ISSN: 2834-7706

Mini Review Article

Journal of Mathematical Techniques and Computational Mathematics

Hyperbolic Rotation with Euclidean Angle Illuminates Space Time Spinors

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Submitted: 2023, Oct 21; **Accepted:** 2023, Nov 23; **Published:** 2023, Nov 27

Citation: Brands, P. J. (2023). Hyperbolic Rotation with Euclidean Angle Illuminates Space Time Spinors. *J Math Techniques Comput Math, 2*(11), 456-478.

Abstract

In this article it will be shown that by introducing a hyperbolic rotation as function of a Euclidean rotation angle, all possible Lorentz group spacetime rotations can be performed by Euclidean rotation parameters. This unification in Lorentz group rotations parameters (all Euclidean) allows for the calculation of the volume of a spacetime region bounded by the light cone of a past event, i.e., a causality-volume with the shape of a spacetime three-sphere. As will be calculated, the surface area of the spacetime three-sphere is the volume of our 3D space. Additionally, the unification in Lorentz group rotation parameters allows for the derivation of a Dirac spinor purely from the spacetime three-sphere symmetries. This derivation from the spacetime symmetries of a geometrical object yields the same result as solving the Dirac equation with quantum mechanical eigenvalue eigenvector complex matrix calculations.

Keywords: Spacetime Spinors, Spacetime Volume Element, Three-Sphere, Lorentz Group, Rotation Parameters, Causality-Volume, Geometric Algebra, Spacetime Algebra.

1. Introduction

The six generators of the Lorentz group divide in two parts: (a) three generators related to a hyperbolic rotation in the three temporal planes (xt, yt and zt) plus (b) three generators related to a Euclidean rotation in the three spatial planes (xy, yz and zx) [1-4]. To perform all possible spacetime rotations three from the six generators are needed. They can be chosen as: one temporal generator (zt plane) with hyperbolic rotation parameter φ and two spatial generators (zx, xy planes) with two Euclidean rotation parameters $\{\theta, \phi\}$, i.e., a mixed (hyperbolic and Euclidean) set of rotation parameters $\{\varphi, \theta, \phi\}$. In section 2 it will be shown that the division in hyperbolic and Euclidean rotation parameters of the Lorentz group can be broken by the introduction of a hyperbolic rotation with Euclidean rotation parameter β. The resulting Euclidean Lorentz group rotation parameters $\{\beta,\theta,\phi\}$ can perform all possible spacetime rotations.

The Euclidean Lorentz group rotation parameters $\{\beta,\theta,\phi\}$ - which form the polar coordinates of a spacetime region bounded by a $4D$ light cone ($4D$ three-sphere) of a past event (Section 3) have the following properties: (a) they are connected to parity-reversal P and time-reversal T (section 4), (b) they can be mapped to a double cube in $\mathbb{R}^{3,0}$ (section 4), and (c) they are related to a Dirac spinor, being derived purely from the $4D$ three- sphere symmetries (section 5) [5-8]. The volume of the $4D$ three-sphere bounds the spacetime positions where potentially an observation can take place, based on embedded event-information from a past event, i.e., a causality-volume (section 4). The surface of the $4D$ three-sphere is a $3D$ two-sphere (section 4) [9].

Section 6 discusses the central line of thought and provides additional meaning. For clarity and focus are most of the details in the various sections (1-6) provided in appendices A-F. However, if necessary, a section will start with relevant information about the algebra or specific methods used in that section.

The mathematical formalism used in this article is based on the geometric algebra (GA) of spacetime as developed by David Hestenes (Appendix A) [10-12]. Foundations of geometric algebra where jointly developed by Grassmann and Clifford in the late 19th century [13,14]. The spacetime algebra (STA) multi-vectors are "geometric numbers" representing scalars (points), vectors (directed line segments), bivectors (directed planes), trivectors (directed $3D$ volumes), and pseudoscalars (directed 4D volumes). These "geometric numbers" are mathematical objects for geometric analysis in the same way as "complex numbers" are for analytical analysis. There are many positive arguments for using GA, especially in physics [15-26]. However, the most decisive argument is the generalization of rotation which can be applied in any dimension, and which can act on any multi-vector by means of the so-called rotors [10,11,15-24,27,28]. Rotors are directly related to spinors and automatically integrate Lie algebra by the GA bivectors [29-31].

In spacetime algebra (STA) a spacetime inertial frame (reference frame) $\{t, x, y, z\}$ is represented by a set of four orthogonal basis vectors $\gamma_\mu = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ [10]. The temporal basis vector γ_0 squares to one, while the spatial basis vectors γ_k square to minus one, i.e., a Lorentz frame in $\mathbb{R}^{1,3}$. The STA orthogonal basis vectors γ_μ satisfy the algebra of the Dirac gamma matrices (Appendix A).

2. Hyperbolic Rotation with Euclidean Angle

#	***STA*** [10, 20-22]	**Description**
1	$\gamma_\mu = \{t, x, y, z\} = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$	The four orthogonal basis vectors γ_μ for the four spacetime axis $\{t, x, y, z\}$.
2	$(\gamma_0)^2 = +1 \quad (\gamma_k)^2 = -1 \quad \gamma_\mu \cdot \gamma_\nu = 0 \mapsto (\mu \neq \nu)$ $metric\ diagonal\ \{+1, -1, -1, -1\}$	The temporal basis vector γ_0 squares to one, while the spatial basis vectors γ_k square to minus one, i.e., $\mathbb{R}^{1,3}$. The basis vectors γ_μ satisfy the algebra of the Dirac gamma matrices.
3	$\gamma_3\gamma_0 = \gamma_3 \cdot \gamma_0 + \gamma_3 \wedge \gamma_0 \quad \gamma_3 \cdot \gamma_0 = 0$ $\gamma_3\gamma_0 = \gamma_3 \wedge \gamma_0 \qquad \gamma_3\gamma_0 = -\gamma_0\gamma_3$	The bivector $\gamma_3\gamma_0$ defines the directed zt-plane. Interchanging the order of the basis vectors reverses the orientation in a tz-plane and introduces a minus sign.
4	$w = cosh(\varphi)\gamma_0 + sinh(\varphi)\gamma_3 \qquad \varphi \in [-\infty, \infty]$ $w^2 = ww = cosh^2(\varphi) - sinh^2(\varphi) = 1$	Hyperbolic unit-vector w as function of the hyperbolic angel $\varphi \in [-\infty, \infty]$ in the plane spanned by the temporal t-axis basis vector γ_0 and the spatial z-axis basis vector γ_3. The norm w^2 is the geometric product of ww.
5	$w\gamma_0 = cosh(\varphi) + sinh(\varphi)\gamma_3\gamma_0 = e^{\gamma_3\gamma_0\varphi}$ $(\gamma_3\gamma_0)^2 = +1$	The geometric product of $w\gamma_0$ gives a multi-vector, with scalar $cosh(\varphi)$ plus bivector $sinh(\varphi)\gamma_3\gamma_0$, which can be written as a Euler relationship $e^{\gamma_3\gamma_0\varphi}$.
6	$R = a + b\gamma_3\gamma_0 \ \rightarrow\ \tilde{R} = a - b\gamma_3\gamma_0$	Revision operator ~ is an invariant type of conjugation.
7	$\beta \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$	Is a range indicator, i.e., Euclidean angle in the range $-\frac{\pi}{2} \leq \beta \leq \frac{3\pi}{2}$.

Table 2.1: Overview of STA relevant for this section.

A more detailed overview of STA is given in Appendix A.

For clarity in this section on hyperbolic rotation with Euclidean angle, we will only consider the tz-plane, as spanned by the temporal γ_0 and spatial γ_3 basis vectors. A generalization follows from section 3 on.

A hyperbolic unit-vector $w(\varphi)$ as function of a hyperbolic angel φ is given by:

$$w(\varphi) = cosh(\varphi)\gamma_0 + sinh(\varphi)\gamma_3 \qquad \varphi \in [-\infty, \infty] \qquad w^2 = cosh^2(\varphi) - sinh^2(\varphi) = 1 \tag{2.1}$$

Unit-vector $w(\varphi)$ describes a hyperbola with an implicit hyperbolic symmetry: $cosh^2(\varphi) - sinh^2(\varphi) = 1$. However, a hyperbola can also be described by a hyperbolic unit-vector $p(\beta)$ as function of a Euclidean angle β:

$$p(\beta) = sec(\beta)\gamma_0 + tan(\beta)\gamma_3 \qquad \beta \in [-\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \qquad p^2 = sec^2(\beta) - tan^2(\beta) = 1 \tag{2.2}$$

unit-vector $p(\beta)$ as function of Euclidean angle β describes a positive and negative hyperbola with an implicit hyperbolic symmetry: $sec^2(\beta) - tan^2(\beta) = 1$. These two hyperbolic unit-vectors $\{w(\varphi), p(\beta)\}$ describe an equal hyperbola $\pm w(\varphi) = p(\beta)$ if the two different angle types $\{\varphi, \beta\}$ have the following implicit relationship:

$$\boldsymbol{tanh(\varphi) = sin(\beta)} \qquad \varphi \in [-\infty, \infty \rightarrow \infty, -\infty] \leftrightarrow \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \rightarrow \frac{\pi}{2}, \frac{3\pi}{2}\right] \tag{2.3}$$

The hyperbolic angle interval $\varphi \in [-\infty, \infty \rightarrow \infty, -\infty]$ is bound by infinities, while the angle $\beta \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ interval is periodic and has a double cover. Substitution of the explicit function $\boldsymbol{\varphi = tanh^{-1}(sin(\beta))}$ (2.3) in $w(\varphi)$ will make the two-hyperbola equal:

$$\left.\begin{aligned} +w\left(tanh^{-1}\left(sin(\beta)\right)\right) &= p(\beta) = sec(\beta)\gamma_0 + tan(\beta)\gamma_3 \quad \beta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \\ -w\left(tanh^{-1}\left(sin(\beta)\right)\right) &= p(\beta) = sec(\beta)\gamma_0 + tan(\beta)\gamma_3 \quad \beta \in [\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \end{aligned}\right\} \ \beta \in [-\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \tag{2.4}$$

Therefore, $p(\beta)$ is under the implicit relationship $\boldsymbol{tanh(\varphi)} = \boldsymbol{sin}(\beta)$ (2.3) a hyperbolic unit-vector $\pm w(\varphi)$ as function of Euclidean angle $\beta \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ (Fig. 2.1), but how to perform a hyperbolic rotation with Euclidean angle?

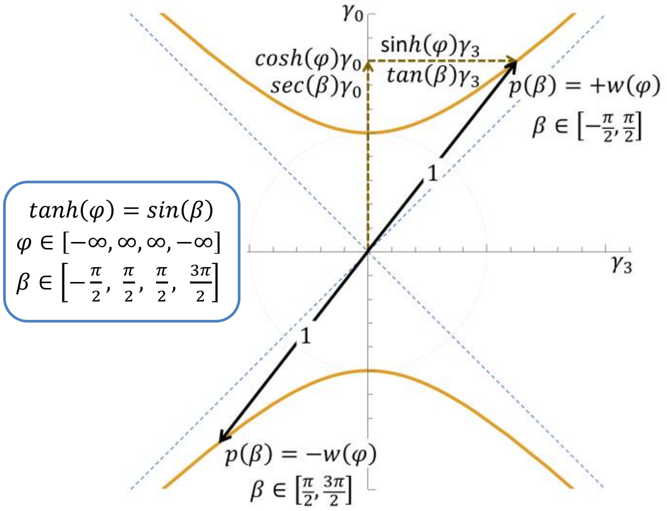


Figure 2.1: Shows the equality between the hyperbolic unit-vectors $\pm w(\varphi)$ as function of a hyperbolic angle φ and the hyperbolic unit-vector $p(\beta)$ as function of a Euclidean angle β (using $tanh(\varphi) = sin(\beta)$).

A general rotor $R = \rho S_R$ is part of the STA even subalgebra (Appendix A) and consists of a scalar density ρ times a spinor S_R [15] (Appendix B):

$$R = \rho S_R \quad R\tilde{R} = 1 \;\mapsto\; \rho = \left(S_R \tilde{S}_R\right)^{-1/2} \qquad \text{Unitarity of } R\tilde{R} = 1 \text{ defines the scalar density } \rho \tag{2.5}$$

An irreducible rotor R can be calculated by taking the square root of the geometric product (GP) of two unit-vectors laying in the plane of rotation [32]. Hence, an irreducible hyperbolic rotor for the $\{\gamma_3\gamma_0 \rightarrow zt\}$ plane can be calculated from the square root of the GP of hyperbolic unit vector $w(\varphi)$ (2.1) and temporal basis vector γ_0:

$$\begin{gathered} R(\varphi) = \sqrt{w\gamma_0} = \sqrt{cosh(\varphi) + sinh(\varphi)\gamma_3\gamma_0} = \sqrt{e^{\gamma_3\gamma_0\varphi}} = e^{\gamma_3\gamma_0\varphi/2} \quad \varphi \in [-\infty, \infty] \;\; (\gamma_3\gamma_0)^2 = 1 \\ R(\varphi) = e^{\gamma_3\gamma_0\varphi/2} = cosh(\varphi/2) + sinh(\varphi/2)\gamma_3\gamma_0 \quad M' = RM\tilde{R} \quad R\tilde{R} = 1 \quad \tilde{R} = e^{-\gamma_3\gamma_0\varphi/2} \end{gathered} \tag{2.6}$$

where $R(\varphi)$ is an irreducible hyperbolic rotor with hyperbolic angle φ and bivector (generator) $\gamma_3\gamma_0$, with a positive $(\gamma_3\gamma_0)^2=+1$ signature (Appendix B). A rotation of a (multi)-vector M takes place by a double-sided GP $M'=RM\tilde{R}$. This is a powerful algebraic generalizing of rotations because it is applicable in any dimension and with any multi-vector M [10,11,15-24,27,28]. The type of rotor (hyperbolic or Euclidean) depends on the squared signature of the bivector. A hyperbolic rotor has a positive $B^2=+1$ signature and a Euclidean rotor has a negative $B^2=-1$ signature. By substitution of the explicit function $\varphi=tanh^{-1}(sin(\beta))$ (2.3) in the irreducible hyperbolic rotor $R(\varphi)$ (2.6) a mixed type of rotor can be calculated:

$$R(\varphi) = e^{\gamma_3\gamma_0\varphi/2} \;\mapsto\; R\left(tanh^{-1}\left(sin(\beta)\right)\right) = L_1(\beta) = \rho L_{u1}(\beta) \qquad \text{Density } \rho \text{ and spinor } L_{u1}(\beta)$$

$$\begin{gathered} L_1(\beta) = \sqrt{sec(\beta)}\left(cos(\beta/2) + sin(\beta/2)\gamma_3\gamma_0\right) \quad \text{Hyperbolic rotor } L_1(\beta) \text{ with Euclidean angle } \beta \\ L_1\tilde{L}_1 = 1 \;\mapsto\; \rho = \left(L_{u1}\tilde{L}_{u1}\right)^{-1/2} = \sqrt{sec(\beta)} \qquad L_{u1}(\beta) = cos(\beta/2) + sin(\beta/2)\gamma_3\gamma_0 \end{gathered} \tag{2.7}$$

where $L_1(\beta)=\rho L_{u1}(\beta)$ is an irreducible hyperbolic rotor ($(\gamma_3\gamma_0)^2=+1$) with Euclidean rotation angle β (Appendix B). Note that temporal spinor $L_{u1}(\beta) \neq Exp(\gamma_3\gamma_0\beta/2)$ is not a Euler relationship $(\gamma_3\gamma_0)^2=+1$.

A connection to spacetime symmetries can be made by mapping the Euclidean angle β to relative speed v/c, in the same way has done with the hyperbolic angle φ (rapidity):

$$
\begin{array}{lll}
tanh(\varphi) = sin(\beta) & \mapsto \; tanh(\varphi) = sin(\beta) = \pm v/c & \text{Mapping to relative speed} \\
\frac{v}{c} \in [-1,+1] & \leftrightarrow \; \varphi \in [-\infty, \infty] & \text{One to one} \\
\frac{v}{c} \in [-1,+1 \rightarrow +1,-1] & \leftrightarrow \; \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \rightarrow \frac{\pi}{2}, \frac{3\pi}{2}\right] & \text{Periodic and double cover}
\end{array}
\tag{2.8}
$$

The mapping of relative speed v/c to hyperbolic angle φ is one to one, while the mapping of relative speed v/c to Euclidean angle β is periodic and has a double cover (Fig. 2.1). The mapping of $tanh(\varphi)$=$sin(\beta)$=$\pm v/c$ results in four trigonometric relativistic proportionality factors with an implicit circular and hyperbolic symmetry:

$$
\begin{array}{lll}
sin(\beta) = \pm v/c & cos(\beta) = \pm\sqrt{1-(v/c)^2} & \mapsto \; sin^2(\beta) + cos^2(\beta) = 1 \;\; Circular \\
sec(\beta) = \frac{1}{\pm\sqrt{1-(v/c)^2}} & tan(\beta) = \frac{\pm v/c}{\pm\sqrt{1-(v/c)^2}} & \mapsto \; sec^2(\beta) - tan^2(\beta) = 1 \;\; Hyperbolic
\end{array}
\tag{2.9}
$$

From the implicit circular and hyperbolic symmetries two explicit vectors $\{x,p\}$ can be derived:

$$
\begin{array}{lll}
sin^2(\beta) + cos^2(\beta) = 1 \; \mapsto \; x = \gamma_0 + sin(\beta)\gamma_3 & Circular\ symmetry & \\
sec^2(\beta) - tan^2(\beta) = 1 \; \mapsto \; p = sec(\beta)\gamma_0 + tan(\beta)\gamma_3 & Hyperbolic\ symmetry & \beta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]
\end{array}
\tag{2.10}
$$

The above given proportionality factors (2.9), symmetries, and vectors (2.10) are visualized in Fig. 2.2:

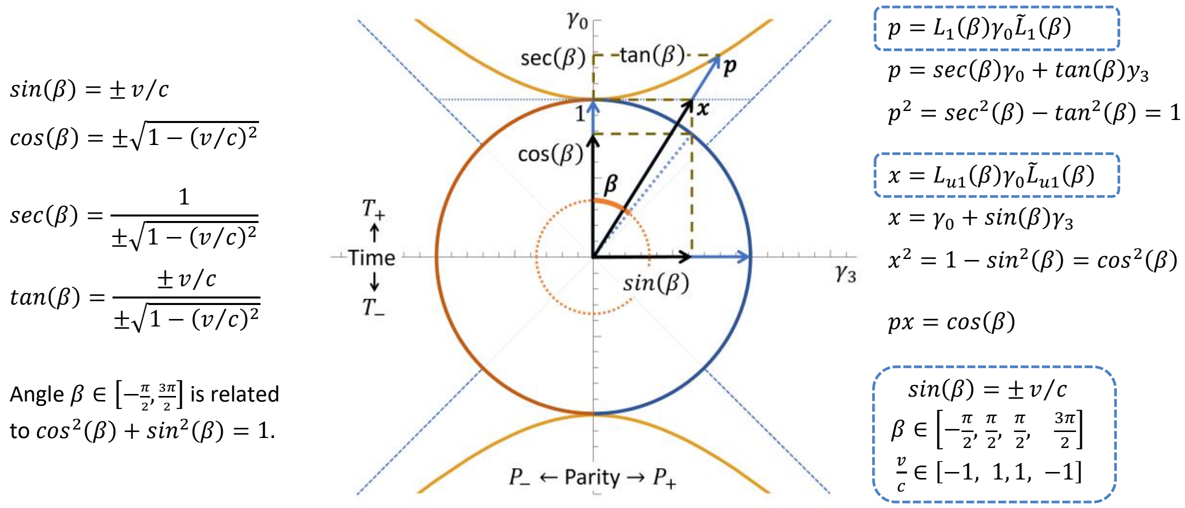


Figure 2.2: Visualization of the relativistic proportionality factors (2.9) together with the two explicit vectors $\{x,p\}$ (2.10). Vector x=$L_{u1}(\beta)\gamma_0\tilde{L}_{u1}(\beta)$ (position vector) is related to a circular symmetry, while vector p=$L_1(\beta)\gamma_0\tilde{L}_1(\beta)$ (momentum vector) is related to a hyperbolic symmetry.

Note that over the full range of $\boldsymbol{\beta}\in\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ vector x is future pointing T_+ while vector p can undergo a time-reversal $T_+ \leftrightarrow T_-$. However, the two vectors $\{x,p\}$ will remain aligned in direction, as reflected in the scalar value of their GP px=$cos(\beta)$=$\pm\sqrt{1-(v/c)^2}$ (Fig. 2.2). Both vectors can have a parity-reversal $P_+ \leftrightarrow P_-$.

3. Spacetime Volume Element

#	*STA* [10, 20-22]	Description
1	$\gamma_\mu = \{t, x, y, z\} = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$	The four orthogonal basis vectors γ_μ for the four spacetime axis $\{t, x, y, z\}$.
2	$(\gamma_0)^2 = +1 \quad (\gamma_k)^2 = -1 \quad \gamma_\mu \cdot \gamma_\nu = 0 \mapsto (\mu \neq \nu)$ $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$ Metric diagonal $\{1, -1, -1, -1\}$	The temporal basis vector γ_0 squares to one, while the spatial basis vectors γ_k square to minus one, i.e., $\mathbb{R}^{1,3}$. The basis vectors γ_μ satisfy the algebra of the Dirac gamma matrices.
3	$\mathbb{i} = \gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \quad \mathbb{i}^2 = -1$	*STA* pseudoscalar $\mathbb{i}$.
4	$\sigma_k = \{x, y, z\} = \{\sigma_1, \sigma_2, \sigma_3\} \quad \sigma_k = \gamma_k\gamma_0$	The three orthogonal basis vectors σ_k for the three spatial axes $\{x, y, z\}$. The spatial basis vectors $\sigma_k = \gamma_k\gamma_0$ are temporal spacetime bivectors which are orthogonal to the temporal basis vector γ_0.
5	$(\sigma_k)^2 = +1 \quad \sigma_k \cdot \sigma_j = 0 \mapsto (k \neq j)$ $\sigma_k \cdot \sigma_j + \sigma_j \cdot \sigma_k = 2\delta_{kj}$	The spatial basis vectors σ_k: square to one, form an even *STA* subalgebra $\{1, \sigma_k, \mathbb{i}\sigma_k, \mathbb{i}\}$, and satisfy the algebra of the Pauli matrices (algebra of $3D$ space).
6	$\mathbb{i} = \sigma_1\sigma_2\sigma_3 = \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \quad \mathbb{i}^2 = -1$ $\mathbb{i} = \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3$	The pseudoscalars of $3D$ space $\sigma_1\sigma_2\sigma_3$ and $4D$ spacetime $\gamma_0\gamma_1\gamma_2\gamma_3$ are equal.
7	$\sigma_k = \gamma_k\gamma_0 \leftrightarrow \gamma_k = \sigma_k\gamma_0$	Changing spacetime basis vector γ_k into a spatial basis vector σ_k and vice versa, can be done via a right *GP* with γ_0.
8	$sin(\beta) = \pm v/c \quad cos(\beta) = \pm\sqrt{1 - (v/c)^2}$	Relativistic proportionality factors with an implicit circular symmetry.

Table 3.1: Overview of STA relevant in this section.

A more detailed overview of STA is given in Appendix A.
In $3D$ space each position x can be marked by three cartesian coordinates $\{x,y,z\}$ or by two polar coordinates $\{\theta,\phi\}$ connected to a symmetrical $3D$ condition:

$$
\begin{aligned}
& r^2 = x^2 + y^2 + z^2 \qquad x(r,\theta,\phi) = r\, u_3(\theta,\phi) && \text{3D position vector} \\
& u_3(\theta,\phi) = sin(\theta)cos(\phi)\sigma_1 + sin(\theta)sin(\phi)\sigma_2 + cos(\phi)\sigma_3 \quad (u_3)^2 = +1 && \text{Spatial unit vector} \\
& e_3(\theta,\phi) = sin(\theta)cos(\phi)\gamma_1 + sin(\theta)sin(\phi)\gamma_2 + cos(\theta)\gamma_3 \quad (e_3)^2 = -1 \quad u_3 = e_3\gamma_0
\end{aligned}
\tag{3.1}
$$

where r^2 is the symmetrical $3D$ condition, an invariant squared spatial distance.

In $4D$ spacetime each position x can be marked by four cartesian coordinates $\{ct,x,y,z\}$:

$$
x = x^\mu\gamma_\mu = ct\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \qquad s^2 = x^2 = c^2t^2 - x^2 - y^2 - z^2 \tag{3.2}
$$

However, by using the Euclidean Lorentz group rotation parameters $\{\beta,\theta,\phi\}$ (section 2), each $4D$ spacetime position x can also be marked by three polar coordinates $\{\beta,\theta,\phi\}$ connected to a symmetrical $4D$ condition (Fig. 3.1) (Appendix C1):

$$
\begin{aligned}
& x(ct,\beta,\theta,\phi) = ct\big(\gamma_0 + sin(\beta)e_3(\theta,\phi)\big) && \text{Spacetime position vector} \\
& x(ct,\beta,\theta,\phi) = x^\mu\gamma_\mu \quad x^0 = ct && x^1 = ct\, sin(\beta)sin(\theta)cos(\phi) \\
& \qquad x^2 = ct\, sin(\beta)sin(\theta)cos(\phi) && x^3 = ct\, sin(\beta)cos(\theta) \\
& s^2 = x^2 = c^2\tau^2 = c^2t^2\big(1 - sin^2(\beta)\big) = c^2t^2cos^2(\beta) && \text{Proper distance}
\end{aligned}
\tag{3.3}
$$

where $s^2 = c^2\tau^2$ is the symmetrical $4D$ condition, an invariant squared spacetime proper distance.

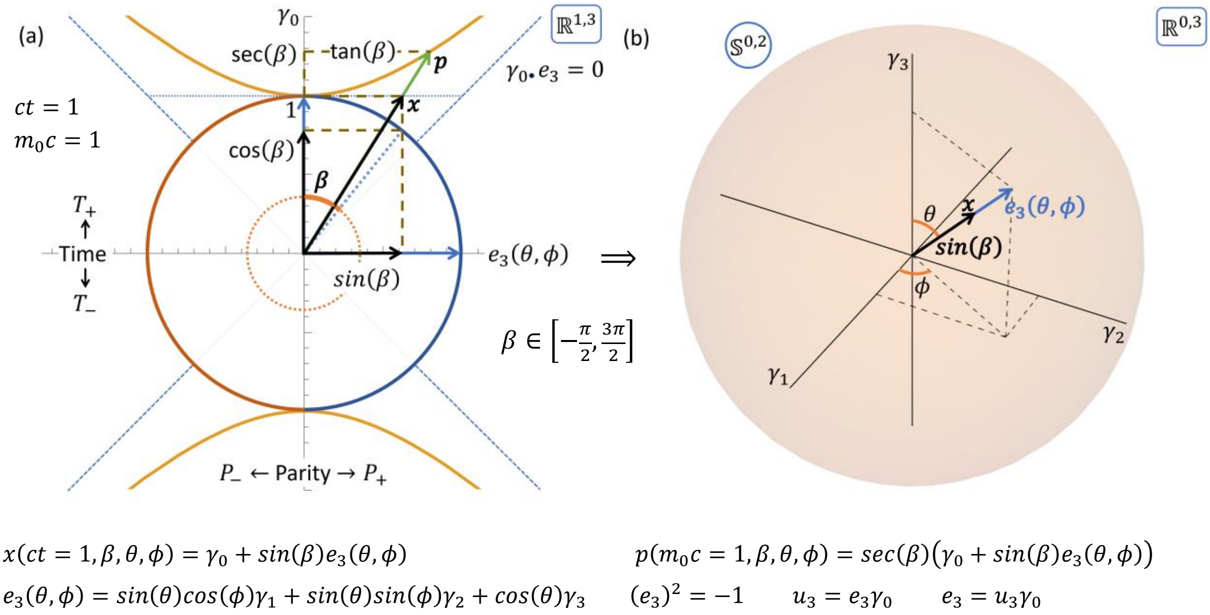


Figure 3.1: Visualizes that each spacetime position x can be marked by three polar coordinates $\{\beta,\theta,\phi\}$ connected to a symmetrical $4D$ condition. In (a) future pointing spacetime position vector $x(ct{=}1,\beta,\theta,\phi)$ (3.3) together with momentum vector $p(m_0c{=}1,\beta,\theta,\phi)$. In (b) two-sphere $\mathbb{S}^{0,2}$ with all possible spatial positions $sin(\beta)e_3(\theta,\phi)$ on and inside two-sphere $\mathbb{S}^{0,2}$.

Figure 3.1a shows all possible spacetime position vectors x in a $2D$ representation, because the temporal basis vector $\gamma_0 \in \mathbb{R}^{1,0}$ and spatial unit vector $e_3(\theta,\phi) \in \mathbb{R}^{0,3}$ (3.1) merge as orthogonal vectors ($\gamma_0 . e_3{=}0$) to $\mathbb{R}^{1,3}$. Figure 3.1b shows all possible spatial positions $sin(\beta)e_3(\theta,\phi)$ on and inside two-sphere $\mathbb{S}^{0,2}$ in $\mathbb{R}^{0,3}$. Note that over the full range of $\beta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ there is a double cover in position. Spatial position $sin(\beta)e_3(\theta,\phi)$ is equal to spatial position $sin(\beta \pm \pi)e_3(\theta \pm \pi,\phi){=}(-sin(\beta))(-e_3(\theta,\phi)){=}sin(\beta)e_3(\theta,\phi)$ (Fig. 3.1b).

By coordinate transformation from cartesian $\{ct,x,y,z\}$ to polar coordinates $\{ct,\beta,\theta,\phi\}$ four spacetime curvilinear basis vectors $\alpha_\mu = \{\alpha_t,\alpha_\beta,\alpha_\theta,\alpha_\phi\}$ can be calculated (Appendix C2):

$$\boxed{\alpha_\mu = \frac{\partial x^\upsilon}{\partial r^\mu}\gamma_\upsilon} \quad \alpha_\mu = \{\alpha_t, \alpha_\beta, \alpha_\theta, \alpha_\phi\} = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} \; r^\mu = \{ct, \beta, \theta, \phi\} \;\; \gamma_\upsilon = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$$

$$x^\upsilon = \{x^0, x^1, x^2, x^3\} \quad x^0 = ct \qquad x^1 = ct\, sin(\beta) sin(\theta) cos(\phi) \tag{3.4}$$

$$x^2 = ct\, sin(\beta) sin(\theta) sin(\phi) \quad x^3 = ct\, sin(\beta) cos(\theta)$$

The metric $g_{\mu\nu}$ of the four curvilinear basis vectors $\alpha_\mu{=}\{\alpha_t,\alpha_\beta,\alpha_\theta,\alpha_\phi\}$ is given in appendix C2. The wedge product $\mathcal{V}\mathbb{i}{=}\alpha_0 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3$ of the four spacetime curvilinear basis vectors α_μ provides via the scalar component $\mathcal{V}$ the spacetime volume element dV:

$$\mathcal{V}\mathbb{i} = \alpha_0 \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \;\rightarrow\; \mathcal{V} = c^3 t^3\, sin^2(\beta)\, cos(\beta)\, sin(\theta) \;\rightarrow\; dV = \mathcal{V}\, cdt\, d\beta\, d\theta\, d\phi$$

$$dV = c^3 t^3\, sin^2(\beta)\, cos(\beta)\, sin(\theta)\, cdt\, d\beta\, d\theta\, d\phi \qquad \text{Spacetime volume element.} \tag{3.5}$$

where dV allows for the calculation of a causality-volume $V_c = \int_\tau^{\cdot} \int_\beta^{\cdot} \int_\theta^{\cdot} \int_\phi^{\cdot} dV$, a spacetime region that is bounded by the $4D$ light cone of a past event (section 4).

4. Causality Volume

This section shows the relation between: (a) causality-volume $V_c = \int_\tau^{\cdot} \int_\beta^{\cdot} \int_\theta^{\cdot} \int_\phi^{\cdot} dV$, (b) position vector (spacetime trajectory [33-35]) $x(c\Delta t,\beta,\theta,\phi)$ (3.3), and (c) the discrete combinations of parity-reversal P and time-reversal T (PT quadrants) as function of hyperbolic rotation angle $\beta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ [5, 8, 36, 37] (Appendix D):

$$V_c = \frac{1}{3}\pi\, c^4 \Delta t^4 cos^4(\beta) \qquad x = c\Delta t\big(\gamma_0 + sin(\beta) e_3(\theta,\phi)\big) \qquad \beta \in \left[\frac{-\pi}{2}, \frac{3\pi}{2}\right]$$

$$(\mathbf{1})\ P_-T_+;\ \beta \in \left[-\tfrac{\pi}{2}, 0\right] \quad (\mathbf{2})\ P_+T_+;\ \beta \in \left[0, \tfrac{\pi}{2}\right] \quad (\mathbf{3})\ P_+T_-;\ \beta \in \left[\tfrac{\pi}{2}, \pi\right] \quad (\mathbf{4})\ P_-T_-;\ \beta \in \left[\pi, \tfrac{3\pi}{2}\right] \tag{4.1}$$

Causality-volume V_c bounds all possible spacetime positions where potentially an observation can take place, based on event-information - embedded in causality-volume V_c - that originated from a past event. The shape of the causality-volume (4D light cone) is related to merging a one-sphere $\mathbb{S}^{1,0}$ with a two-sphere $\mathbb{S}^{0,2}$ into a three-sphere $\mathbb{S}^{1,2}$ (Fig. 3.1) (Fig. 4.1).

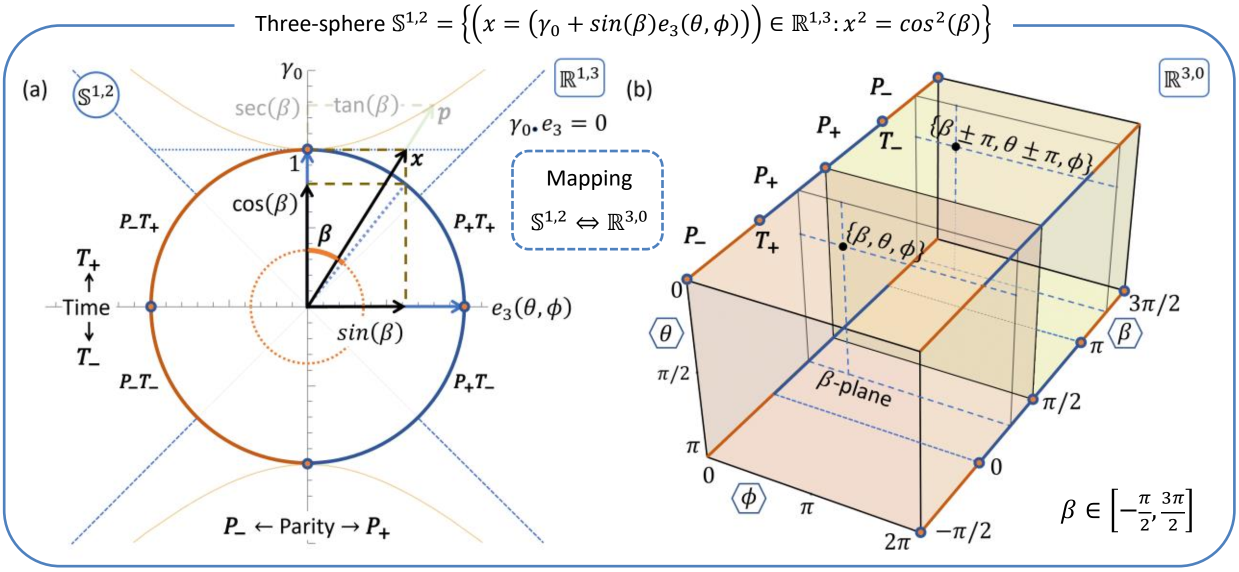


Figure 4.1: Visualizes the relation between all causal position vectors $x(c\Delta t{=}1,\beta,\theta,\phi)$ in three-sphere $\mathbb{S}^{1,2}$ and the discrete PT quadrants. In (a) all causal position vectors $x(c\Delta t{=}1,\beta,\theta,\phi)$ with polar coordinates $\{\beta,\theta,\phi\}$ forming the boundary of three-sphere $\mathbb{S}^{1,2}$. In (b) the mapping of the polar coordinates $\{\beta,\theta,\phi\}$ of position vector x, momentum vector p and the PT quadrants to a double cube in $\mathbb{R}^{3,0}$.

To give insight in the shape of three-sphere $\mathbb{S}^{1,2}$ a comparison between: (a) a two-sphere $\mathbb{S}^{2,0}$ in $\mathbb{R}^{3,0}$, (b) a three-sphere $\mathbb{S}^{3,0}$ in $\mathbb{R}^{4,0}$ [38, 39] and (c) spacetime three-sphere $\mathbb{S}^{1,2}$ in $\mathbb{R}^{1,3}$:

$$\mathbb{S}^{2,0} = \left\{\left(x_a = ct\, u_3(\theta,\phi)\right) \in \mathbb{R}^{3,0}: x_a^2 = c^2t^2; \boldsymbol{r = ct}\right\} \quad u_3 = e_3\gamma_0 \quad (e_3)^2 = -1 \quad V = \frac{4\pi r^3}{3}$$

$$\mathbb{S}^{3,0} = \left\{\left(x_b = ct\left(cos(\beta)\gamma_0 + sin(\beta)u_3(\theta,\phi)\right)\right) \in \mathbb{R}^{4,0}: x_b^2 = c^2t^2; \boldsymbol{r = ct}\right\} \qquad V = \frac{\pi^2 r^4}{2} \tag{4.2}$$

$$\mathbb{S}^{1,2} = \left\{\left(x = ct\left(\gamma_0 + sin(\beta)e_3(\theta,\phi)\right)\right) \in \mathbb{R}^{1,3}: x^2 = c^2t^2cos^2(\beta); \boldsymbol{r = ct\,cos(\beta)}\right\} \quad V = \frac{\pi r^4}{3}$$

where V is the volume of the spheres in comparison. The radius $r{=}ct\,cos(\beta)$ of three-sphere $\mathbb{S}^{1,2}$ depends on the relative motion of an observer. Hence, it will stay a three-sphere independent of the direction of motion [9].

The volume of three-sphere $\mathbb{S}^{1,2}$ forms a boundary for proper distance $c\Delta\tau{=}c\Delta t\,cos(\beta)$ (3.3), i.e., a causality-volume V_c that can be calculated via:

$$V_c = \int_\tau^{\cdot}\int_\beta^{\cdot}\int_\theta^{\cdot}\int_\phi^{\cdot} dV = \int_{\tau=0}^{\Delta\tau}\int_{\beta=0}^{\pi/2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} c^3t^3 sin^2(\beta)\,cos(\beta)\,sin(\theta)\,cdt\,d\beta\,d\theta\,d\phi$$

$$V_c = \frac{1}{3}\pi\, c^4\Delta\tau^4\ [m^4] \quad \Delta\tau = \Delta t\,cos(\beta) \quad \text{Spacetime causality-volume } V_c \text{ of three-sphere } \mathbb{S}^{1,2} \tag{4.3}$$

The β integrand in the calculation of causality-volume V_c must be in the range of $\beta\in\left[0,\frac{\pi}{2}\right]$, because there is a double cover in spatial position ($sin(\beta)e_3(\theta,\phi)=sin(\beta\pm\pi)e_3(\theta\pm\pi,\phi)$) (Fig. 3.1).

The event-information in causality-volume $\boldsymbol{V_c}$ with a $4D$ "radial" expansion as function of time $c\Delta\tau$ [9, 40] can potentially be observed by an observer (measurement device) following a spacetime trajectory $x(c\Delta t,\beta,\theta,\phi)$ with a proper distance $c\Delta\tau$ (Appendix C1). Here the term spacetime trajectory is meant in the sense of a quantum mechanical path integral interpretation [33-35, 41]. The event-information in causality-volume Vc - originated from a past event $c\Delta\tau$ - expands with the flow of time $c\Delta\tau$ while an observer moves along a proper distance $c\Delta\tau$. Hence, the event-information in causality-volume V_c will be observed on the surface of V_c (Appendix D). The surface A_c of V_c ($4D$ light cone) can be calculated via the expansion rate $c\Delta\tau$ of causality-volume V_c:

$$A_c=\frac{d}{c\,d\Delta\tau}V_c=\frac{4}{3}\pi\, c^3\Delta\tau^3\ [m^3] \qquad \Delta\tau=\Delta t\,cos(\beta) \qquad \text{Surface } A_c \text{ of three-sphere } \mathbb{S}^{1,2} \tag{4.4}$$

The surface A_c of causality-volume V_c is the volume of $\boldsymbol{3D}$ space, i.e., the volume of a two-sphere $\mathbb{S}^{2,0}$ (4.2). For observers in relative motion A_c will remain a two-sphere [9, 40] (independent of the direction of motion).

The relation between the relativistic proportionality factors (2.9) and the discrete P and T reversals ($\boldsymbol{PT}$ quadrants in $\mathbb{S}^{1,2}$) (4.1) (Fig. 4.1) as function of hyperbolic rotation angle $\beta\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$ are shown in (Table 4.1):

PT quadrants as function of hyperbolic rotation angle β	PT reversal ratio $R=P/T$	Parity-reversal P	Time-reversal T
	$tan(\beta)=\dfrac{sin(\beta)}{cos(\beta)}=\dfrac{\pm v/c}{\pm\sqrt{1-(v/c)^2}}$	$sin(\beta)=\pm v/c$	$cos(\beta)=\pm\sqrt{1-(v/c)^2}$ $sec(\beta)=\dfrac{1}{\pm\sqrt{1-(v/c)^2}}$
$P_-T_+;\ \beta\in\left[-\frac{\pi}{2},0\right]$	R_-	P_-	T_+
$P_+T_+;\ \beta\in\left[0,\frac{\pi}{2}\right]$	R_+	P_+	T_+
$P_+T_-;\beta\in\left[\frac{\pi}{2},\pi\right]$	R_-	P_+	T_-
$P_-T_-;\beta\in\left[\pi,\frac{3\pi}{2}\right]$	R_+	P_-	T_-

Table 4.1: Discrete PT quadrants (4.1) as function of hyperbolic rotation angle $\beta\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$, i.e., four $\pi/2$ regions.

Each reference frame $\{ct,x,y,z\}$ will contain its own angle β dependent discrete parity-reversal $P_+\leftrightarrow P_-$ and time-reversal $T_+\leftrightarrow T_-$ (PT quadrants) (4.1) (Fig. 4.1). Hence, in case of a difference in temporal and/or spatial orientation (boost and/or spatial rotation) there is a difference in time-reversal T and/or parity-reversal P between the reference frames, i.e., the PT quadrants of the reference frames will not be aligned.

The result of an observation - based on event-information in causality-volume V_c - arises from the interaction between observer trajectory x and event-information in the four possible PT quadrants:

$$\left.\begin{array}{lll} P_-T_+ & \beta\in\left[-\frac{\pi}{2},0\right] & x(c\Delta t,\beta,\theta\pm\pi,\phi)=c\Delta t\big(\gamma_0-sin(\beta)e_3(\theta,\phi)\big) \\ P_+T_+ & \beta\in\left[0,\frac{\pi}{2}\right] & x(c\Delta t,\beta,\theta,\phi)=c\Delta t\big(\gamma_0+sin(\beta)e_3(\theta,\phi)\big) \end{array}\right\} T_+ \text{ in } \mathbb{R}^{3,0}$$

Fig. 4.1b

$$\left.\begin{array}{lll} P_+T_- & \beta\in\left[\frac{\pi}{2},\pi\right] & x(c\Delta t,\beta\pm\pi,\theta,\phi)=c\Delta t\big(\gamma_0-sin(\beta)e_3(\theta,\phi)\big) \\ P_-T_- & \beta\in\left[\pi,\frac{3\pi}{2}\right] & x(c\Delta t,\beta\pm\pi,\theta\pm\pi,\phi)=c\Delta t\big(\gamma_0+sin(\beta)e_3(\theta,\phi)\big) \end{array}\right\} T_- \text{ in } \mathbb{R}^{3,0} \tag{4.5}$$

Each of the four spacetime trajectories $x(c\Delta t,\beta,\theta,\phi)$ is related to a PT quadrant (4.1). Spatial rotation angle θ provides a parity-reversal P by $\theta\pm\pi$ acting on $e_3(\theta,\phi)$. Temporal rotation angle β provides a time-reversal T by $\beta\pm\pi$ acting on $cos(\beta)$ plus also a parity-reversal by $\beta\pm\pi$ acting on $sin(\beta)$ (Table 4.1).

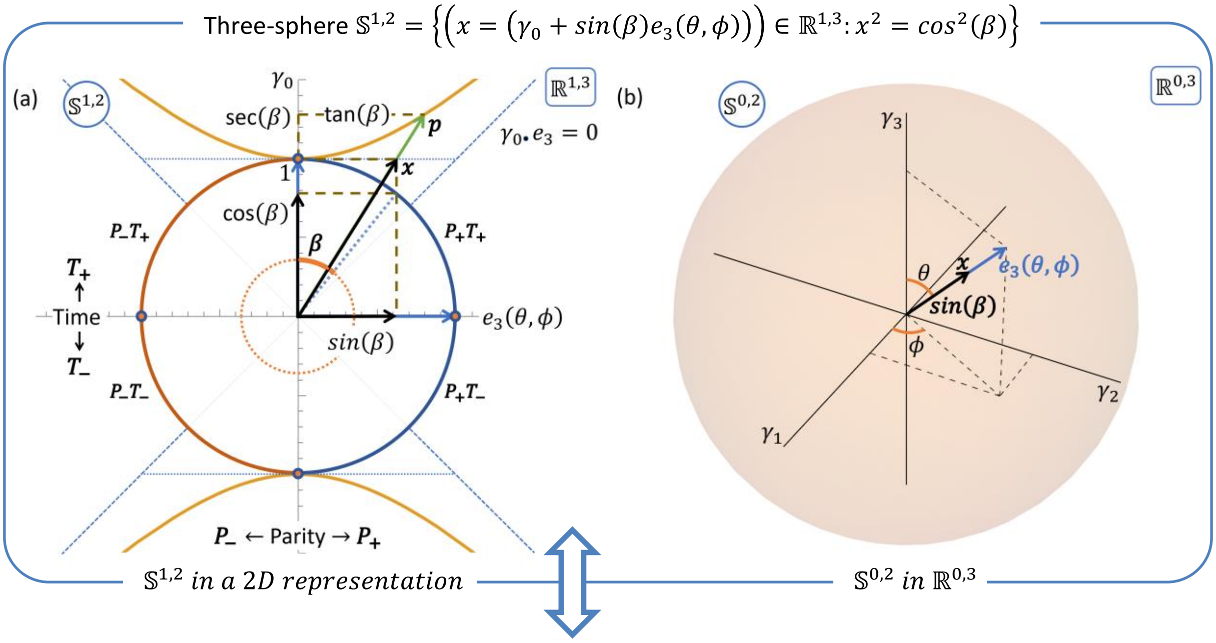
So, an observer can perform an observation moving along spacetime trajectory $x(c\Delta t,\beta,\theta,\phi)$ or spacetime trajectory $x(c\Delta t,\beta\pm\pi,\theta\pm\pi,\phi)$ which are equal, but different in PT quadrant ($P+T+$ or $P-T-$) (4.5). The same holds for the spacetime trajectories $x(c\Delta t,\beta,\theta\pm\pi,\phi)$ and $x(c\Delta t,\beta\pm\pi,\theta,\phi)$ which are equal, but also different in PT quadrant (P_-T_+ or P_+T_-) (4.5). The result of an observation manifests itself on the surface A_c (4.1) of causality-volume V_c (4.3). However, interactions (observer trajectory $\leftrightarrow$ event-information) take place during ($c\Delta t$) a spacetime trajectory in causality-volume V_c (4.3) (duration expressed in coordinate time $c\Delta t$).

The puzzling and fascinating character of an quantum mechanical observation is the manifestation of event-information in a particle (fermion or a boson) [4, 42, 43], i.e. it are particles that are being observed. This is very noticeable in the sound of a Geiger counter. An observation contains an initial state, a transition phase, and a final state. In the initial state definite information (energy, momentum, spin, etc …) about a particle is known. The transition phase takes place during an interaction between observer (measurement device) and the event-information in causality-volume V_c. Where the event-information in causality-volume V_c originates from both the initial state of the particle (past event) and the observer. The final state provides the manifestation of a particle - at present time - on the surface A_c of causality-volume V_c. The initial state is connected to the final state by a spacetime trajectory $x(c\Delta t,\beta,\theta,\phi)$ [33-35, 41] through causality-volume V_c.

The discrete PT quadrants as function of hyperbolic rotation angle $\beta\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$ (4.1) are connected to the reference frame of an observer as well as to the reference frame of the event-information (initial state reference frame). Both reference frames (observer and event-information) are bounded by the causality-volume V_c. The observer via a spacetime trajectory $x(c\Delta t,\beta,\theta,\phi)$ with proper distance $c\Delta\tau$. The event-information - expanding with the flow of time $c\Delta\tau$ - distributed over the causality-volume V_c (4.3). An PT quadrant is discrete, but the orientation between the reference frames (observer and event-information) can differ continuously by a temporal rotation over angle β (boost) and/or by a spatial rotation given by the polar angles $\{\theta,\phi\}$. For the latter we must distinguish between two types of spatial rotation: (1) a spatial rotation defining the direction of a boost and (2) a spatial rotation defining the spatial spin direction (vector). In this article we use an equal boost and spin direction (vector), known as helicity.

5. Spacetime Spinors

This section shows the derivation of a Dirac spinor ψ_1, purely from three-sphere $\mathbb{S}^{1,2}$ symmetries (Fig. 5.1).



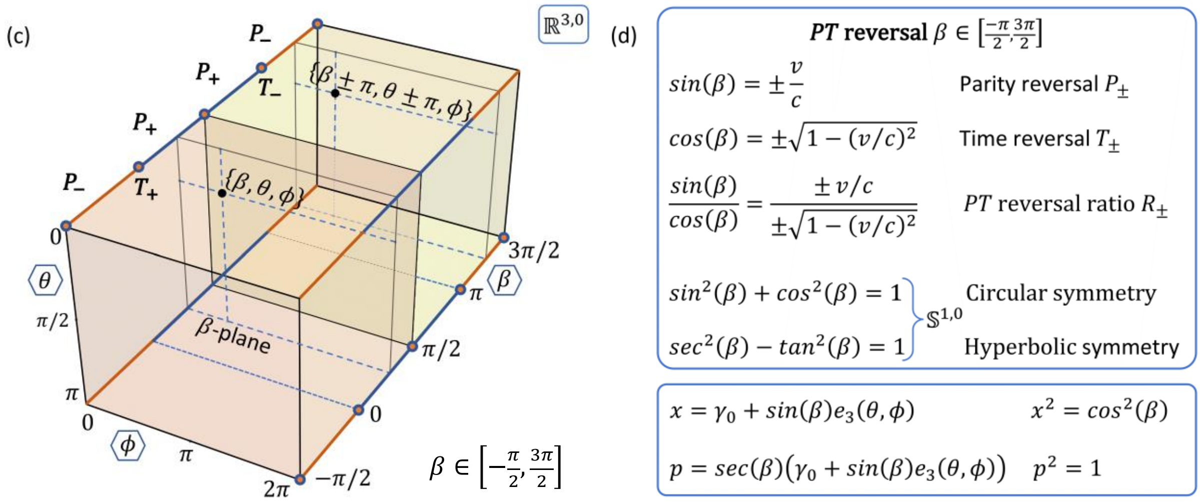


Figure 5.1: Three-sphere $\mathbb{S}^{1,2}$ represents all possible spacetime trajectories $x(c\Delta t{=}1,\beta,\theta,\phi)$ parameterized by three Euclidean polar coordinates $\{\beta,\theta,\phi\}$, which can be mapped to a double cube in $\mathbb{R}^{3,0}$ (c). In (a) three-sphere $\mathbb{S}^{1,2}$ in a $2D$ representation with: (1) the four PT quadrants as function of angle β, (2) circular symmetry related position vector x, and (3) hyperbolic symmetry related momentum vector p. In (b) two-sphere $\mathbb{S}^{0,2}$ in $\mathbb{R}^{0,3}$ with all possible spatial positions $sin(\beta)e_3(\theta,\phi)$. In (d) discrete parity-reversal P and time-reversal T as function of angle $\beta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, in relation to the circular symmetry of position vector x and the hyperbolic symmetry of momentum vector p.

The Lorentz group consist of six generators, represented by the six spacetime bivectors $\{\sigma_k, \mathbb{i}\sigma_k\}$ (Appendix A). To perform all possible spacetime rotations three of the six bivectors $\{\sigma_k, \mathbb{i}\sigma_k\}$ are needed. They can be chosen as: (a) temporal bivector σ_3 (zt plane) and (b) two spatial bivectors $\{\mathbb{i}\sigma_2, \mathbb{i}\sigma_3\}$ (zx, xy plane) [22, 28]. From this set of three orthogonal planes represented by spacetime bivectors $\{\sigma_3, \mathbb{i}\sigma_2, \mathbb{i}\sigma_3\}$, three unitary irreducible rotors $\{L_1(\beta), Q_\theta(\theta), Q_\phi(\phi)\}$ can be calculated [32] (2.6) (2.7) (Appendix B):

$$L_1(\beta) = \sqrt{sec(\beta)}\left(cos\left(\frac{\beta}{2}\right) + sin\left(\frac{\beta}{2}\right)\sigma_3\right) \quad \text{Temporal rotor in: } \sigma_3 = \gamma_3\gamma_0 \text{ plane} \quad (\sigma_3)^2 = +1$$

$$Q_\theta(\theta) = 1\left(cos\left(\frac{\theta}{2}\right) - sin\left(\frac{\theta}{2}\right)\mathbb{i}\sigma_2\right) \quad \text{Spatial rotor in: } -\mathbb{i}\sigma_2 = \gamma_3\gamma_1 \text{ plane} \quad (\mathbb{i}\sigma_2)^2 = -1 \tag{5.1}$$

$$Q_\phi(\phi) = 1\left(cos\left(\frac{\phi}{2}\right) - sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_3\right) \quad \text{Spatial rotor in: } -\mathbb{i}\sigma_3 = \gamma_1\gamma_2 \text{ plane} \quad (\mathbb{i}\sigma_3)^2 = -1$$

where $L_1(\beta)=\sqrt{sec(\beta)}L_{u1}(\beta)$ is an unitary irreducible temporal hyperbolic rotor with scalar density $\sqrt{sec(\beta)}$ times temporal spinor $L_{u1}(\beta)$ (2.7) and $\{Q_\theta(\theta), Q_\phi(\phi)\}$ are two unitary irreducible spatial rotors. Rotors are part of the STA even subalgebra $M+$ (Appendix A). So, the GP of $S_1(\theta,\phi)=Q_\phi(\phi)Q_\theta(\theta)$ provides by means of the two orthogonal spatial planes $\{\mathbb{i}\sigma_2, \mathbb{i}\sigma_3\}$ a unitary spatial rotor $S_1(\theta,\phi)$:

$$S_1(\theta,\phi) = \lambda_1 W_1 = 1\left(cos\left(\frac{\phi}{2}\right) - sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_3\right)\left(cos\left(\frac{\theta}{2}\right) - sin\left(\frac{\theta}{2}\right)\mathbb{i}\sigma_2\right) \quad \lambda_1 = \left(W_1\widetilde{W}_1\right)^{-1/2} = 1$$
$$\mathbb{S}^{0,2} = \left\{e_k = S_1\gamma_k\tilde{S}_1 \in R^{0,3} : e_k^2 = -1\right\} \qquad e_k.e_j = diag(-1,-1,-1) \tag{5.2}$$

Spatial rotor/spinor $S_1(\theta,\phi)$ is equal to a complex Pauli spinor [15, 20, 44, 45] (Appendix E1).

The geometrical product of spatial spinor $S_1(\theta,\phi)$ with irreducible temporal hyperbolic rotor $L_1(\beta)$ (5.1) - acting in temporal plane σ_3, which is orthogonal to each of the two spatial planes $\{\mathbb{i}\sigma_2, \mathbb{i}\sigma_3\}$ - results in a spacetime rotor $R_1(\beta,\theta,\phi)=S_1(\theta,\phi)L_1(\beta)$ (Appendix E2):

$$S_1 = \left(cos\left(\frac{\phi}{2}\right) - sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_3\right)\left(cos\left(\frac{\theta}{2}\right) - sin\left(\frac{\theta}{2}\right)\mathbb{i}\sigma_2\right) \quad L_1 = \sqrt{sec(\beta)}\left(cos\left(\frac{\beta}{2}\right) + sin\left(\frac{\beta}{2}\right)\sigma_3\right)$$

$$R_1 = \sqrt{sec(\beta)}\left(cos\left(\frac{\phi}{2}\right) - sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_3\right)\left(cos\left(\frac{\theta}{2}\right) - sin\left(\frac{\theta}{2}\right)\mathbb{i}\sigma_2\right)\left(cos\left(\frac{\beta}{2}\right) + sin\left(\frac{\beta}{2}\right)\sigma_3\right) \tag{5.3}$$

In spacetime rotor $R_1(\beta,\theta,\phi)=S_1(\theta,\phi)L_1(\beta)\tilde{S}_1(\theta,\phi)S_1(\theta,\phi)=S_1(\theta,\phi)L_1(\beta)$ are the boost direction and the spin direction vector aligned, i.e., helicity. The general case with a separation in boost and spin direction is research in progress $R_1(\beta,\theta_b,\phi_b,\theta_s,\phi_s)=S_1(\theta_b,\phi_b)L_1(\beta)$ $S_1(\theta_b,\phi_b)S_1(\theta_s,\phi_s)$.

Spacetime rotor $R_1=\eta_1U_1$ has due to the demand for unitarity $R_1\tilde{R}_1=1$ a density factor of $\eta_1=\sqrt{sec(\beta)}$, which is true for the helicity (this article) and general (in progress) formulation of spacetime spinor R_1. The spacetime spinor part $U_1=S_1L_{u1}$ consist of a spatial spinor S_1 times irreducible temporal spinor L_{u1}:

$$R_1(\beta,\theta,\phi) = \eta_1 U_1(\beta,\theta,\phi) \quad U_1(\beta,\theta,\phi) = S_1(\theta,\phi)L_{u1}(\beta) \quad L_{u1}(\beta) = cos\left(\frac{\beta}{2}\right) + sin\left(\frac{\beta}{2}\right)\sigma_3$$

$$U_1(\beta,\theta,\phi) = \left(cos\left(\frac{\phi}{2}\right) - sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_3\right)\left(cos\left(\frac{\theta}{2}\right) - sin\left(\frac{\theta}{2}\right)\mathbb{i}\sigma_2\right)\left(cos\left(\frac{\beta}{2}\right) + sin\left(\frac{\beta}{2}\right)\sigma_3\right) \tag{5.4}$$

$$R_1\tilde{R}_1 = 1 \;\mapsto\; \eta_1 = \left(U_1\tilde{U}_1\right)^{-1/2} = \sqrt{sec(\beta)} \qquad U_1\tilde{U}_1 = cos(\beta) = \pm\sqrt{1-(v/c)^2}$$

Spacetime rotor R_1 (5.3) and spacetime spinor U_1 (5.4) can perform all possible spacetime rotations while preserving circular and hyperbolic symmetries in $\mathbb{R}^{1,3}$ (Fig. 5.1) (2.9). This symmetry preserving property is reflected in the two spacetime frames $\mathcal{P}_\mu=R_1\gamma_\mu\tilde{R}_1$ and $\chi_\mu=U_1\gamma_\mu\tilde{U}_1$, as shown in table 5.1.

μ	$\mathcal{P}_\mu = \boldsymbol{R_1\gamma_\mu\tilde{R}_1} \in R^{1,3}: \mathcal{P}_0^2 = 1, \mathcal{P}_k^2 = -1$ $R_1 = \sqrt{sec(\beta)}S_1L_{u1} \quad \mathcal{P}_\mu = \boldsymbol{R_1\gamma_\mu\tilde{R}_1}$		$\chi_\mu = \boldsymbol{U_1\gamma_\mu\tilde{U}_1} \in R^{1,3}: \chi_0^2 = cos^2(\beta), \chi_k^2 = -cos^2(\beta)$ $U_1 = S_1L_{u1} \quad \chi_\mu = \boldsymbol{U_1\gamma_\mu\tilde{U}_1}$
0	$\mathcal{P}_0 = p = sec(\beta)(\gamma_0 + sin(\beta)e_3)$		$\chi_0 = x = \gamma_0 + sin(\beta)e_3$
1	$\mathcal{P}_1 = e_1 = S_1\gamma_1\tilde{S}_1$	Spacetime bivector $\gamma_1\gamma_2 = -\mathbb{i}\sigma_3$	$\chi_1 = cos(\beta)e_1$
2	$\mathcal{P}_2 = e_2 = S_1\gamma_2\tilde{S}_1$		$\chi_2 = cos(\beta)e_2$
3	$\mathcal{P}_3 = sec(\beta)(sin(\beta)\gamma_0 + e_3)$		$\chi_3 = sin(\beta)\gamma_0 + e_3$
	Hyperbolic symmetry: $sec^2(\beta) - tan^2(\beta) = 1$		**Circular symmetry**: $sin^2(\beta) + cos^2(\beta) = 1$

Table 5.1: Circular and hyperbolic $\mathbb{R}^{1,3}$ symmetries as reflected in the spacetime frames $\mathcal{P}_\mu$ and χ_μ.

The emerging pattern in the spacetime frames $\{\mathcal{P}_\mu,\chi_\mu\}$ is independent of the three selected orthogonal spacetime bivectors $\{\sigma_3,\mathbb{i}\sigma_2,\mathbb{i}\sigma_3\}$ (5.1) from which the three irreducible rotors where calculated $\{L_1(\beta),Q_\theta(\theta),Q_\phi(\phi)\}$. If the choice had been a different temporal plane with two other spatial planes, the emerging pattern would have been the same.

The emerging pattern from the spacetime frames $\mathcal{P}_\mu(\beta,\theta,\phi)$ and $\chi_\mu(\beta,\theta,\phi)$ can be summarized as:

- The sum of the vectors $\mathcal{P}_0+\mathcal{P}_3$ and $\chi_0+\chi_3$ are null vectors, i.e., $\mathcal{P}_\mu$ and χ_μ are Lorentz frames.
- The GP of $\mathcal{P}_1\mathcal{P}_2$ and $\chi_1\chi_2$ are related to spacetime bivector $\gamma_1\gamma_2=-\mathbb{i}\sigma_3$ (Table 5.1) via bivector $e_1e_2=(R_1\gamma_1\tilde{R}_1)(R_1\gamma_2\tilde{R}_1)=R_1\gamma_1\gamma_2\tilde{R}_1=-\mathbb{i}(sin(\theta)cos(\phi)\sigma_1+sin(\theta)sin(\phi)\sigma_2+cos(\theta)\sigma_3)$ [21] (Venzo de Sabbata; page 116 – page 122). So, spacetime bivector $e_1e_2=R_1\gamma_1\gamma_2\tilde{R}_1=-\mathbb{i}u_3$ is the dual of the spatial unit vector $u_3=S_1\sigma_3\tilde{S}_1$, i.e., the dual of the boost and spin direction (vector).

A phase $S/\hbar$ (action S divided by Planck's constant $\hbar$) - to induce a phase in bivector $\gamma_1\gamma_2$=$-\mathbb{i}\sigma_3$ - can be calculated from the GP of momentum and position S=px. This GP of action S=px [46, 47] will unify the hyperbolic symmetry of momentum $\boldsymbol{p}$ with the circular symmetry of spacetime trajectory $\boldsymbol{x}$ (Appendix E2):

$$S = px = m_0c^2\Delta t\, cos(\beta) = \pm m_0c^2\Delta t\sqrt{1-(v/c)^2} \quad [kg\ m^2/s] \quad px = p.x \ \mapsto \ p\wedge x = 0$$
$$S/\hbar = \frac{m_0c^2}{\hbar}\Delta t\, cos(\beta) \quad [-] \quad \Rightarrow \quad S/\hbar = \omega_0\Delta\tau \qquad \beta \in \left[\frac{-\pi}{2},\frac{3\pi}{2}\right] \tag{5.5}$$

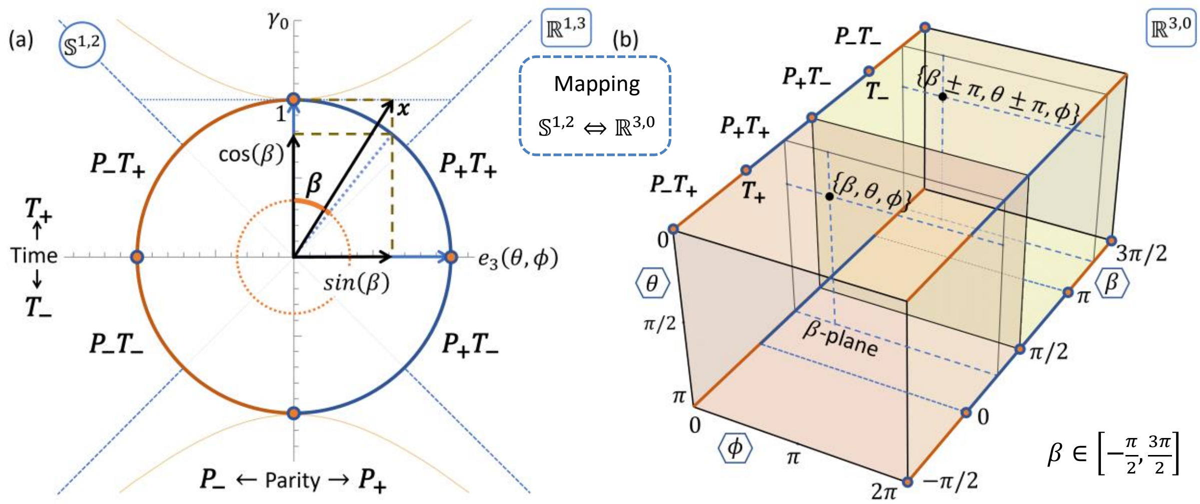
Bringing together: (a) spacetime spinor U_1 (5.4), (b) spacetime bivector $\gamma_1\gamma_2$=$-\mathbb{i}\sigma_3$ (Table 5.1) and (c) phase $S/\hbar$ = $\omega_0\Delta\tau$ (5.5), results in a wavefunction $\psi_1(x)$=$\psi_1(c\Delta t,\beta,\theta,\phi)$:

$$\psi_1(c\Delta t,\beta,\theta,\phi) = U_1\ Exp(-\mathbb{i}\sigma_3\omega_0\Delta\tau) \quad \omega_0 = \frac{m_0c^2}{\hbar} \quad \Delta\tau = \Delta t\, cos(\beta) = \pm\Delta t\sqrt{1-(v/c)^2} \tag{5.6}$$

Wavefunction $\psi_1(c\Delta t,\beta,\theta,\phi)$ derived purely from $\mathbb{S}^{1,2}$ symmetries (Fig. 5.1) is a solution of the Dirac equation [4, 15, 22, 27, 48, 49] (Appendix E3):

$$\hbar\nabla\psi_1(x)\mathbb{i}\sigma_3 - m_0c^2\psi_1(x)\gamma_0 = 0 \quad \psi_1(x) = \psi_1(c\Delta t,\beta,\theta,\phi) = U_1(\beta,\theta,\phi)\ Exp(-\mathbb{i}\sigma_3\omega_0\Delta\tau) \tag{5.7}$$

Consequently, $\boldsymbol{\psi_1(x)}$ is a Dirac spinor representing information that at observation will manifest in one of the four fermion properties: a particle or anti-particle with spin-up or spin-down. An observer (measurement device) can perform a measurement along a spacetime trajectory x in four possible $\boldsymbol{PT}$ quadrants (4.5) (Fig. 5.2).



(c)

***PT* quadrant**	$P_-T_+;\ \beta \in \left[-\frac{\pi}{2}, 0\right]$ $\quad P_+T_+;\ \beta \in \left[0, \frac{\pi}{2}\right]$ $\quad P_+T_-;\beta \in \left[\frac{\pi}{2}, \pi\right]$ $\quad P_-T_-;\beta \in \left[\pi, \frac{3\pi}{2}\right]$
Relative orientation between the frames	The discrete PT quadrants are connected to a $\{ct, x, y, z\}$ reference frame (inertial frame). These reference frames can have a difference in temporal and/or spatial orientation.
Related causality-volume.	$V_c = \frac{1}{3}\pi\, c^4\Delta\tau^4\ [m^4]$ has an $4D$ "radial" expansion as function of $c\Delta\tau = c\Delta t\, cos(\beta)$
Related observer (measurement device) spacetime trajectories $x(c\Delta t, \beta, \theta, \phi)$ (4.5) with proper distance $c\Delta\tau$.	$R = P/T\,; Ratio$ (giving $c\Delta\tau$): $R_-P_-T_+$ $\quad x(c\Delta t, \beta, \theta \pm \pi, \phi) = c\Delta t(\gamma_0 - sin(\beta)e_3(\theta, \phi))$ $R_+P_+T_+$ $\quad x(c\Delta t, \beta, \theta, \phi) = c\Delta t(\gamma_0 + sin(\beta)e_3(\theta, \phi))$ $R_-P_+T_-$ $\quad x(c\Delta t, \beta \pm \pi, \theta, \phi) = c\Delta t(\gamma_0 - sin(\beta)e_3(\theta, \phi))$ $R_+P_-T_-$ $\quad x(c\Delta t, \beta \pm \pi, \theta \pm \pi, \phi) = c\Delta t(\gamma_0 + sin(\beta)e_3(\theta, \phi))$
Related Dirac spinor $\psi_1(x)$. $\psi_1(c\Delta t, \beta, \theta, \phi)$ with the four possible fermion properties at observation. Note the connection with the spacetime trajectories.	$R_-P_-T_+$; Particle S_{down} $\quad \psi_1(c\Delta t, \beta, \theta \pm \pi, \phi) = \pm S_1(\theta \pm \pi, \phi)\gamma_0 L_{u1}(\beta)\gamma_0 e^{-\mathbb{i}\sigma_3\omega_0\Delta\tau}$ $R_+P_+T_+$; Particle S_{up} $\quad \psi_1(c\Delta t, \beta, \theta, \phi) = \pm S_1(\theta, \phi)L_{u1}(\beta)e^{-\mathbb{i}\sigma_3\omega_0\Delta\tau}$ $R_-P_+T_-$; Anti-Particle S_{down} $\quad \psi_1(c\Delta t, \beta \pm \pi, \theta, \phi) = \pm S_1(\theta, \phi)\gamma_0 L_{u1}(\beta \pm \pi)\gamma_0 e^{+\mathbb{i}\sigma_3\omega_0\Delta\tau}$ $R_+P_-T_-$; Anit-Particle S_{up} $\quad \psi_1(c\Delta t, \beta \pm \pi, \theta \pm \pi, \phi) = \pm S_1(\theta \pm \pi, \phi)L_{u1}(\beta \pm \pi)e^{+\mathbb{i}\sigma_3\omega_0\Delta\tau}$ $Exp(-\mathbb{i}\sigma_3\omega_0\Delta t\, cos(\beta)) = e^{-\mathbb{i}\sigma_3\omega_0\Delta\tau}$ $\quad Exp(-\mathbb{i}\sigma_3\omega_0\Delta t\, cos(\beta \pm \pi)) = e^{+\mathbb{i}\sigma_3\omega_0\Delta\tau}$
$\mathbb{S}^{1,2}, R;\ \boldsymbol{\gamma_0 L_{u1}\gamma_0}$ Spin-reversal via PT ratio	$\mathbb{S}^{1,2}, P;\ \boldsymbol{\theta \pm \pi}$ Parity-reversal $\quad \mathbb{S}^{1,2}, T;\ \boldsymbol{\beta \pm \pi}$ Time-reversal via $cos(\beta)$

Figure 5.2: Three-sphere $\mathbb{S}^{1,2}$ representing all possible spacetime trajectories $x(c\Delta t{=}1,\beta,\theta,\phi)$ parameterized by three polar coordinates $\{\beta,\theta,\phi\}$. In (a) three-sphere $\mathbb{S}^{1,2}$ with the four PT quadrants together with future pointing position vector x. In (b) the mapping of the PT quadrants in $\mathbb{S}^{1,2}$ to a double cube in $\mathbb{R}^{3,0}$. In (c) the relation between: PT quadrants, causality-volume V_c, related observer spacetime trajectory x, and related Dirac spinor $\psi_1(x)$ with the four possible fermion properties at observation on surface A_c of causality-volume V_c.

Dirac spinor $\psi_1(x)$ holds all possible information that at observation will manifest in one of the four fermion properties: one of the four eigenvalues of the Dirac equation a particle or anti-particle with spin-up or spin-down. Hence, Dirac spinor $\psi_1(x)$ implicitly holds all four eigenvectors $\psi_j=\{\psi_1,\psi_2,\psi_3,\psi_4\}$ of the Dirac equation. The eigenvectors ψ_j whitin ψ_1 can be made explicit by geometric derivation acting on ψ_1 (Fig. 5.2c) (Table 5.2). The geometric derivation provides a connection between the eigenvectors ψ_j and the PT quadrants via a parity-reversal ($P_+\leftrightarrow P_-$) provided by $\theta\pm\pi$, a time-reversal ($T_+\leftrightarrow T_-$) provided by $\beta\pm\pi$ and a spin-reversal (up $\leftrightarrow$ down) provided by $\gamma_0 L_{u1}\gamma_0$, i.e., spin-up at P_+T_+,P_-T_- and spin-down at P_-T_+,P_+T_- (Table 5.2).

Eigenvectors $\psi_j(c\Delta t, \beta, \theta, \phi)$	**Geometric derivation** of U_k and V_k with $U_1 = S_1 L_{u1}$ as a base.		$U_j(\beta, \theta, \phi)$ and $V_j(\beta, \theta, \phi)$ $S_2(\theta, \phi) = S_1(\theta + \pi, \phi)$	
$\psi_1 = U_1 e^{-\mathbb{i}\sigma_3\omega_0\Delta\tau}$	$U_1 = \pm S_1(\theta, \phi)L_{u1}(\beta)$	$R_+P_+T_+$	$U_1 = \pm S_1(\theta, \phi)(+cos(\beta/2) + sin(\beta/2)\sigma_3)$	S_{up}
$\psi_2 = U_2 e^{-\mathbb{i}\sigma_3\omega_0\Delta\tau}$	$U_2 = \pm S_1(\theta \pm \pi, \phi)\gamma_0 L_{u1}(\beta)\gamma_0$	$R_-P_-T_+$	$U_2 = \pm S_2(\theta, \phi)(+cos(\beta/2) - sin(\beta/2)\sigma_3)$	S_{down}
$\psi_3 = V_1 e^{+\mathbb{i}\sigma_3\omega_0\Delta\tau}$	$V_1 = \pm S_1(\theta \pm \pi, \phi)L_{u1}(\beta \pm \pi)$	$R_+P_-T_-$	$V_1 = \pm S_2(\theta, \phi)(-sin(\beta/2) + cos(\beta/2)\sigma_3)$	S_{up}
$\psi_4 = V_2 e^{+\mathbb{i}\sigma_3\omega_0\Delta\tau}$	$V_2 = \pm S_1(\theta, \phi)\gamma_0 L_{u1}(\beta \pm \pi)\gamma_0$	$R_-P_+T_-$	$V_2 = \pm S_1(\theta, \phi)(-sin(\beta/2) - cos(\beta/2)\sigma_3)$	S_{down}
$Exp(-\mathbb{i}\sigma_3\omega_0\Delta t\, cos(\beta)) = e^{-\mathbb{i}\sigma_3\omega_0\Delta\tau}$	$Exp(-\mathbb{i}\sigma_3\omega_0\Delta t\, cos(\beta \pm \pi)) = e^{+\mathbb{i}\sigma_3\omega_0\Delta\tau}$		Positive and negative energy states	
$\mathbb{S}^{1,2}, R;\ \boldsymbol{\gamma_0 L_{u1}\gamma_0}$ Spin-reversal PT ratio	$\mathbb{S}^{1,2}, P;\ \boldsymbol{\theta \pm \pi}$ Parity-reversal		$\mathbb{S}^{1,2}, T;\ \boldsymbol{\beta \pm \pi}$ Time-reversal via $cos(\beta)$	

Table 5.2: Relation between Dirac spinor $\psi_1(x)$ and the eigenvectors $\psi_j(x)$ of the Dirac equation.

Here U_1=S_1L_{u1} is a base for the geometric derivation. However, each of the spinors U_k or V_k can serve as a base because each of them separately can be derived purely from $\mathbb{S}^{1,2}$ symmetries. The spatial spinors S_1 (5.2) and S_2 in U_k,V_k are directly related to the complex Pauli spinors [50] ($\pm S_2(\theta,\phi)$=$S_1(\theta\pm\pi,\phi)$) (Appendix E1).

Each of the eigenvectors ψ_j (Table 5.2) of the Dirac equation (5.7) are directly related to the four well known complex eigenvectors [4] of the Dirac equation [15,22,27,48,49] (Appendix E3). Figure 5.2c and table 5.2 show that the four eigenvectors ψ_j={ψ_1,ψ_2,ψ_3,ψ_4} are connected to the PT quadrants (Table 4.1) and the spacetime trajectories $x(c\Delta t,\beta,\theta,\phi)$ (4.5) of an observer. The two reference frames (observer and event-information) will interact in the transition phase. If they are aligned in PT quadrants (4.1) the initial state is with a 100% probability detected in the final state. If there is a difference in temporal and/or spatial orientation an $\boldsymbol{PT}$ quadrant information mixing will occur providing final state outcome probabilities for: particle and anti-particles with spin-up or spin-down. An experiment to proof or falsify this logic is given in appendix F.

6. Discussion

The introduction of a hyperbolic rotor $L_1(\beta)$ (2.7) with Euclidean angle β unite the mixed Lorentz group rotation parameters into a set of Euclidean rotation parameters. This unification allows for: (a) a periodic and double cover connection with spacetime symmetries (2.8) (2.9), (b) the calculation of a causality-volume, i.e., the volume of a three-sphere $\mathbb{S}^{1,2}$ ($4D$ light cone) (Fig. 4.1) (4.3), and (c) the derivation of a Dirac spinor (5.6) purely from three-sphere $\mathbb{S}^{1,2}$ symmetries (Fig. 5.1). The connection with spacetime symmetry (2.9) is based on the mapping of $\boldsymbol{tanh(\varphi)}$=$\boldsymbol{sin(\beta)}$=$\pm v/c$ to relative speed (2.8). It is this implicit $\boldsymbol{tanh(\varphi)}$=$\boldsymbol{sin(\beta)}$ relationship which allows for the decomposition of irreducible hyperbolic rotor $R(\varphi)$ (2.6) in a scalar density $\sqrt{sec(\beta)}$ and a temporal spinor $L_{u1}(\beta)$ (2.7). An important factor for the calculation of causality-volume V_c ($4D$ light cone) is the spacetime curvilinear volume element dV (3.5). This volume element dV (3.5) also offers the possibility to calculate spacetime densities, spacetime Green's functions [51, 52] and spacetime Fourier transformations.

The Euclidean Lorentz group rotation parameters {β,θ,ϕ}, form the polar coordinates {β,θ,ϕ} of three-sphere $\mathbb{S}^{1,2}$ ($4D$ light cone) (Fig. 4.1). The volume of three-sphere $\mathbb{S}^{1,2}$ bounds the spacetime trajectories $x(c\Delta t,\beta,\theta,\phi\ )$ (4.2) where potentially an observation can take place based on event-information from a past event $c\Delta\tau$, i.e., a causality-volume V_c. The event-information in causality-volume $\boldsymbol{V_c}$ has a $4D$ "radial" expansion as function of time $c\Delta\tau$ [9,40] while an observer follows a spacetime trajectory $x(c\Delta t,\beta,\theta,\phi)$ in V_c with proper distance $c\Delta\tau$ [33-35,41]. Hence, an observation takes place - at present time - on surface A_c (4.4) of V_c. However, the interaction between event-information and observer will take place during the transition phase in V_c (4.3). Where the event-information is connected to the refence frame of the initial state.

Spacetime rotor R_1 (5.3) and spacetime spinor U_1 (5.4) can perform all possible spacetime rotations while preserving circular and hyperbolic symmetries in $\mathbb{R}^{1,3}$ (2.9) (Fig. 5.1) (Table 5.1). Due to the use of a hyperbolic rotation with Euclidean angle $L_1(\beta)$ (2.7) (5.1) is it possible to decompose spacetime rotor R_1=$\eta_1S_1L_{u1}$=η_1U_1 (5.3) in a scalar density η_1=$\sqrt{sec(\beta)}$ and a spacetime spinor U_1=S_1L_{u1}. This decomposition shows clearly that spacetime spinor U_1=S_1L_{u1} consist of the GP of a spatial spinor S_1 (5.2) and a temporal spinor L_{u1} (2.7) (5.1).

Temporal spinor L_{u1} with the characteristic half Euclidean angle β is related to a boost, while the spatial spinor S_1 is related to the well-known complex Pauli spinors [15,20,44,45,50] (Appendix E1). Although U_1=S_1L_{u1} is STA even, the GP $U_1\tilde{U}_1$=$cos(\beta)$ is a scalar value. So, the demand for unitarity of spacetime rotor $R_1\tilde{R}_1$=1 introduces the density factor η_1=$\sqrt{sec(\beta)}$ (5.4), which has a singularity at $\beta = \pm\ \pi/2 \mapsto v = \pm c$. Dirac spinor $\psi_1(x)$=$U_1(\beta,\theta,\phi)\ Exp(-\mathbb{i}\sigma_3\omega_0\Delta\tau)$ (5.7) derived purely from $\mathbb{S}^{\mathbf{1,2}}$ symmetries (Fig. 5.1) contains all possible information to manifest at observation in one of the four fermion properties (Fig. 5.2c) (Table 5.2), i.e., one of the four eigenvalues of the Dirac equation (Appendix E3). Hence, the geometric derivation (Table 5.2) - acting on Dirac spinor $\psi_1(x)$ - yields the same result as solving the Dirac equation with complex quantum mechanical eigenvalue eigenvector matrix equations.

The process of an observation contains an initial state, a transition phase, and a final state. The transition phase takes place during ($c\Delta t$) an interaction between observer (measurement device) and the event-information in causality-volume V_c. The observer as well as the event-information have discrete PT quadrant (4.1) connected reference frames (Fig. 5.2c) (Table 5.2). Where the event-information is connected to the reference frame of the initial state. If there is a difference in temporal and/or spatial orientation between the two reference frames (observer, event-information) a discrete PT quadrant information mixing will occur. An experiment to proof or falsify this logic is given in appendix F.

The idea to connect relative speed to a hyperbolic and trigonometric function $\boldsymbol{tanh(\varphi)}$=$\boldsymbol{sin(\beta)}$ = ± v/c (2.8) was already in my notebook many years ago. This connection opened a view on unknown territory, but by using common math/physics it was difficult to put my ideas on paper. At the end it was the use of geometric algebra that allowed me to write this article on "Hyperbolic rotation with Euclidean angle illuminates spacetime spinors". Hopefully it will encourage you to embark for a stimulating research journey through unexplored territory.

Appendix A: Overview of spacetime algebra (STA)

#	Spacetime Algebra (*STA*) [10, 20-22]	Description
1	$\gamma_\mu = \{t, x, y, z\} = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$	The four orthogonal basis vectors γ_μ for the four spacetime axis $\{t, x, y, z\}$.
2	$(\gamma_0)^2 = +1 \quad (\gamma_k)^2 = -1 \quad \gamma_\mu . \gamma_\nu = 0 \mapsto (\mu \neq \nu)$ $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$ Metric diagonal $\{1, -1, -1, -1\}$	The temporal basis vector γ_0 squares to one, while the spatial basis vectors γ_k square to minus one, i.e., $\mathbb{R}^{1,3}$. The basis vectors γ_μ satisfy the algebra of the Dirac gamma matrices.
3	$\mathbb{i} = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \quad \mathbb{i}^2 = -1$	*STA* pseudoscalar $\mathbb{i}$.
4	$a = a^0 \gamma_0 + a^1 \gamma_1 \quad b = b^0 \gamma_0 + b^1 \gamma_1$	Two *STA* vectors a, b in the $\{\gamma_0 \gamma_1 \to tx\}$ plane.
5	$ab = a.b + a \wedge b$ $ab = (a^0 \gamma_0 + a^1 \gamma_1)(b^0 \gamma_0 + b^1 \gamma_1)$ $ab = a^0 b^0 \gamma_0 \gamma_0 + a^0 b^1 \gamma_0 \gamma_1 + a^1 b^0 \gamma_1 \gamma_0 + a^1 b^1 \gamma_1 \gamma_1$ $ab = (a^0 b^0 - a^1 b^1)1 + (a^0 b^1 - a^1 b^0)\gamma_0 \gamma_1$ $a.b = a^0 b^0 - a^1 b^1$ Scalar $a \wedge b = (a^0 b^1 - a^1 b^0)\gamma_0 \gamma_1$ Bivector	The *GP* of two vectors $ab = a.b + a \wedge b$ is the dot product $a.b$ plus the wedge $a \wedge b$ product, resulting in a scalar plus a bivector. The dot product $a.b$ is the symmetrical part of the *GP* ab while the wedge product $a \wedge b$ is the asymmetrical part of the *GP* ab.
6	4*D* space | 3*D* space, M_+ Scalars | 1 | $\langle M \rangle_0$ | 1 Vectors | γ_μ | $\langle M \rangle_1$ | Spacetime Bivectors | $\gamma_k \gamma_0$, $\mathbb{i}\gamma_k \gamma_0$ | $\langle M \rangle_2$ | $\sigma_k = \gamma_k \gamma_0$, $\mathbb{i}\sigma_k = \mathbb{i}\gamma_k \gamma_0$ Trivectors | $\mathbb{i}\gamma_\mu$ | $\langle M \rangle_3$ | Pseudovectors | $\mathbb{i}$ | $\langle M \rangle_4$ | $\mathbb{i}$	The full *STA* is a graded $\langle M \rangle_k$ linear space with dimension $2^4 = 16$. The *STA* multi-vectors are "geometric numbers" being scalars, vectors, bivectors, trivectors, and pseudoscalars. The spacetime bivectors $\langle M \rangle_2$ consist of three temporal bivectors σ_k and three spatial bivectors $\mathbb{i}\sigma_k$. The temporal spacetime bivectors form the basis vectors σ_k for the three spatial axes of 3*D* space $\{x, y, z\}$ (see A.9). General *STA* multi-vector M: $M = \langle M \rangle_0 + \langle M \rangle_1 + \langle M \rangle_2 + \langle M \rangle_3 + \langle M \rangle_4$ 4*D* space. $M_+ = \frac{1}{2}(M - \mathbb{i}M\mathbb{i}) = \langle M \rangle_0 + \langle M \rangle_2 + \langle M \rangle_4$ 3*D* space
7	$M = \langle M \rangle_0 + \langle M \rangle_1 + \langle M \rangle_2 + \langle M \rangle_3 + \langle M \rangle_4$ $\widetilde{M} = \langle M \rangle_0 + \langle M \rangle_1 - \langle M \rangle_2 - \langle M \rangle_3 + \langle M \rangle_4$	Revision operator ~ is an invariant type of conjugation.
8	$M_+ = \frac{1}{2}(M - \mathbb{i}M\mathbb{i}) = \langle M \rangle_0 + \langle M \rangle_2 + \langle M \rangle_4$ $M_+ M_+ = M_+ = \langle M \rangle_0 + \langle M \rangle_2 + \langle M \rangle_4$ $M_+ \widetilde{M}_+ = \langle M \rangle_0 + \langle M \rangle_4$	The even subalgebra M_+ of *STA* is closed under a *GP*, i.e., the *GP* of $M_+ M_+ = M_+$. The *GP* of $M_+ \widetilde{M}_+ = \langle M \rangle_0 + \langle M \rangle_4$ gives a scalar plus a pseudo scalar.
9	$\sigma_k = \{x, y, z\} = \{\sigma_1, \sigma_2, \sigma_3\}$ $\sigma_k = \gamma_k \gamma_0 \quad (\sigma_k)^2 = +1$ $\mathbb{i}\sigma_k = \{\sigma_2 \sigma_3, \sigma_3 \sigma_1, \sigma_1 \sigma_2\}$ $\mathbb{i}\sigma_k = \mathbb{i}\gamma_k \gamma_0 \quad (\mathbb{i}\sigma_k)^2 = -1$	The three orthogonal basis vectors σ_k for the three spatial axes $\{x, y, z\}$. The spatial basis vectors $\sigma_k = \gamma_k \gamma_0$ are **temporal spacetime bivectors** with each of them orthogonal to the temporal basis vector γ_0. The **spatial bivectors** $\mathbb{i}\sigma_k$ square to minus one and are together with σ_k part of the even subalgebra M_+ of STA (see A.6).
10	$(\sigma_k)^2 = +1 \quad \sigma_k . \sigma_j = 0 \mapsto (k \neq j)$ $\sigma_k . \sigma_j + \sigma_j . \sigma_k = 2\delta_{kj}$	The spatial basis vectors σ_k square to one and satisfy the algebra of the Pauli matrices (algebra of 3*D* space).
11	$\mathbb{i} = \sigma_1 \sigma_2 \sigma_3 = \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \quad \mathbb{i}^2 = -1$ $\mathbb{i} = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \sigma_1 \sigma_2 \sigma_3$	The pseudoscalars of 3*D* space $\sigma_1 \sigma_2 \sigma_3$ and 4*D* spacetime $\gamma_0 \gamma_1 \gamma_2 \gamma_3$ are equal.
12	$\sigma_k = \gamma_k \gamma_0 \quad \Leftrightarrow \quad \gamma_k = \sigma_k \gamma_0$	Changing spacetime basis vector γ_k into a spatial basis vector σ_k and vice versa, can be done via a right *GP* with the temporal basis vector γ_0.

Appendix B: Irreducible rotors

#	**Irreducible rotors**: an irreducible rotor $R \equiv \langle M \rangle_0 + \langle M \rangle_2$ is an even subalgebra M_+ unitary $\left(R\tilde{R} = +1\right)$ geometrical number with zero pseudoscalar $\langle M \rangle_4 = 0$. Where $\langle M \rangle_2$ is a *STA* temporal bivector $\sigma_k = \gamma_k \gamma_0$ or a *STA* spatial bivector $\mathbb{i}\sigma_k$ (Appendix A). The type of irreducible rotor depends on the squared signature of the bivector (hyperbolic $(\sigma_k)^2 = +1$ or Euclidean $(\mathbb{i}\sigma_k)^2 = -1$).	
1	$R = \rho S_R \quad R\tilde{R} = 1 \mapsto \rho = \left(S_R \tilde{S}_R\right)^{-1/2} \quad R \equiv \langle M \rangle_0 + \langle M \rangle_2 + \langle M \rangle_4$ A general rotor $R = \rho S_R$ consist of a scalar density ρ times a spinor S_R and is part of the *STA* even subalgebra M_+. Even if pseudoscalar $\langle M \rangle_4$ in R is non-zero the rotor should be unitary $R\tilde{R} = 1$, i.e., the pseudoscalar part in $R\tilde{R}$ should be zero.	
2	$R(\varphi) = +1(cosh(\varphi/2) + sinh(\varphi/2)\sigma_3) = e^{+\sigma_3 \varphi/2}$ $\tilde{R}(\varphi) = +1(cosh(\varphi/2) - sinh(\varphi/2)\sigma_3) = e^{-\sigma_3 \varphi/2}$ $w = \boldsymbol{R\gamma_0\tilde{R}} = cosh(\varphi)\gamma_0 + sinh(\varphi)\gamma_3 \quad \varphi \in [-\infty, \infty]$	Irreducible hyperbolic rotor $R(\varphi) \mapsto (\sigma_3)^2 = +1$ with hyperbolic angle φ acting in the $\sigma_3 \to zt$ plane. A rotation of a (multi)-vector M takes place by a double-sided *GP* $M' = RM\tilde{R}$.

3	$Q_\theta(\theta) = +1(cos(\theta/2) - sin(\theta/2)\mathbb{i}\sigma_2) = e^{-\mathbb{i}\sigma_2\theta/2}$ $\tilde{Q}_\theta(\theta) = +1(cos(\theta/2) + sin(\theta/2)\mathbb{i}\sigma_2) = e^{+\mathbb{i}\sigma_2\theta/2}$ $u = \boldsymbol{Q_\theta \gamma_3 \tilde{Q}_\theta} = cos(\theta)\gamma_3 + sin(\theta)\gamma_1 \quad \theta \in [0, 2\pi]$	Irreducible spatial rotor $Q_\theta(\theta) \mapsto (\mathbb{i}\sigma_2)^2 = -1$ with Euclidean angle θ acting in the $\mathbb{i}\sigma_2 \to zx$ plane. Rotor and spinor are equal.
4	$Q_\phi(\phi) = +1(cos(\phi/2) - sin(\phi/2)\mathbb{i}\sigma_3) = e^{-\mathbb{i}\sigma_3\phi/2}$ $\tilde{Q}_\phi(\phi) = +1(cos(\phi/2) + sin(\phi/2)\mathbb{i}\sigma_3) = e^{+\mathbb{i}\sigma_3\phi/2}$ $u = \boldsymbol{Q_\phi \gamma_1 \tilde{Q}_\phi} = cos(\phi)\gamma_1 + sin(\phi)\gamma_2 \quad \phi \in [0, 2\pi]$	Irreducible spatial rotor $Q_\phi(\phi) \mapsto (\mathbb{i}\sigma_3)^2 = -1$ with Euclidean angle ϕ acting in the $\mathbb{i}\sigma_3 \to xy$ plane. Rotor and spinor are equal.
5	$R(\varphi) = e^{\sigma_3\varphi/2} \;\mapsto\; R\left(tanh^{-1}(sin(\beta))\right) = L_1(\beta) = \rho L_{u1}$ $L_1(\beta) = \sqrt{sec(\beta)}(cos(\beta/2) + sin(\beta/2)\sigma_3)$ $\tilde{L}_1(\beta) = \sqrt{sec(\beta)}(cos(\beta/2) - sin(\beta/2)\sigma_3)$ $\rho = \sqrt{sec(\beta)} \;\; L_{u1} = cos(\beta/2) + sin(\beta/2)\sigma_3 \quad \beta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$	Irreducible hyperbolic rotor $L_1(\beta) \mapsto (\sigma_3)^2 = +1$ with Euclidean angle β acting in the $\sigma_3 \to zt$ plane. Rotor and spinor are different. The substitution of $\varphi = tanh^{-1}(sin(\beta))$ (2.3) in the hyperbolic rotor $R(\varphi)$ (2.6) (B.2) gives a **mixed type of rotor** $L_1(\beta)$. **Temporal spinor**: $L_{u1}(\beta) = cos(\beta/2) + sin(\beta/2)\sigma_3 \neq \boldsymbol{e^{+\sigma_3\beta/2}}$
6	$\boldsymbol{u = L_1\gamma_0\tilde{L}_1} = sec(\beta)\gamma_0 + tan(\beta)\gamma_3 \quad \beta \in [0, 2\pi] \;\; u^2 = +1$ $\boldsymbol{x = L_{u1}\gamma_0\tilde{L}_{u1}} = \gamma_0 + sin(\beta)\gamma_3 \quad \beta \in [0, 2\pi] \;\; x^2 = cos^2(\beta)$ $u = sec(\beta)x = sec(\beta)(\gamma_0 + sin(\beta)\gamma_3) \qquad ux = cos(\beta)$	Vector x is future pointing while vector u can have a time-reversal for $\beta \pm \pi$. Note that $\beta \pm \pi$ will also induce a parity-reversal in vector x. Both vectors will remain aligned, as reflected in the scalar value of their GP $ux = cos(\beta)$ (**Fig. 2.2**).
7	$sin(\beta) = \pm v/c \qquad cos(\beta) = \pm\sqrt{1-(v/c)^2} \;\mapsto sin^2(\beta) + cos^2(\beta) = 1 \;\; Circular\ symmetry \quad \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}, \to \frac{\pi}{2}, \frac{3\pi}{2}\right]$ $sec(\beta) = \frac{1}{\pm\sqrt{1-(v/c)^2}} \quad tan(\beta) = \frac{\pm v/c}{\pm\sqrt{1-(v/c)^2}} \;\mapsto sec^2(\beta) - tan^2(\beta) = 1 \;\; Hyperbolic\ symmetry \;\; \frac{v}{c} \in [-1, 1, \to 1, -1]$ The mapping of $tanh(\varphi) = sin(\beta) = \pm v/c \;\; \beta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ results in four trigonometric relativistic proportionality factors with an implicit circular and hyperbolic symmetry.	

Appendix C1: Spacetime position vector x and momentum vector p

#	Spacetime **position vector** x and **momentum vector** p.	
1	$e_3(\theta, \phi) = sin(\theta)cos(\phi)\gamma_1 + sin(\theta)sin(\phi)\gamma_2 + cos(\theta)\gamma_3$	Spacetime spatial unit vector $e_3(\theta, \phi)$; $(e_3)^2 = -1$; $u_3 = e_3\gamma_0$.
2	$x(ct, \beta, \theta, \phi) = ct\left(\gamma_0 + sin(\beta)e_3(\theta, \phi)\right)$	Spacetime position vector $x(ct, \beta, \theta, \phi)$ (**Fig. 3.1**).
3	$x = x^\mu\gamma_\mu$ $tanh(\varphi) = sin(\beta) = \pm v/c \;\Rightarrow$ $x = ct\gamma_0 + ct\, sin(\beta)sin(\theta)cos(\phi)\gamma_1 + ct\, sin(\beta)sin(\theta)sin(\phi)\gamma_2 + ct\, sin(\beta)cos(\theta)\gamma_3$ $x = ct\gamma_0 + vt\, sin(\theta)cos(\phi)\gamma_1 + vt\, sin(\theta)sin(\phi)\gamma_2 + vt\, cos(\theta)\gamma_3 = ct\gamma_0 + vt\, e_3(\theta, \phi)$ $x^2 = c^2t^2 - v^2t^2 = c^2t^2\, cos^2(\beta) \;\to\; s = c\tau = ct\, cos(\beta) \;\to\; \tau = t\, cos(\beta)$ The different vector components $x^\mu = \{ct, \lvert\vec{x}\rvert e_3(\theta, \phi)\}$ of spacetime position vector $x(ct, \beta, \theta, \phi)$. Proper distance $s = c\tau = ct\, cos(\beta)$ with proper time $\tau = t\, cos(\beta) = \pm t\sqrt{1-(v/c)^2}$.	
4	$p(m_0c, \beta, \theta, \phi) = m_0c\, sec(\beta)\left(\gamma_0 + sin(\beta)e_3(\theta, \phi)\right)$	Spacetime momentum vector $p(m_0c, \beta, \theta, \phi)$ (**Fig. 3.1**).
5	$p = p^\mu\gamma_\mu = p^0\gamma_0 + p^1\gamma_1 + p^2\gamma_2 + p^3\gamma_3$ $p = m_0c\, sec(\beta)\gamma_0 + m_0c\, tan(\beta)sin(\theta)cos(\phi)\gamma_1 + m_0c\, tan(\beta)sin(\theta)sin(\phi)\gamma_2 + m_0c\, tan(\beta)cos(\theta)\gamma_3$ $p = m_0c\, sec(\beta)\gamma_0 + m_0v\, sec(\beta)sin(\theta)cos(\phi)\gamma_1 + m_0v\, sec(\beta)sin(\theta)sin(\phi)\gamma_2 + m_0v\, sec(\beta)cos(\theta)\gamma_3$ $E/c = m_0c\, sec(\beta) \quad \lvert\vec{p}\rvert = m_0v\, sec(\beta) \quad p = \frac{E}{c}\gamma_0 + \lvert\vec{p}\rvert e_3(\theta, \phi) \quad p^2 = (m_0c)^2\left(sec^2(\beta) - tan^2(\beta)\right) = (m_0c)^2$ The different vector components $p^\mu = \{E/c, \lvert\vec{p}\rvert e_3(\theta, \phi)\}$ of spacetime momentum vector $p(m_0c, \beta, \theta, \phi)$.	

Appendix C2: Curvilinear basis vectors α_μ and spacetime volume element dV

#	Curvilinear spacetime basis vectors α_μ and **spacetime volume element** dV.
1	$x = x^0\gamma_0 + x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3 \qquad x = ct\gamma_0 + ct\, sin(\beta)sin(\theta)cos(\phi)\gamma_1 + ct\, sin(\beta)sin(\theta)sin(\phi)\gamma_2 + ct\, sin(\beta)cos(\theta)\gamma_3$ Spacetime position vector $x(ct, \beta, \theta, \phi)$
2	$\alpha_\mu = \frac{\partial x^\nu}{\partial r^\mu}\gamma_\nu \quad \alpha_\mu = \{\alpha_t, \alpha_\beta, \alpha_\theta, \alpha_\phi\} \quad x^\nu = \{x^0, x^1, x^2, x^3\} \quad r^\mu = \{ct, \beta, \theta, \phi\} \quad \gamma_\nu = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ $x^0 = ct \quad x^1 = ct\, sin(\beta)sin(\theta)cos(\phi) \quad x^2 = ct\, sin(\beta)sin(\theta)sin(\phi) \quad x^3 = ct\, sin(\beta)cos(\theta)$ $\alpha_0 = \frac{\partial x^\nu}{\partial r^0}\gamma_\nu \;\mapsto\; \alpha_0 = \gamma_0 + sin(\beta)sin(\theta)cos(\phi)\gamma_1 + sin(\beta)sin(\theta)sin(\phi)\gamma_2 + sin(\beta)cos(\theta)\gamma_3$

	$\alpha_1 = \frac{\partial x^v}{\partial r^1}\gamma_v \;\mapsto\; \alpha_1 = ct\,cos(\beta)sin(\theta)cos(\phi)\gamma_1 + ct\,cos(\beta)sin(\theta)sin(\phi)\gamma_2 + ct\,cos(\beta)cos(\theta)\gamma_3$ $\alpha_2 = \frac{\partial x^v}{\partial r^2}\gamma_v \;\mapsto\; \alpha_2 = ct\,sin(\beta)cos(\theta)cos(\phi)\gamma_1 + ct\,sin(\beta)cos(\theta)sin(\phi)\gamma_2 - ct\,sin(\beta)sin(\theta)\gamma_3$ $\alpha_3 = \frac{\partial x^v}{\partial r^3}\gamma_v \;\mapsto\; \alpha_3 = -ct\,sin(\beta)sin(\theta)sin(\phi)\gamma_1 + ct\,sin(\beta)sin(\theta)cos(\phi)\gamma_2$ The four curvilinear spacetime basis vectors $\alpha_\mu = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$.
3	$g_{\mu\nu} = \alpha_\mu.\alpha_\nu = \begin{pmatrix} cos^2(\beta) & -ct\,sin(2\beta)/2 & 0 & 0 \\ -ct\,sin(2\beta)/2 & -c^2t^2\,cos^2(\beta) & 0 & 0 \\ 0 & 0 & -c^2t^2\,sin^2(\beta) & 0 \\ 0 & 0 & 0 & -c^2t^2\,sin^2(\beta)sin^2(\theta) \end{pmatrix}$ The metric $g_{\mu\nu}$ of the four curvilinear basis vectors $\alpha_\mu = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$.
4	$ds^2 = cos^2(\beta)c^2dt^2 - ct\,sin(\beta)cos(\beta)dtd\beta - ct\,sin(\beta)cos(\beta)d\beta dt - c^2t^2\,cos^2(\beta)d\beta^2 - c^2t^2\,sin^2(\beta)d\theta^2$ $-c^2t^2\,sin^2(\beta)sin^2(\theta)d\phi^2$ Line-element form of the $g_{\mu\nu} = \alpha_\mu.\alpha_\nu$ metric.
5	$dV = c^3t^3\,sin^2(\beta)\,cos(\beta)\,sin(\theta)\,cdt\,d\beta\,d\theta\,d\phi$ — Spacetime volume element dV (3.5).

Appendix D: Causality-volume of three-sphere $\mathbb{S}^{1,2}$ ($4D$ light cone)

#	**Causality-volume** of three-sphere $\mathbb{S}^{1,2}$ in relation to **position vector x** and discrete ***PT* quadrants**.
1	$\mathbb{S}^{1,2} = \left\{\left(x = c\Delta t(\gamma_0 + sin(\beta)e_3(\theta,\phi))\right) \in \mathbb{R}^{1,3}\colon x^2 = c^2\Delta t^2cos^2(\beta);\ \boldsymbol{r = c\Delta\tau = c\Delta t\,cos(\beta)}\right\}$ Spacetime three-sphere $\mathbb{S}^{1,2}$ with radius $\boldsymbol{r = c\Delta\tau}$.
2	$x = c\Delta t(\gamma_0 + sin(\beta)e_3(\theta,\phi))$ $\quad x^2 = c^2\Delta t^2\,cos^2(\beta) \;\mapsto\; s = c\Delta\tau = c\Delta t\,cos(\beta)$ Spacetime position vector $x(c\Delta t,\beta,\theta,\phi)$ with proper distance $\boldsymbol{s = c\Delta\tau}$.
3	The radius of three-sphere $\mathbb{S}^{1,2}$ ($r = c\Delta\tau$) is equal to the proper distance $s = c\Delta\tau$ (3.3) of position vector x. Hence, position vector $x(c\Delta t,\beta,\theta,\phi\,)$ is pointing at the surface of three-sphere $\mathbb{S}^{1,2}$.
4	Note that the radius of $\mathbb{S}^{1,2}$ and proper distance of position vector x are positive because ct and $cos(\beta)$ keep an equal sign under time-reversal $T_\pm\colon\{c\Delta\tau = (\pm c\Delta t)(\pm cos(\beta))\}$ (**Fig. 4.1**).
5	$V_C = \int_\tau\int_\beta\int_\theta\int_\phi dV = \int_{\tau=0}^{\Delta\tau}\int_{\beta=0}^{\pi/2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} c^3t^3sin^2(\beta)\,cos(\beta)\,sin(\theta)\,cdt\,d\beta\,d\theta\,d\phi = \frac{1}{3}\pi\,c^4\Delta\tau^4 \qquad \Delta\tau = \Delta t\,cos(\beta)$ Spacetime causality-volume V_c of three-sphere $\mathbb{S}^{1,2}$ ($4D$ light cone) (4.3). Causality-volume V_c has a $4D$ "radial" expansion as function of time $c\Delta\tau$ and bounds the spacetime positions x where potentially an observation can take place based on information from a past event $c\Delta\tau$.
6	**(1)** $P_-T_+;\ \beta \in \left[-\frac{\pi}{2}, 0\right]$ **(2)** $P_+T_+;\ \beta \in \left[0, \frac{\pi}{2}\right]$ **(3)** $P_+T_-;\beta \in \left[\frac{\pi}{2}, \pi\right]$ **(4)** $P_-T_-;\beta \in \left[\pi, \frac{3\pi}{2}\right]$ The discrete combinations of parity-reversal $P_+ \leftrightarrow P_-$ and time-reversal $T_+ \leftrightarrow T_-$ (PT quadrants) in relation to the four quadrants of hyperbolic rotation angle $\beta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ [5, 8, 36, 37] (**Table 4.1**) (**Fig. 5.2**). The discrete PT quadrants are connected to a reference frame (inertial frame) $\{ct, x, y, z\}$. Hence, the different reference frames in an interaction can via a difference in temporal and/or spatial orientation contain PT quadrant information mixing, i.e., parity-reversal $P_+ \leftrightarrow P_-$ and time-reversal $T_+ \leftrightarrow T_-$ between the reference will not be aligned.
7	$x(c\Delta t,\beta,\theta,\phi) = c\Delta t(\gamma_0 + sin(\beta)e_3(\theta,\phi)) \qquad s = c\Delta\tau = c\Delta t\,cos(\beta)$ The **event-information in causality-volume V_c** has a 4D "radial" expansion as function of time $c\Delta\tau$ while an **observer follows a spacetime trajectory** $x(c\Delta t,\beta,\theta,\phi)$ through V_c over a proper distance $c\Delta\tau$. Hence, the event-information in V_c will be observed - at present time - on surface A_c of causality-volume V_c. $A_c = \frac{d}{c\,d\Delta\tau}V_c = \frac{4}{3}\pi\,c^3\Delta\tau^3 \qquad \Delta\tau = \Delta t\,cos(\beta) \qquad \beta \in \left[\frac{-\pi}{2}, \frac{3\pi}{2}\right]$ The surface A_c of causality-volume V_c is **the volume of $3D$ space**, i.e., the volume of a two-sphere $\mathbb{S}^{2,0}$ (4.2). Surface A_c can be negative. However, due to the future pointing character of the spacetime trajectories an observation will take place on the positive side of surface A_c. However, the double cover of position vector x allows for the contribution of all PT quadrants to an observation.

Appendix E1: Spatial spinor $S^1(\theta,\phi)$ is equal to a spin-up complex Pauli spinor

#	**Spatial spinor $S_1(\theta,\phi)$** is equal to a spin-up complex **Pauli spinor**.
1	$S_1(\theta,\phi) = \left(\cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_3\right)\left(\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\mathbb{i}\sigma_2\right) = e^{-\mathbb{i}\sigma_3\phi/2}e^{-\mathbb{i}\sigma_2\theta/2}$ $S_1(\theta,\phi) = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_1 - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_2 - \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_3$ $S_1(0,0) = 1$ Spin – Up $S_1(\theta,\phi) = a^0 + a^1\mathbb{i}\sigma_1 - a^2\mathbb{i}\sigma_2 + a^3\mathbb{i}\sigma_3 \leftrightarrow \mathrm{I}S_1(\theta,\phi)\rangle = \begin{pmatrix}+a^0 + ia^3\\ -a^2 + ia^1\end{pmatrix} = \begin{pmatrix}\cos(\theta/2)e^{-\frac{i\phi}{2}}\\ \sin(\theta/2)e^{+\frac{i\phi}{2}}\end{pmatrix}$ $\mathrm{I}S_1(0,0)\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ Spin – Up Spatial spinor $S_1(\theta,\phi)$ (5.2) is equal to a **spin-up complex Pauli spinor** $\mathrm{I}S_1\rangle$ [15, 44] ($i = \sqrt{-1}$).
2	$S_2(\theta,\phi) = S_1(\theta+\pi,\phi) = \left(\cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_3\right)\left(-\sin\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right)\mathbb{i}\sigma_2\right)$ $S_2(0,0) = -\mathbb{i}\sigma_2$ Spin – Down $\mathrm{I}S_2(\theta,\phi)\rangle = \mathrm{I}S_1(\theta+\pi,\phi)\rangle = \begin{pmatrix}-\sin(\theta/2)e^{-i\phi/2}\\ +\cos(\theta/2)e^{+i\phi/2}\end{pmatrix}$ $\mathrm{I}S_2(0,0)\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ Spin – Down Spatial spinor $S_2(\theta,\phi)$ is equal to a **spin-down complex Pauli spinor** $\mathrm{I}S_2\rangle$.
3	$k\pi = \{0, \pi, 2\pi, 3\pi, 4\pi\}$ $S_1(\theta + k\pi,\phi) = \{+S_1, +S_2, -S_1, -S_2, +S_1\}$ Typical spinor characteristics: picking up a minus sign at a 2π rotation and being back original after a 4π rotation. **Important**: note that $\theta \pm \pi$ gives a spin-reversal (up ↔ down), plus also a minus sign: $S_1(\theta \pm \pi,\phi) = \{+S_2, -S_2\}$ and $S_2(\theta \pm \pi,\phi) = \{-S_1, +S_1\}$ However, $\theta \pm \pi$ will also induce a parity-reversal in the spacetime spatial unit vector $e_3(\theta,\phi)$, $(e_3)^2 = -1$. $e_3(\theta,\phi) = \sin(\theta)\cos(\phi)\gamma_1 + \sin(\theta)\sin(\phi)\gamma_2 + \cos(\theta)\gamma_3 \rightarrow e_3(\theta\pm\pi,\phi) = -\sin(\theta)\cos(\phi)\gamma_1 - \sin(\theta)\sin(\phi)\gamma_2 - \cos(\theta)\gamma_3$
4	$\langle\tilde{S}_1(\theta,\phi)S_1(\theta_0,\phi_0)\rangle_q = \langle\tilde{S}_1(\theta,\phi)S_1(\theta_0,\phi_0)\rangle_0 - \langle\tilde{S}_1(\theta,\phi)S_1(\theta_0,\phi_0)\mathbb{i}\sigma_3\rangle_0\mathbb{i}\sigma_3$ **The inner product** $\langle\tilde{S}_1(\theta,\phi)S_1(\theta_0,\phi_0)\rangle_q$ to calculate the probability amplitude for detecting spin-up in the $u_3(\theta,\phi)$ direction given spin-up in the $u_3(\theta_0,\phi_0)$ direction. The inner product $\langle\tilde{S}_1(\theta,\phi)S_1(\theta_0,\phi_0)\rangle_q$ projects out the $\{1, \mathbb{i}\sigma_3\}$ components from the *GP* $\tilde{S}_1(\theta,\phi)S_1(\theta_0,\phi_0)$ [15, 44]. $u_3(\theta,\phi) = \sin(\theta)\cos(\phi)\sigma_1 + \sin(\theta)\sin(\phi)\sigma_2 + \cos(\theta)\sigma_3$ $u_3 = e_3\gamma_0$ $(u_3)^2 = +1$ Spatial unit vector.
5	The probability $\mathbb{P}$ for detecting spin-up in the $u_3(\theta,\phi)$ direction given spin-up in the $u_3(\theta_0,\phi_0)$ direction can be calculated via: $\mathbb{P}((\theta,\phi),(\theta_0,\phi_0)) = \left\|\langle\tilde{S}_1(\theta,\phi)S_1(\theta_0,\phi_0)\rangle_q\right\|^2 = \frac{1}{2}(1 + u_3(\theta,\phi).u_3(\theta_0,\phi_0))$ $\mathbb{P}((\theta,\phi),(0,0)) = \frac{1}{2}(1+\cos(\theta)) = \cos^2(\theta/2)$

Appendix E2: Spacetime rotor $R_1(\beta,\theta,\phi)$, spacetime spinor $U_1(\beta,\theta,\phi)$ and Dirac spinor $\psi_1(c\Delta t,\beta,\theta,\phi)$

#	**Spacetime rotor** $R_1(\beta,\theta,\phi)$, **spacetime spinor** $U_1(\beta,\theta,\phi)$ and **Dirac spinor** $\psi_1(c\Delta t,\beta,\theta,\phi)$.	
1	$S_1 = \left(\cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_3\right)\left(\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\mathbb{i}\sigma_2\right)$	Spatial spinor $S_1(\theta,\phi)$ (5.2).
2	$L_1(\beta) = \sqrt{\sec(\beta)}(\cos(\beta/2) + \sin(\beta/2)\sigma_3)$	Irreducible hyperbolic rotor $L_1(\beta) = \sqrt{\sec(\beta)}L_{u1}(\beta)$.
3	$R_1(\beta,\theta,\phi) = S_1(\theta,\phi)L_1(\beta) = \eta_1 U_1(\beta,\theta,\phi)$	Spacetime rotor $R_1(\beta,\theta,\phi)$ and spacetime spinor $U_1(\beta,\theta,\phi)$.
4	$R_1(\beta,\theta,\phi) = \eta_1 U_1(\beta,\theta,\phi)$ $R_1\tilde{R}_1 = 1 \mapsto \eta_1 = (U_1\tilde{U}_1)^{-1/2} = \sqrt{\sec(\beta)}$ Unitarity of $R\tilde{R} = 1$ defines the scalar density η_1. $R_1 = \sqrt{\sec(\beta)}\left(\cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_3\right)\left(\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\mathbb{i}\sigma_2\right)\left(\cos\left(\frac{\beta}{2}\right) + \sin\left(\frac{\beta}{2}\right)\sigma_3\right)$ $R_1 \equiv \langle M\rangle_0 + \langle M\rangle_2 + \langle M\rangle_4$ Rotor R_1 (5.3) and spinor U_1 (5.4) can perform **all possible spacetime rotations** while preserving circular and hyperbolic symmetries in $\mathbb{R}^{1,3}$ (**Fig. 5.1**) (Table 5.1) (2.9). Where rotor R_1 is related to hyperbolic symmetry and spinor U_1 is related to circular symmetry.	
5	$U_1(\beta,\theta,\phi) = S_1(\theta,\phi)L_{u1}(\beta) = S_1(\theta,\phi)\left(\cos\left(\frac{\beta}{2}\right) + \sin\left(\frac{\beta}{2}\right)\sigma_3\right)$ $U_1\tilde{U}_1 = \cos(\beta)$ $U_1 \equiv \langle M\rangle_0 + \langle M\rangle_2 + \langle M\rangle_4$ $U_1(\beta,\theta,\phi) = \cos\left(\frac{\beta}{2}\right)\left(+\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_1 - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_2 - \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_3\right)$ $+\sin\left(\frac{\beta}{2}\right)\left(-\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\phi}{2}\right)\mathbb{i} + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi}{2}\right)\sigma_1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\phi}{2}\right)\sigma_2 + \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\phi}{2}\right)\sigma_3\right)$ Spacetime spinor $U_1(\beta,\theta,\phi)$. Note that although spacetime spinor U_1 is *STA* even, the *GP* $U_1\tilde{U}_1 = \cos(\beta)$ is a scalar value [15].	
6	$p(m_0c,\beta,\theta,\phi) = m_0c\,\boldsymbol{R_1\gamma_0\tilde{R}_1} = m_0c\sec(\beta)(\gamma_0 + \sin(\beta)e_3(\theta,\phi))$ $p^2 = m_0^2c^2(\sec^2(\beta) - \tan^2(\beta)) = m_0^2c^2$ $x(c\Delta t,\beta,\theta,\phi) = c\Delta t\,\boldsymbol{U_1\gamma_0\tilde{U}_1} = c\Delta t(\gamma_0 + \sin(\beta)e_3(\theta,\phi))$ $x^2 = c^2\Delta t^2(1-\sin^2(\beta)) = c^2\Delta t^2\cos^2(\beta)$ Preserving spacetime symmetries $\{p^2;\ x^2\}$ by spacetime rotor R_1 and spacetime spinor U_1.	

7	$S = px = m_0c^2\Delta t\, \boldsymbol{R_1\gamma_0\tilde{R}_1U_1\gamma_0\tilde{U}_1} = m_0c^2\Delta t\, sec(\beta)\boldsymbol{U_1\gamma_0\tilde{U}_1U_1\gamma_0\tilde{U}_1} = m_0c^2\Delta t\, sec(\beta)cos^2(\beta) = m_0c^2\Delta t\, cos(\beta)$ $S = px = m_0c^2\Delta t\, cos(\beta) = m_0c^2\Delta t\sqrt{1-(v/c)^2}\quad [kg\, m^2/s]\qquad px = p.x \ \mapsto\ p \wedge x = 0\qquad \frac{S}{\hbar} = \frac{m_0c^2}{\hbar}\Delta t\, cos(\beta)\quad [-]$ **Action** $S = px = p.x$ provides a unification between the hyperbolic symmetry of momentum p and the circular symmetry of spacetime trajectory $x(c\Delta t, \beta, \theta, \phi)$. Resulting in a phase $S/\hbar$: $\frac{S}{\hbar} = \frac{m_0c^2}{\hbar}\Delta t\, cos(\beta) = \omega_0\Delta t\, cos(\beta) = \omega_0\Delta\tau \ \mapsto\ \boldsymbol{\frac{S}{\hbar} = \omega_0\Delta\tau}\qquad \omega_0 = \frac{m_0c^2}{\hbar}$
8	Bringing together: (a) spacetime spinor U_1 (E2.5), (b) spacetime bivector $\gamma_1\gamma_2 = -\mathbb{i}\sigma_3$ (Table 5.1) and (c) phase $S/\hbar = \omega_0\Delta\tau$ (5.5), results in a wavefunction $\psi_1(x) = \psi_1(c\Delta t, \beta, \theta, \phi)$: $\psi_1(x) = \psi_1(c\Delta t, \beta, \theta, \phi) = U_1Exp(-\mathbb{i}\sigma_3\, S/\hbar) = U_1\, Exp(-\mathbb{i}\sigma_3\omega_0\Delta\tau)\quad x(c\Delta t, \beta, \theta, \phi) = c\Delta t\big(\gamma_0 + sin(\beta)e_3\big)\quad \beta \in \left[\frac{-\pi}{2}, \frac{3\pi}{2}\right]$
9	$\boldsymbol{\psi_1(x)} = U_1Exp(-\mathbb{i}\sigma_3\, p.x/\hbar) = \boldsymbol{U_1\, Exp(-\mathbb{i}\sigma_3\omega_0\Delta\tau)}$ Wavefunction $\psi_1(x)$ **is a solution of the Dirac equation** (E3). So, $\psi_1(x)$ is a **Dirac spinor** derived purely from three-sphere $\mathbb{S}^{1,2}$ symmetries.

Appendix E3: Dirac equation

#	Dirac equation.	
1	$\hbar\nabla\psi_1(x)\mathbb{i}\sigma_3 - m_0c^2\psi_1(x)\gamma_0 = 0$	Dirac equation.
2	$\nabla = \gamma^\mu\partial_\mu = \gamma^0\frac{\partial}{c\partial t} + \gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^2\frac{\partial}{\partial z}\qquad \gamma^\mu.\gamma_\nu = \delta^\mu_\nu$	Spacetime vector derivative ∇. Where the γ^μ are a reciprocal frame of vectors to the basis vectors γ_μ, defined via $\gamma^\mu.\gamma_\nu = \delta^\mu_\nu$.
3	$\psi_1(x) = \psi_1(c\Delta t, \beta, \theta, \phi) = U_1Exp(-\mathbb{i}\sigma_3\, S/\hbar) = U_1Exp(-\mathbb{i}\sigma_3\, p.x/\hbar) = U_1\, Exp(-\mathbb{i}\sigma_3\omega_0\Delta\tau)$	Dirac spinor (E2.8) (E2.9).
4	$\nabla\psi_1 = -\omega_0\big(sec(\beta)\gamma_0 + tan(\beta)e_3\big)\mathbb{i}\sigma_3\psi_1\qquad \nabla U_1 = 0$ $m_0c^2\big(sec(\beta)\gamma_0 + tan(\beta)e_3\big)\psi_1 = m_0c^2\psi_1\gamma_0$ $\big(sec(\beta)\gamma_0 + tan(\beta)e_3\big)U_1 = U_1\gamma_0$ $\big(sec(\beta)\gamma_0 + tan(\beta)e_3\big)U_1\tilde{U}_1 = U_1\gamma_0\tilde{U}_1\qquad U_1\tilde{U}_1 = cos(\beta)$ $\gamma_0 + sin(\beta)e_3 = \gamma_0 + sin(\beta)e_3$	Vector derivative of Dirac spinor: $\nabla\psi_1$. Outcome of the vector derivative substituted in the Dirac equation. Becomes an algebraic equation with U_1 **derived purely from** $\mathbb{S}^{1,2}$. Solving by multiplying both sides on the right with $\tilde{U}_1$ $\psi_1(x) = U_1Exp(-\mathbb{i}\sigma_3\, p.x/\hbar) = U_1\, Exp(-\mathbb{i}\sigma_3\omega_0\Delta\tau)$ is a solution of the Dirac equation.
5	**Dirac spinor $\boldsymbol{\psi_1(x)}$** holds all possible information that at observation will manifest in one of the four fermion properties: one of the four eigenvalues of the Dirac equation particle spin-up, particle spin-down, anti-particle spin-down or anti-particle spin-up.	
6	$(\mathbf{1})\ R_-P_-T_+;\ \beta \in \left[-\frac{\pi}{2}, 0\right]\quad (\mathbf{2})\ R_+P_+T_+;\ \beta \in \left[0, \frac{\pi}{2}\right]\quad (\mathbf{3})\ R_-P_+T_-; \beta \in \left[\frac{\pi}{2}, \pi\right]\quad (\mathbf{4})\ R_+P_-T_-; \beta \in \left[\pi, \frac{3\pi}{2}\right]\qquad R = P/T\, ; Ratio$	
7	$R_-P_-T_+$ Particle S_{down} $\psi_1(c\Delta t, \beta, \theta \pm \pi, \phi) = \pm S_1(\theta \pm \pi, \phi)\gamma_0L_{u1}(\beta)\gamma_0e^{-\mathbb{i}\sigma_3\omega_0\Delta\tau} \ \mapsto\ \psi_2 = U_2e^{-\mathbb{i}\sigma_3\omega_0\Delta\tau}$ $R_+P_+T_+$ Particle S_{up} $\psi_1(c\Delta t, \beta, \theta, \phi) = \pm S_1(\theta, \phi)L_{u1}(\beta)e^{-\mathbb{i}\sigma_3\omega_0\Delta\tau} \ \mapsto\ \psi_1 = U_1e^{-\mathbb{i}\sigma_3\omega_0\Delta\tau}$ $R_-P_+T_-$ Anti-Particle S_{up} $\psi_1(c\Delta t, \beta \pm \pi, \theta, \phi) = \pm S_1(\theta, \phi)\gamma_0L_{u1}(\beta \pm \pi)\gamma_0e^{+\mathbb{i}\sigma_3\omega_0\Delta\tau} \ \mapsto\ \psi_4 = V_1e^{+\mathbb{i}\sigma_3\omega_0\Delta\tau}$ $R_+P_-T_-$ Anti-Particle S_{down} $\psi_1(c\Delta t, \beta \pm \pi, \theta \pm \pi, \phi) = \pm S_1(\theta \pm \pi, \phi)L_{u1}(\beta \pm \pi)e^{+\mathbb{i}\sigma_3\omega_0\Delta\tau} \ \mapsto\ \psi_3 = V_2e^{+\mathbb{i}\sigma_3\omega_0\Delta\tau}$ The four well known eigenvectors $\psi_j = \{\psi_1, \psi_2, \psi_3, \psi_4\}$ [4] of the Dirac equation are all implicit contained in $\psi_1(x)$ (**Table 5.2**). The **geometric derivation** provides a connection between the eigenvectors ψ_j and the PT quadrants via a parity-reversal ($P_+ \leftrightarrow P_-$) related to $\theta \pm \pi$, a time-reversal ($T_+ \leftrightarrow T_-$) related to $\beta \pm \pi$ and a spin-reversal (up ↔ down) related to $\gamma_0L_{u1}\gamma_0$. Where $\gamma_0L_{u1}\gamma_0$ is connected to the ratio $R = P/T$ between the PT quadrants (clockwise or counterclockwise temporal spinor (**Table 5.2**)).	
8	$R_-P_-T_+$ Particle S_{down} $x(c\Delta t = 1, \beta, \theta \pm \pi, \phi) = \gamma_0 + \big(+sin(\beta)\big)\big(-e_3(\theta, \phi)\big) = \gamma_0 - sin(\beta)e_3(\theta, \phi)$ $R_+P_+T_+$ Particle S_{up} $x(c\Delta t = 1, \beta, \theta, \phi) = \gamma_0 + \big(+sin(\beta)\big)\big(+e_3(\theta, \phi)\big) = \gamma_0 + sin(\beta)e_3(\theta, \phi)$ $R_-P_+T_-$ Anti-Particle S_{up} $x(c\Delta t = 1, \beta \pm \pi, \theta, \phi) = \gamma_0 + \big(-sin(\beta)\big)\big(+e_3(\theta, \phi)\big) = \gamma_0 - sin(\beta)e_3(\theta, \phi)$ $R_+P_-T_-$ Anti-Particle S_{down} $x(c\Delta t = 1, \beta \pm \pi, \theta \pm \pi, \phi) = \gamma_0 + \big(-sin(\beta)\big)\big(-e_3(\theta, \phi)\big) = \gamma_0 + sin(\beta)e_3(\theta, \phi)$ **Event-information in causality-volume $\boldsymbol{V_c}$** expands with the flow of time $c\Delta\tau$ while an observer (measurement device) follows a spacetime trajectory $x(c\Delta t, \beta, \theta, \phi)$ in V_c with a proper distance of $c\Delta\tau = c\Delta t\, cos(\beta) = \pm c\Delta t\sqrt{1-(v/c)^2}$. The full range of $\beta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ results in a double cover in position (4.5). Hence, an observer can perform a measurement at **equal spacetime trajectories**, which are different in ***PT* quadrant** (Fig. **5.2**c). The geometric derivation gives both: an eigenvector ψ_j and spacetime trajectory connection to the PT quadrants.	

Appendix F: Experiment

The discrete PT quadrants are connected to a reference frame (Fig. 5.2). An PT quadrant is discrete, but the orientation between reference frames can differ continuously. So, the different reference frames (observer and event-information) can undergo a PT quadrant related information mixing.

#	**Experiment** to proof or falsify PT quadrant related information mixing.	
1	$\psi_1(x) = \psi_1(c\Delta t, \beta, \theta, \phi) = U_1(\beta, \theta, \phi) Exp(-\mathbb{i}\sigma_3 S/\hbar)$	Dirac spinor derived purely from three-sphere $\mathbb{S}^{1,2}$ (E2.7) (E2.9)
2	**The inner product** $\langle \widetilde{U}_1(\beta = 0, \theta, \phi) U_1(\beta_0, \theta_0, \phi_0)\rangle_q$ to calculate the **probability amplitude** for observing - in the lab frame (relative speed $\beta = 0$) - a particle in a spin-up state in the $u_3(\theta, \phi)$ direction given a particle prepared in a spin-up state in the $u_3(\theta_0, \phi_0)$ direction with relative speed β_0 in the $u_3(\theta_0, \phi_0)$ direction is given by: $\langle \widetilde{U}_1(0, \theta, \phi) U_1(\beta_0, \theta_0, \phi_0)\rangle_q = \langle \widetilde{U}_1(0, \theta, \phi) U_1(\beta_0, \theta_0, \phi_0)\rangle_0 - \langle \widetilde{U}_1(0, \theta, \phi) U_1(\beta_0, \theta_0, \phi_0)\mathbb{i}\sigma_3\rangle_0 \mathbb{i}\sigma_3 \qquad \mathbb{i}\sigma_3 = -\gamma_1\gamma_2$ The inner product $\langle \widetilde{U}_1(\beta = 0, \theta, \phi) U_1(\beta_0, \theta_0, \phi_0)\rangle_q$ projects out the $\{1, \mathbb{i}\sigma_3\}$ components from *GP* $\widetilde{U}_1(0, \theta, \phi) U_1(\beta_0, \theta_0, \phi_0)$ [15, 44].	
3	The probability $\mathbb{P}_2$ for detecting a particle in the spin-up state in the $u_3(\theta, \phi)$ direction given a particle prepared in a spin-up state in the $u_3(\theta_0, \phi_0)$ direction with a relative speed β_0 in the $u_3(\theta_0, \phi_0)$ direction can be calculated as: $\mathbb{P}_2\big((0, \theta, \phi), (\beta_0, \theta_0, \phi_0)\big) = \left|\langle \widetilde{U}_1(0, \theta, \phi) U_1(\beta_0, \theta_0, \phi_0)\rangle_q\right|^2 = \frac{1}{2} cos^2(\beta_0/2)\big(1 + u_3(\theta, \phi).u_3(\theta_0, \phi_0)\big)$ $\mathbb{P}_2\big((0, \theta, \phi), (\beta_0, \theta_0, \phi_0)\big) = \left|\langle \widetilde{U}_1(0, \theta, \phi) U_1(\beta_0, \theta_0, \phi_0)\rangle_q\right|^2 = cos^2(\beta_0/2)\left|\langle \tilde{S}_1(\theta, \phi) S_1(\theta_0, \phi_0)\rangle_q\right|^2$ (E1.4-5)	
4	The above (F.3) allows to calculate the probabilities for detecting the four fermion properties: particle spin-up, particle spin-down, anti-particle spin-up or anti-particle spin-down in the $u_3(\theta, \phi)$ direction given a particle prepared in a spin-up state in the z-direction with relative speed β_0 in the z-direction $u_3(\theta_0 = 0, \phi_0 = 0)$: $\mathbb{P}_1\big((0, \theta \pm \pi, \phi), (\beta_0, 0, 0)\big) = cos^2(\beta_0/2) sin^2(\theta/2)$ Particle S_{down} P_-T_+ $\mathbb{P}_2\big((0, \theta, \phi), (\beta_0, 0, 0)\big) = cos^2(\beta_0/2) cos^2(\theta/2)$ Particle S_{up} P_+T_+ $\mathbb{P}_3\big((0, \theta, \phi), (\beta_0 \pm \pi, 0, 0)\big) = sin^2(\beta_0/2) cos^2(\theta/2)$ Anti-Particle S_{up} P_+T_- $\mathbb{P}_4\big((0, \theta \pm \pi, \phi), (\beta_0 \pm \pi, 0, 0)\big) = sin^2(\beta_0/2) sin^2(\theta/2)$ Anti-Particle S_{down} P_-T_- $\mathbb{P}_1 + \mathbb{P}_2 + \mathbb{P}_3 + \mathbb{P}_4 = 1$ Each of the probabilities $\mathbb{P}_j$ is related to a: fermion property, PT quadrant and a spacetime trajectory $x(c\Delta t, -\beta_0, \theta, \phi)$ of the measurement device. The particle is moving with relative speed β_0 through the measurement device, but due to relativity we can also state that the measurement device is moving with a relative speed $-\beta_0$ along the particle. The PT quadrants are discrete, but the orientation between the reference frames (observer and event-information) can differ continuously by a temporal rotation over angle β_0 or by a spatial rotation over the polar angles $\{\theta, \phi\}$. So, the reference frames of observer and event-information (initial state reference frame) can undergo a PT quadrant related information mixing. This PT quadrant related information mixing can be proven or falsified by performing the following experiment.	
Experimental setup: to proof or falsify PT quadrant information mixing.		
6	**Spin measurement device** in the γ_0 frame (lab-frame $\beta = 0$) with one degree of freedom, a rotation in the $\{\gamma_3\gamma_1 \to zx\}$ plane of the lab frame as function of the spatial rotation angle θ. The spatial rotation angle θ will provide a parity-reversal mixing between the reference frame of the event-information (initial state reference frame) and the reference frame of the observer (lab frame).	
7	Prepare a beam of particles in a **spin-up state in the z-direction** with a given **relative speed β_0 in the z-direction** $u_3(\theta_0 = 0, \phi_0 = 0)$. The relative speed β_0 will provide a time-reversal mixing between the reference frame of the event-information (initial state reference frame) and the reference frame of the observer (lab frame).	
8	Perform spin measurements under different spatial rotation angle θ orientations on a beam of particles prepared in a spin-up state in the z-direction having different relative speeds β_0 in the z-direction. The calculated probabilities $\mathbb{P}_j$ (F.4) for measuring fermion properties are given in **table F.1**.	
9	Already my thanks and respect to the researchers who want to conduct this experiment (please contact me). You will need a great deal of inventiveness and creativity. Crunching my head on how this experiment can be done? Maybe via a Stern-Gerlach device to prepare the spin-up state followed by a cathode ray tube to induce a relative speed followed by a Stern-Gerlach device to measure?	

Table F.1: Probabilities for measuring fermion properties, under different spatial angle θ orientations, given a beam of particles prepared in a **spin-up state in the z-direction** with different **relative speeds β_0 in the z-direction**.

Event-information mixing		$\mathbb{S}^{1,2}, T; \boldsymbol{\beta}$ Time-reversal (**Table 4.1**) (**Fig. 5.2**)		$\mathbb{S}^{1,2}, P; \boldsymbol{\theta}$ Parity-reversal (**Table 4.1**) (**Fig. 5.2**)		
PT quadrants		P_-T_+	P_+T_+	P_+T_-	P_-T_-	
Relative speed β_0	Spatial angle θ	Probability $\mathbb{P}_1$ detecting particle spin-down %	Probability $\mathbb{P}_2$ detecting particle spin-up %	Probability $\mathbb{P}_3$ detecting anti-particle spin-up %	Probability $\mathbb{P}_4$ detecting anti-particle spin-down %	**(F.4)** Sum %
≈ 0 $\frac{v}{c} \approx 0$	0	0	100	0	0	100
	$\pi/3$	25	75	0	0	100
	$\pi/2$	50	50	0	0	100

Probabilities for a beam of particles prepared in a spin-up state having a relative speed of $\beta_0 \approx 0$ (time-reversal angle). Rotating the spin measurement device at different spatial angle θ orientations (parity-reversal angle) give the well-known Stern Gerlach spin measurements of particle spin-up and particle spin-down probabilities.

$\pi/4$ $\frac{v}{c} = \frac{\sqrt{2}}{2}$	0	0	85.36	14.64	0	100
	$\pi/3$	21.34	64.02	10.98	3.66	100
	$\pi/2$	42.68	42.68	7.32	7.32	100

Probabilities for a beam of particles prepared in a spin-up state having a relative speed of $\beta_0 = \pi/4$ (time-reversal angle). At an $\theta = 0$ orientation of the spin measurement device there is a 14.64% probability of detecting anti-particles in a spin-up state because of the time-reversal event-information mixing of $\{T_+, T_-\}$ caused by the relative speed of $\beta_0 = \pi/4$. At an $\theta = \pi/3$ and $\theta = \pi/2$ orientation of the spin measurement device there will also arise probabilities for anti-particle spin-down and particle spin-down states because of the parity-reversal event-information mixing $\{P_+, P_-\}$.

$\pi/3$ $\frac{v}{c} = \frac{\sqrt{3}}{2}$	0	0	75	25	0	100
	$\pi/3$	18.75	56.25	18.75	6.25	100
	$\pi/2$	12.5	37.5	37.5	12.5	100

Probabilities for a beam of particles prepared in a spin-up state having a relative speed of $\beta_0 = \pi/3$ (time-reversal angle). At an $\theta = 0$ orientation of the spin measurement device there is a 25% probability of detecting anti-particles in a spin-up state because of the time-reversal event-information mixing of $\{T_+, T_-\}$ caused by the relative speed of $\beta_0 = \pi/3$. At an $\theta = \pi/3$ and $\theta = \pi/2$ orientation of the spin measurement device there will also arise probabilities for anti-particle spin-down and particle spin-down states because of the parity-reversal event-information mixing $\{P_+, P_-\}$.

At a relative speed of $\beta = \pm\pi/2$ where time-reversal is indefinite the Dirac spinor $\psi_1(x)$ will have the following form:

$$\psi_1\left(c\Delta t, \beta = \pm\tfrac{\pi}{2}, \theta, \phi\right) = \sqrt{2}\, S_1(\theta, \phi)\left(\tfrac{1}{2}(1 \pm \sigma_3)\right) \qquad \psi_1\tilde{\psi}_1 = cos(\pm\pi/2) = 0 \qquad \left(\tfrac{1}{2}(1 \pm \sigma_3)\right)^2 = \tfrac{1}{2}(1 \pm \sigma_3)$$

$$\psi_1\left(c\Delta t, \beta = \pm\tfrac{\pi}{2}, \theta, \phi\right) = \tfrac{1}{\sqrt{2}}\left(+cos\left(\tfrac{\theta}{2}\right) cos\left(\tfrac{\phi}{2}\right) + sin\left(\tfrac{\theta}{2}\right) sin\left(\tfrac{\phi}{2}\right) \mathbb{i}\sigma_1 - sin\left(\tfrac{\theta}{2}\right) cos\left(\tfrac{\phi}{2}\right) \mathbb{i}\sigma_2 - cos\left(\tfrac{\theta}{2}\right) sin\left(\tfrac{\phi}{2}\right) \mathbb{i}\sigma_3\right)$$
$$\pm\tfrac{1}{\sqrt{2}}\left(-cos\left(\tfrac{\theta}{2}\right) sin\left(\tfrac{\phi}{2}\right) \mathbb{i} + sin\left(\tfrac{\theta}{2}\right) cos\left(\tfrac{\phi}{2}\right) \sigma_1 + sin\left(\tfrac{\theta}{2}\right) sin\left(\tfrac{\phi}{2}\right) \sigma_2 + cos\left(\tfrac{\theta}{2}\right) cos\left(\tfrac{\phi}{2}\right) \sigma_3\right)$$

Photons are massless and particle/anti-particle in one. Hence, Dirac spinor $\psi_1(x)$ also contains a connection between fermions and bosons.

Acknowledgment
I want to thank my wife, Mireille, for her continued support and encouragement.

Recommendation: Start teaching Geometric Algebra to high school students.

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