F)^{++}, v) * \boldsymbol{Z}$. Thus, $\pi_1(E(F)^+,v) = \boldsymbol{Z}$ if and only if $\pi_1(E(F)^{++}, v) = \{1\}$ and Conjecture $\boldsymbol{Z}$ is equivalent to the claim that $\pi_1(E(F)^{++}, v) = \{1\}$. The following observation is used.

Lemma 3

$w(\pi_1(E(F)^+, v)) = w(\pi_1(E(F)^{++}, v) * Z) = 1.$

Proof of Lemma 3

Because the fundamental group $\pi_1(E(F)^+,v)$ is a non-trivial quotient group of $\pi_1(E,v)$ and $w(\pi_1(E,v)) = 1$, the desired result is obtained. This completes the proof of Lemma 3.

The following lemma is proved in Section 2.

Lemma 4

The fundamental group $\pi_1(E(F)+, v)$ is a torsion-free group.

By assuming Lemma 4, the proof of Theorem 1 is completed as follows:

Proof of Theorem 1

Klyachko Theorem says that if G is a torsion-free group and $w(G * \boldsymbol{Z}) = 1$, then $G = \{1\}$. Hence by this theorem and Lemmas 3, 4, $\pi_1(E(F)^{++},v) \cong \{1\}$ and $\pi_1(E(F)^+,v) \cong \boldsymbol{Z}$. This completes the proof of Theorem 1.

In the first draft of this research, the author tried to show that every finitely presented group G with $w(G * \boldsymbol{Z}) = 1$ is torsion-free. This trial succeeds for a group G of deficiency 0, but failed for a group G of negative deficiency. The main point of this failure is the attempt to construct a finitely presented group of deficiency 0 from the group of negative deficiency, which forced the author to show that G is a torsion-free group while the deficiency remains negative. Fortunately, the fundamental group $\pi_1(E(F)^+,v)$ of the 3-complex $E(F)^+$ was an excellent object to this consideration, so it could be done.

2. Proof of Lemma 4

The proof of Lemma 4 is done as follows by using the concept of collapse in [7].

Proof of Lemma 4

Collapse F into a triangulated graph γ by using that F is a bounded surface. Enlarge the fiber I of a normal line bundle $F \times I$ of F in E into a fiber J of a normal line bundle $F \times J$ of F in E so that $I \subset J \setminus \partial J$. Let $J^c = \mathrm{cl}(J \setminus I)$. Let $E(F)^- = \mathrm{cl}(E \setminus F \times J)$. Collapse $F \times J^c$ into $\gamma \times J^c$. Triangulate $\gamma \times J^c$ without introducing new vertexes. The 3-complex $E(F)^+$ is collapsed into a finite 3-complex

$$E(F)^- \cup_\gamma \times J_c \cup \mathrm{Cone}(v, \gamma \times \partial I)$$

and thus collapsed into a finite 2-complex

$$P = P^- \cup \gamma \times J^c \cup \mathrm{Cone}\,(v, \gamma \times \partial I)$$

obtained by taking any 2-complex P^- collapsed from $E(F)^-$. This 2-complex P is a subcomplex of a 3-complex

$$Q = \mathrm{Cone}\,(v, P^- \cup_\gamma \times J^c).$$

Since every 2-complex of $\gamma \times J^c$ contains at most one 1-simplex of $\gamma \times \partial I$, every 3-simplex of Cone $(v, \gamma \times J^c)$ contains at most one 2-simplex of Cone$(v, \gamma \times \partial I)$. Collapse every 3-simplex of Cone$(v, \gamma \times J^c)$ from a 2-face containing v and not belonging to Cone$(v, \gamma \times \partial I)$. Then collapse every 3-simplex of Cone(v,P^-) from any 2-face containing the vertex v. Thus, the 3-complex Q is collapsed to a finite 2-complex C containing the 2-complex P as a subcomplex. Since Q is collapsed to the vertex v, C is a finite contractible 2-complex. It is shown that every connected subcomplex of a finite contractible 2-complex is aspherical [8,9]. Since the fundamental group of a connected aspherical complex is a torsion-free group, the group $\pi_1(P,v)$ is a torsion-free group. Note that this torsion-freeness comes from the torsion-freeness of the fundamental group of a ribbon S^2-link in the 4-sphere S^4, as it is discussed in [8], where the free product of $\pi_1(P,v)$ and a free group is shown to be isomorphic to the fundamental group π of a ribbon S^2-link in S^4 and then the group π is shown to be a torsion-free group. Since $\pi_1(E(F)^+,v)$ is isomorphic to $\pi_1(P,v)$, the group $\pi_1(E(F)^+,v)$ is a torsion-free group. This completes the proof of Lemma 4.

Acknowledgements

This work was partly supported by JSPS KAKENHI Grant Numbers JP19H01788, JP21H00978 and MEXT Promotion of Distinctive Joint Research Center Program JPMXP0723833165.

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