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Review Article

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Hyper Exponential Function

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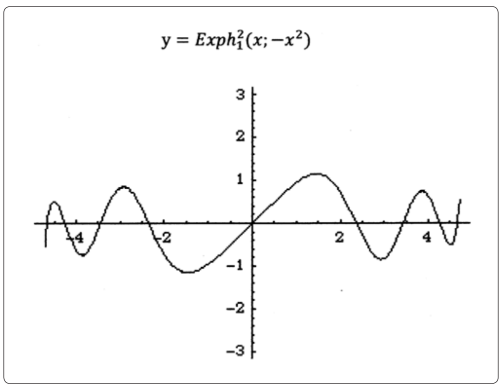


Hyper exponential function, which was created by Uchida, is a group of special functions.

The form of Hyper exponential functions of n-order.

$$Exph_j^n(x; f(x))$$

Hyper exponential functions of n-order generated by using any function $f(x)$.
***n*:** order.
***j*:** the number of seed.
***x*:** variable.
***f*(*x*):** any function that is defined in an interval that contains zero.

seed (*x*; *j*) $= \frac{x^j}{j!}\,(j = 0, 1, 2, 3 \ldots n-1)$

The seed of the Hyper exponential function means the first term of the series.

The main feature of the Hyper exponential functions of n-order.

$$x \in R, y \in R$$
$$y = Exph_j^n(x; f(x))$$

The n-order derivative of y is the product of $f(x)$ and y.

$$\frac{d^n y}{dx^n} = f(x)y$$

The Hyper exponential functions are defined by the characteristics of the derivative rather than defined according to its method of generation.

The feature of Hyper exponential functions of second-order.

$$\frac{d^2 y}{dx^2} = f(x)y$$

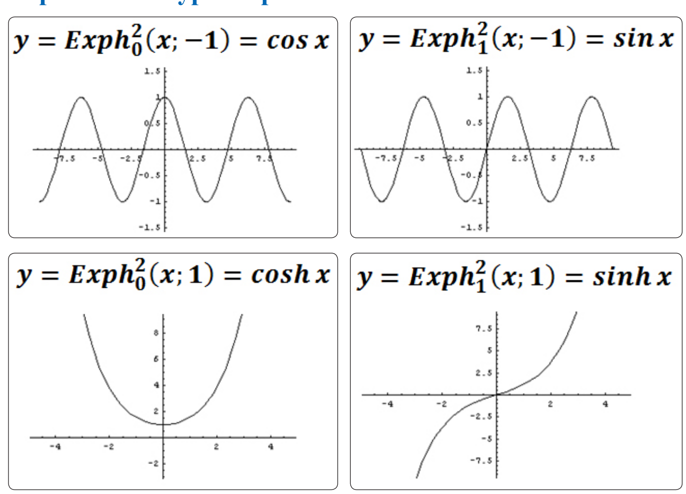
Two Hyper exponential functions of second-order that are generated using a function $f(x)$, one is the first term 1, and the other is the first term x.

$Exph_0^2$ --- **The number of seed is 0. --- Seed(*x*)=1**

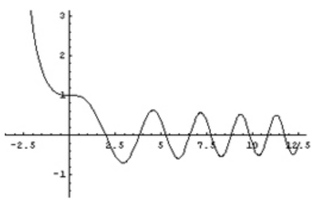
$Exph_1^2$ --- **The number of seed is 1. --- Seed(*x*)=*x***

These two functions are linearly independent.

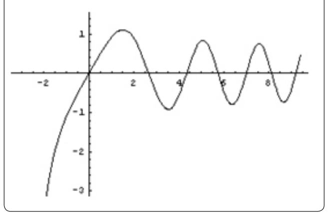
Graphs of the Hyper exponential functions of second-order.



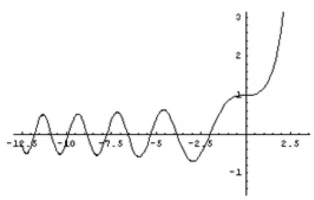
$y = Exph_0^2(x;-x)$



$y = Exph_1^2(x;-x)$



$y = Exph_0^2(x;x)$



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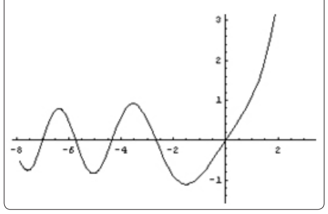
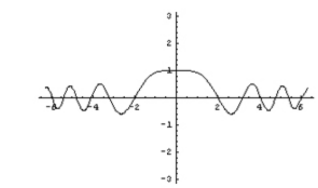


$y = Exph_0^2(x;-x^2)$



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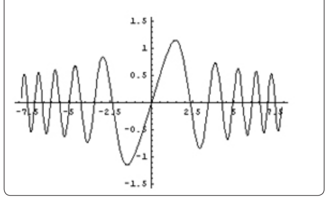

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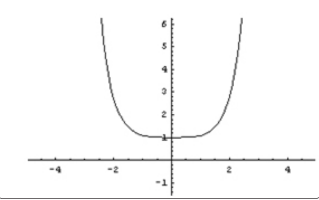
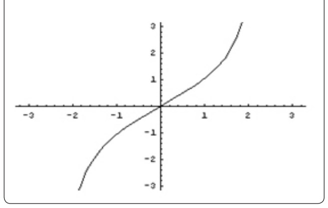
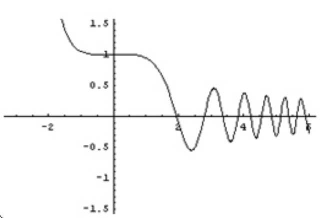


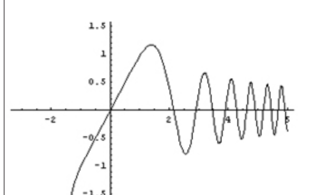
$y = Exph_1^2(x;x^2)$



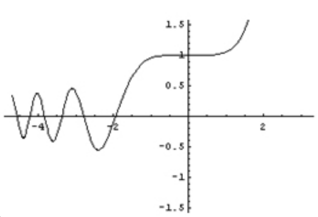
$y = Exph_0^2(x;-x^3)$



$y = Exph_1^2(x;-x^3)$



$y = Exph_0^2(x;x^3)$



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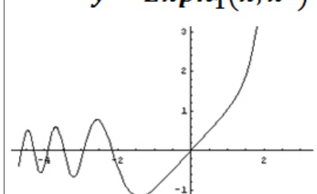
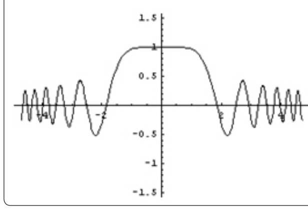
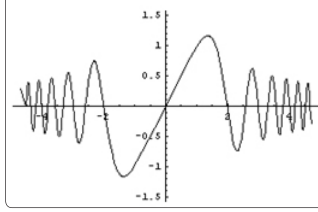


$y = Exph_0^2(x;-x^4)$



$y = Exph_1^2(x;-x^4)$



All of the graphs listed above were drawn by Mathematica.

The formula for the solution of the second order linear homogeneous equation with variable coefficients.

$$x \in R, y \in R$$
$$u(x) = Exph_0^1(x; f(x))$$
$$v(x) = Exph_j^2(x; g(x)) \,(j = 0, 1)$$

$$u' = f(x)u(x)$$
$$v'' = g(x)v(x)$$

$$e^{\int_0^x f(x)dx} = Exph_0^1(x; f(x))$$

$$y = uv$$
$$A(x) = -2f(x)$$
$$B(x) = f(x)^2 - f'(x) - g(x)$$
$$y'' + A(x)y' + B(x)y = 0$$

$$y = Exph_0^1(x; f(x)) \cdot \left(c_1 Exph_0^2(x; g(x)) + c_2 Exph_1^2(x; g(x))\right)$$

An example.

$$y'' + (5x+1)y' + \left(6x^2 + \frac{1}{2}\right)y = 0$$

The initial conditions are as follows:

$$y(0) = 12, y'(0) = 10$$

The answer is as follows:

$$f(x) = -\frac{5x+1}{2}$$
$$g(x) = \frac{(5x+1)^2}{4} + \frac{(5x+1)'}{2} - \left(6x^2 + \frac{1}{2}\right) = \frac{1}{4}(x+1)(x+9)$$
$$u(x) = Exph_0^1(x; -\frac{5x+1}{2})$$
$$v(x) = Exph_0^2(x; \frac{1}{4}(x+1)(x+9))$$
$$w(x) = Exph_1^2(x; \frac{1}{4}(x+1)(x+9))$$

C_1 and C_2 as arbitrary constants.

$$y = u(c_1 v + c_2 w)$$

From the initial conditions.

$$y(0) = u(0)(c_1 v(0) + c_2\, w(0)) = 12$$
$$u(0) = v(0) = 1, w(0) = 0$$
$$y(0) = c_1 = 12$$

$$y' = u'(c_1 v + c_2 w) + u(c_1 v' + c_2 w')$$
$$y'(0) = u'(0)\big(c_1 v(0) + c_2 w(0)\big) + u(0)\big(c_1 v'(0) + c_2 w'(0)\big) = 10$$
$$v'(0) = 0,\ w'(0) = 1$$
$$y'(0) = 12u'(0) + c_2 = 10 \longrightarrow u'(x) = -\frac{5x+1}{2}u(x)$$
$$y'(0) = c_2 - 6 = 10$$
$$c_2 = 16$$
$$\therefore$$
$$y = u(x)\big(12v(x) + 16w(x)\big)$$

The list of the differential equations that the solutions by using the Hyper exponential functions.

Hermite Differential Equations

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$
$$y = Exph_0^1 \cdot \left(c_1 Exph_0^2(x; x^2 - 2n - 1) + c_2 Exph_1^2(x; x^2 - 2n - 1)\right)$$

Bessel Differential Equations

$$\frac{d^2y}{du^2} - \frac{1}{u}\frac{dy}{du} + \left(1 - \frac{n^2}{u^2}\right)y = 0$$
$$u = \pm e^{-x}$$
$$y = c_1 Exph_0^2\left(x; n^2 - e^{-2x}\right) + c_2 Exph_1^2\left(x; n^2 - e^{-2x}\right)$$

Legendre Differential Equations

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$
$$Y = \int_\alpha^x y\, dx$$

$x \neq \pm 1$

$$\frac{d^2Y}{dx^2} = \frac{n(n+1)}{(x^2-1)}Y$$
$$Y = c_1 Exph_0^2\left(x; \frac{n(n+1)}{(x^2-1)}\right) + c_2 Exph_1^2\left(x; \frac{n(n+1)}{(x^2-1)}\right)$$

Solution to satisfy the wave equation.

The Hyper exponential functions of second-order are used to describe the solution that satisfies the wave equation.

$$x \in R, y \in R, z \in R, t \in R$$
$$F(v) = Exph_j^2\big(v; f(v)\big) \qquad (j = 0, 1)$$
$$v = lx + my + nz \pm ct$$
$$l^2 + m^2 + n^2 = 1$$

The c is constant.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{c^2 \partial t^2}\right)F(v) = (l^2 + m^2 + n^2 - 1)\frac{d^2F(v)}{dv^2}$$
$$\therefore$$
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{c^2 \partial t^2}\right)F(v) = 0$$

$$\frac{\partial^2}{\partial x^2} = l^2\frac{d^2}{dv^2}$$
$$\frac{\partial^2}{\partial y^2} = m^2\frac{d^2}{dv^2}$$
$$\frac{\partial^2}{\partial z^2} = n^2\frac{d^2}{dv^2}$$
$$\frac{\partial^2}{\partial t^2} = c^2\frac{d^2}{dv^2}$$

Handling of a singular point by the division by zero calculus.

The singular point of the following Bessel differential equations is considered.

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$$
$$y = y_1 y_2$$

However, y_1 and y_2 satisfy the following differential equations.

$$-2xy_1' = y_1 \qquad \text{---} \quad ①$$
$$\frac{4x^2}{4n^2 - 1 - 4x^2}y_2'' = y_2 \qquad \text{---} \quad ②$$

x=0

From ①
$$y_1(0) = 0$$

From ②
$$y_2(0) = 0$$
$$\therefore$$
$$y(0) = y_1(0)y_2(0) = 0$$ → By the division by zero calculus.

Supplement:

1. The Hyper exponential functions of n-order can be generated using repetitive integrals.
2. The Hyper exponential functions of n-order are uniform convergence in the wider sence.
3. The n-order Hyper-exponential functions generated using a certain $f(x)$ are linearly independent one another.
4. The domain and the range of the Hyper exponential functions of second-order are extended to a complex number. In addition, The formula for the solution of the second order linear homogeneous equation with variable coefficients is also extended to a complex number.

Recent Publications:

1. Kumahara K, Saitoh S, Uchida K(2009) Normal solutions of linear ordinary differential equations of the second order,International Journal of Applied Mathematics,Volume 22 No.6 2009, 981-996.
2. Uchida K(2017)「Introduction to Hyper exponential function and differential equation revised first edition」, eBookland. (In Japanese).
   In this book, the method to generate the hyper exponential functions is described concretely.