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Review Article

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Complex Polytopic Fuzzy Sets and their Geometric Aggregation Operators

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Abstract
This paper is concerned to the study of complex Polytopic fuzzy sets (CPFSs), some of their basic operational laws and their corresponding aggregation operators, which extends the notion of q-Rung orthopair fuzzy sets (q-ROFSs) and Polytopic fuzzy sets (PFSs). We introduce some basic operations, such as union, intersection and complement under complex Polytopic fuzzy numbers (CPFNs). Moreover, we establish some new operators such as, complex Polytopic fuzzy ordered weighted geometric (CPFOWG) operator, complex Polytopic fuzzy weighted geometric (CPFWG) operator, complex Polytopic fuzzy hybrid geometric (CPFHG) operator, induced complex Polytopic fuzzy ordered weighted geometric (I-CPFOWG) operator and induced complex Polytopic fuzzy hybrid geometric (I-CPFHG) operator coupled with the boundedness, idempotency and monotonicity of their structure properties. The efficiency and effectiveness this method is shown with an illustrative example.

Keywords: CPFOWG Operator, CPFWG Operator, CPFHG Operator, I-CPFOWG Operator, I-CPFHG Operator, MAGDM Problem.

1. Introduction

Emergency decision-making is a crucial and valuable tool for communities and nations. It increases the dependability and efficacy of emergency response, reducing the number of casualties, ecological damage, and financial losses. Making decisions is one of the best ways to sort through all of your options and choose the best one. In the past, it has been widely believed that all the data pertaining to the alternative's criteria and accompanying weights are given as clear numbers. However, most judgments in real-life circumstances are made in a setting where the objectives and restrictions are generally not well-defined or ambiguous. In order to handle these situations in real life problems different theories were developed with passage of time such as, fuzzy set (FS) theory [1], rough sets (RS) theory [2] and soft sets (SS) theory [3]. All of the aforementioned theories have unique, noteworthy applications.

It is important to keep in mind that Zadeh [1], the author of fuzzy set theory, was the one who began this journey by introducing the idea of fuzzy set (FS), with one element known as the membership function and only describes satisfaction's degree of an object without mentioning the object's dissatisfaction. Later on, the theory fails to handle data that acquire unsatisfactory and satisfactory information of interest. In order to address this deficiency, Atanassov [4] developed the concept of IFS by presenting each set element as an ordered pair, (α,β), where β and α stand for the degree of non-membership and membership, respectively. Later on, this field used in various situations where the IFS was not appropriate since the data was interval-based. In order to get over the aforementioned challenges, mathematician requires some more potent techniques. Following that, Atanassov and Gargov [5] expanded on the notion of an intuitionistic fuzzy set and introduced the concept of an interval-valued intuitionistic fuzzy set, in which the grades of membership and non-membership are intervals instead of real values with constraints that bounded by 0 and 1. According to the criteria of addressing the problem, each technique requires some operators that serve as a backbone, as demonstrated in the aforementioned literature. As a result, aggregation operators are crucial instruments for gathering the provided data on various alternatives in various sectors. Many scholars have examined at the utility of operators related to intuitionistic fuzzy environments and interval valued IFS in [6–10], and they have developed a variety of operators with these different domains and applied them on decision-making problems. Wei [11] presented the concept of Hamacher and image fuzzy aggregation operators and constructed a multi-attribute group decision-making issue based on the recommended operators while taking into account the importance of these operators. Wang et al. [12] proposed the use of image fuzzy numbers to create new operators called Muirhead mean operators. Based on intuitionistic fuzzy numbers, Rahman [13] established the idea of various geometric logarithmic and averaging aggregation operators. Shapley fuzzy measure was applied by Tian et al. [14] who developed the concepts of power operator and weighted geometric operator. Many aggregation operators have been developed using intuitionistic fuzzy environments by Ding et al. [15], Yang et al. [16], He et al. [17], Zhang et al. [18], Li et al. [19], Meng et al. [20].

Yager [21] developed a model idea of q-rung orthopair fuzzy set (qROFS) in response to the increasing of scientific knowledge modelling in relation to decision-making problems of theories, which needed additional refinement to handle the maximum powers of IFS. In q-ROFS, q-th power of non-membership with the sum of q-th power of membership is equal to or less than one which covers the excluded information of IFS. Furthermore, it is shown that IFS and PFS are particular instances of q-ROFS. It demonstrates that qROFS is an extension of the approaches outlined before, covering a wide range of information and handling uncertain environments more effectively. Furthermore, Yager examined certain fundamental q-ROFS characteristics in [22] and employed them in additional research pertaining to this class. Liu et al. [23] then investigated the q-rung technique for operators such orthopair fuzzy weighted averaging (q-ROFWA) and orthopair fuzzy weighted geometric (q-ROFWG), and based on these, they resolved further decision-making issues. The concept of q-ROFS information measures, such as entropy, similarity measure, inclusion measure, and distance measure, was recently developed by Peng et al. [24].

The aforementioned approaches fail miserably of producing the desired outcome since there are so many real-world issues where the computation of the neutral membership degree is crucial. Cuong et al. [25] presented the idea of an image fuzzy set to deal with this kind of problem. Three terms membership degree, neutral membership degree, and nonmembership degree were utilised in their work, and their combined sum did not surpass 1. Ashraf [26] extended this idea to the spherical fuzzy set in order to address the drawbacks of the picture fuzzy set. Based on this concept, Kutlu and Kahraman [27, 28] expanded the aforementioned methods by creating TOPSIS and VIKOR. Additionally, the authors expanded the idea of spherical fuzzy sets to spherical soft fuzzy sets in [29], and on the basis of these cutting-edge techniques, they presented several new aggregation operations. In consideration of the above mentioned methods, there were some problems which cannot be handled by these methods such as, in situation like when the sum of squares of the membership degree, neutral membership and nonmembership degree (0.7, 0.5, 0.9) exceeds 1. To address such complicated issues, Ismat et al. [30] developed the concept of polytopic fuzzy sets as an extension of q-rung orthopair fuzzy sets and spherical fuzzy sets. It is significant to highlight that Polytopic fuzzy set theory may be used to solve

According to the research described above, these approaches have flaws and are unable to represent the partial ignorance of the data and its fluctuations during a particular time period. As a result, only non-periodic information may be handled by any of the FS theory extensions indicated above. But in complicated data sets, ambiguity and vagueness also coexist with changes to the data's periodicity. For instance, large volumes of data from image analysis, facial recognition, audio, medical research, and government biometric databases are included in complex data sets. These databases also have a lot of contradictory and incomplete information in them. In order to circumvent these circumstances, Ramot et al. [31] proposed the concept of complex fuzzy set (CFS) theory, in which the degree of alternatives is expressed as a complex fuzzy number (CFN) rather than a real integer from the unit circle. Later on Alkouri and Salleh [32] generalized this idea by introducing the idea of CFS into complex intuitionistic fuzzy sets (CIFS), based on the membership grade $\hbar e^{ia}$ and non-membership grade λe^{ib} with some conditional restrictions $0 \leq \hbar + \lambda \leq 1$, $0 \leq \frac{a}{2\pi} + \frac{b}{2\pi} \leq 1$, where $\hbar, \lambda \in [0,1]$ and $a, b \in [0,\ 2\pi]$. Complex fuzzy numbers (CFNs) were applied by Ma et al. [33] to develop the CFS-based technique, a unique approach for solving multiperiodic factors. Several operational laws for CFS for decision-making issues were provided by Dick et al. [34]. The results of [34] were altered by Liu and Zhang [35] and presented in a brand-new, sophisticated manner. Different types of aggregation operators in complex intuitionistic fuzzy environments were studied by Garg and Rani [36], Kumar and Bajaj [37], and Rani and Garg [38]. Greenfield et al. [39] also introduced the idea of complex interval-valued fuzzy set theory. Although the CIFS concept has been used in several contexts, due to its structural limitations, the principle only has a restricted range of applications. The CIFS concept will fail in instances where the total of the real part and imaginary part of the duplet exceeds the unit interval, for example, and neither the membership grade nor the non-membership grade with the restricted requirement will be given any information in such cases.

i. To introduce the notion of complex Polytopic fuzzy set theory. CPFS is an extension of Polytopic fuzzy set. Thus CPFS is more effective, broad, and efficient than existing theories like CIFS and CPyFS in dealing with uncertain information throughout the decision-making process. Moreover, there is no research work up to date on the CPFS.
ii. To provide some fundamental operational principles for complex Polytopic fuzzy numbers.
iii. To introduce the aggregation operators, such as the CPFWG operator, CPFOWG operator, CPFHG operator, I-CPFOWG operator and I-CPFHG operator.
iv. To establish a decision-making technique on the basis of this novel model.
v. To demonstrate the validity and usefulness of the new method through an example.

The remaining paper is organized as: Section 2 presents some fundamental definitions. PFSs are discussed in Section 3 along with some of their basic operation laws. In Section 4 different aggregation operators under CPF environment are studied. Moreover, Section 5 includes emergency decision-making model under the novel approach. Section 6 contains illustrative example under different techniques. Section 7 and Section 8 presents the comparative analysis sensitivity analysis, respectively. Next, Section 9 and 10 contains the limitations and conclusion of the novel model.

2. Preliminaries

Some fundamental ideas and terminology that are necessary later on are presented in this section.

Definition 1: [42] Let C be a non-empty complex fuzzy set (CFS) defined on a finite set X by $C = \left\{ \chi, \hbar_C(\chi) e^{i a_C(\chi)} : \chi \in X \right\}$ with $\hbar_C(\chi) : X \to [0,1]$ is called the complex membership degree (MED) of χ with $i = \sqrt{-1}$ in the complex plane.

Definition 2: [43] Let X be a fixed set; a complex intuitionistic fuzzy set I in a finite X is given by

$I=\left\{\left\langle \chi,\hbar_I(\chi)e^{ia_I(\chi)},\lambda_I(\chi)e^{ib_I(\chi)}\right\rangle:\chi\in X\right\}$ with $0\le \hbar_I(\chi),\lambda_I(\chi)\le 1$, $i=\sqrt{-1}$ and $a_I(\chi),b_I(\chi)\in[0,2\pi]$ such that $0\le\hbar_I(\chi)+\lambda_I(\chi)\le 1,\forall\chi\in X$, $0\le \frac{a_I(\chi)}{2\pi}+\frac{b_I(\chi)}{2\pi}\le 1$ and $\hbar_I(\chi),\lambda_I(\chi)$ represents MED and non-membership degree (NOMED) respectively.

Definition 3: [44] Let X be a fixed set, then the Complex Pythagorean fuzzy set is defined as $P=\left\{\left\langle \chi,\hbar_P(\chi)e^{ia_P(\chi)},\lambda_P(\chi)e^{ib_P(\chi)}\right\rangle:\chi\in X\right\}$ with $0\le(\hbar_P(\chi))^2+(\lambda_P(\chi))^2\le 1,\forall\chi\in X$, $0\le\left(\frac{a_P(\chi)}{2\pi}\right)^2+\left(\frac{b_P(\chi)}{2\pi}\right)^2\le 1$ and $\hbar_I(\chi),\lambda_I(\chi)$ represents MED and NOMED respectively.

In addition, its hesitancy or indeterminacy can be find out by using the following formula: $G_P(\chi)=\sqrt{1-\left((\hbar_P(\chi))^2+(\lambda_P(\chi))^2\right)}e^{\sqrt{1-\left((\hbar_P(\chi))^2+(\lambda_P(\chi))^2\right)}}$, then the term $G_P(\chi)$ is called the grade of indeterminacy or hesitancy of the element $\chi\in X$.

Definition 4: [45] Let X be a fixed set, then the complex q-Rung orthopair fuzzy set (q-CROFS) is defined as $Q=\left\{\left\langle \chi,\hbar_q(\chi)e^{ia_q(\chi)},\lambda_q(\chi)e^{ib_q(\chi)}\right\rangle:\chi\in X\right\}$ with $0\le(\hbar_q(\chi))^q+(\lambda_q(\chi))^q\le 1,\forall\chi\in X,q\ge 1$,

$0\le\left(\frac{a_q(\chi)}{2\pi}\right)^q+\left(\frac{b_q(\chi)}{2\pi}\right)^q\le 1$ and $\hbar_q(\chi),\lambda_q(\chi)$ represents MED and NOMED respectively.

In addition, its hesitancy or indeterminacy can be find out by using the following formula: $G_q(\chi)=\left(1-\left((\hbar_q(\chi))^q+(\lambda_q(\chi))^q\right)\right)^{\frac{1}{q}}e^{\left(1-\left((\hbar_q(\chi))^q+(\lambda_q(\chi))^q\right)\right)^{\frac{1}{q}}}$, then the term $G_q(\chi)$ is called the grade of indeterminacy or hesitancy of the element $\chi\in X$.

3. Complex Polytopic Fuzzy Sets

This section introduces the concept of Complex Polytopic fuzzy sets (CPFS), Complex Polytopic fuzzy numbers, and some of its fundamental operational laws, score function and accuracy function are investigated. Furthermore, various aggregation operators and their features are described, which will be useful in tackling MADM problems. The notion of Polytopic fuzzy set (PFS) extends the concepts of q-rung orthopair fuzzy set and spherical fuzzy set.

Definition 5: Let X be a fixed set, then the mathematical form CPFS ρ is written as $\rho=\left\{\left\langle \chi,\hbar_\rho(\chi)e^{ia_\rho(\chi)},\lambda_\rho(\chi)e^{ib_\rho(\chi)},\xi_\rho(\chi)e^{in_\rho(\chi)}\right\rangle:\chi\in X\right\}$ with $\hbar_\rho(\chi):X\to[0,\ 1]$, $\lambda_\rho(\chi):X\to[0,\ 1],\xi_\rho(\chi):X\to[0,\ 1]$, $a_\rho(\chi)\in[0,2\pi],b_\rho(\chi)\in[0,2\pi],n_\rho(\chi)\in[0,2\pi]$, and $0\le\left(\hbar_\rho(\chi)e^{ia_\rho(\chi)}\right)^q+\left(\lambda_\rho(\chi)e^{ib_\rho(\chi)}\right)^q+\left(\xi_\rho(\chi)e^{in_\rho(\chi)}\right)^q\le 1,\ \forall\chi\in X,\ q\ge 1,\ 0\le\left(\frac{a_\rho(\chi)}{2\pi}\right)^q+\left(\frac{b_\rho(\chi)}{2\pi}\right)^q+\left(\frac{n_\rho(\chi)}{2\pi}\right)^q$

≤ 1 and $\hbar_\rho(\chi),\lambda_\rho(\chi),\xi_\rho(\chi)$ represents membership degree (MED), neutral membership degree (NMED) and non-membership degree (NOMED) respectively.

Remark 1: Some special cases of Definitions 1-4 are stated as follows:

i) Complex fuzzy set: If we consider the phase term such as: $a_C(\chi)=0$, then it can be reduces to fuzzy set.

ii) Complex intuitionistic fuzzy: If we consider the phase terms such as: $a_I(\chi)=0, b_I(\chi)=0$, then it can be reduces to intuitionistic fuzzy set.

iii) Complex Pythagorean fuzzy: If we consider the phase terms such as: $a_p(\chi)=0$, $b_p(\chi)=0$, then it can be reduces to Pythagorean fuzzy set.

iv) Complex q-Rung orthopair fuzzy set: If we consider the phase terms such as: $a_q(\chi)=0$, $b_q(\chi)=0$, then it can be reduces to q-Rung orthopair fuzzy set.

v) Complex Polytopic fuzzy set: If we consider the phase terms such as: $a_\rho(\chi)=0$, $b_\rho(\chi)=0$ and $n_\rho(\chi)=0$, then it can be reduces to Polytopic fuzzy set.

Definition 6: Let a family of CPF numbers be such that $\upsilon_j=\left(\hbar_j e^{ia_j}, \lambda_j e^{ib_j}, \xi_j e^{in_j}\right)(j=1,2)$, and $\gamma>0$, be a real number, then

i) $$\upsilon_1\oplus\upsilon_2=\left[\begin{array}{l}\left(\hbar_1^q+\hbar_2^q-\hbar_1^q\hbar_2^q\right)^{\frac{1}{q}}.e^{i2\pi\left(\left(\frac{a_1}{2\pi}\right)^q+\left(\frac{a_2}{2\pi}\right)^q-\left(\frac{a_1}{2\pi}\right)^q\left(\frac{a_2}{2\pi}\right)^q\right)^{\frac{1}{q}}},\\ \left(\lambda_1\lambda_2\right).e^{i2\pi\left(\frac{b_1}{2\pi}\right)\left(\frac{b_2}{2\pi}\right)},\left(\xi_1\xi_2\right).e^{i2\pi\left(\frac{n_1}{2\pi}\right)\left(\frac{n_2}{2\pi}\right)}\end{array}\right]$$

ii) $$\upsilon_1\otimes\upsilon_2=\left[\begin{array}{l}\left(\hbar_1\hbar_2\right).e^{i2\pi\left(\frac{a_1}{2\pi}\right)\left(\frac{a_2}{2\pi}\right)},\left(\lambda_1\lambda_2\right).e^{i2\pi\left(\frac{b_1}{2\pi}\right)\left(\frac{b_2}{2\pi}\right)},\\ \left(\xi_1^q+\xi_2^q-\xi_1^q\xi_2^q\right)^{\frac{1}{q}}.e^{i2\pi\left(\left(\frac{n_1}{2\pi}\right)^q+\left(\frac{n_2}{2\pi}\right)^q-\left(\frac{n_1}{2\pi}\right)^q\left(\frac{n_2}{2\pi}\right)^q\right)^{\frac{1}{q}}}\end{array}\right]$$

iii) $$\gamma\upsilon_1=\left(\left(1-\left(1-\hbar_1^q\right)^\gamma\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\left(1-\left(\frac{a}{2\pi}\right)^q\right)^\gamma\right)^{\frac{1}{q}}},\ \left(\lambda_1\right)^\gamma.e^{i2\pi\left(\frac{b}{2\pi}\right)^\gamma},\left(\xi_1\right)^\gamma.e^{i2\pi\left(\frac{n}{2\pi}\right)^\gamma}\right)$$

iv) $$\left(\upsilon_1\right)^\gamma=\left(\left(\hbar_1\right)^\gamma.e^{i2\pi\left(\frac{a}{2\pi}\right)^\gamma},\left(\lambda_1\right)^\gamma.e^{i2\pi\left(\frac{b}{2\pi}\right)^\gamma}\left(1-\left(1-\xi_1^q\right)^\gamma\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\left(1-\left(\frac{n}{2\pi}\right)^q\right)^\gamma\right)^{\frac{1}{q}}}\right)$$

Definition 7: Let $\upsilon = \left(\hbar e^{ia}, ƛ e^{ib}, \xi e^{in}\right)$, be a CPF number, then its score function $S(\upsilon)$ and accuracy function $A(\upsilon)$ is defined in the following mathematical form:

$$S(\upsilon) = \frac{1}{3}\left[\left(1+\hbar^q + ƛ^q - \xi^q\right) + \left(1 + a^q + b^q - n^q\right)\right] \text{ with } S(\upsilon) \in [-2, 2], \text{ and}$$

$$A(\upsilon) = \frac{1}{2}\left[\left(1+\max\left(\hbar^q, ƛ^q\right) - \xi^q\right) + \left(1 + \max\left(a^q, b^q\right) - n^q\right)\right] \text{with } A(\upsilon) \in [0, 2].$$

Definition 8: Let $\varphi = \left\{\left\langle \chi, \hbar_{\varphi}(\chi) e^{ia_{\varphi}(\chi)}, ƛ_{\varphi}(\chi) e^{ib_{\varphi}(\chi)}, \xi_{\varphi}(\chi) e^{in_{\varphi}(\chi)} \right\rangle : \chi \in X\right\}$ and $\phi = \left\{\left\langle \chi, \hbar_{\phi}(\chi) e^{ia_{\phi}(\chi)}, ƛ_{\phi}(\chi) e^{ib_{\phi}(\chi)}, \xi_{\phi}(\chi) e^{in_{\phi}(\chi)} \right\rangle : \chi \in X\right\}$, be the two CPFNs then, some of its basic operations are stated as follows:

1) Union: $\varphi \cup \phi = \left\{\left\langle \chi, \hbar_{\varphi\cup\phi}(\chi) e^{ia_{\varphi\cup\phi}(\chi)}, ƛ_{\varphi\cup\phi}(\chi) e^{ib_{\varphi\cup\phi}(\chi)}, \xi_{\varphi\cup\phi}(\chi) e^{in_{\varphi\cup\phi}(\chi)} \right\rangle : \chi \in X\right\}$

   where $\hbar_{\varphi\cup\phi}(\chi) = \max\left\{\hbar_{\varphi}(\chi), \hbar_{\phi}(\chi)\right\}, ƛ_{\varphi\cup\phi}(\chi) = \max\left\{ƛ_{\varphi}(\chi), ƛ_{\phi}(\chi)\right\},$ $\xi_{\varphi\cup\phi}(\chi) = \min\left\{\xi_{\varphi}(\chi), \xi_{\phi}(\chi)\right\}$

   and $a_{\varphi\cup\phi}(\chi) = \max\left\{a_{\varphi}(\chi), a_{\phi}(\chi)\right\}, b_{\varphi\cup\phi}(\chi) = \max\left\{b_{\varphi}(\chi), b_{\phi}(\chi)\right\},$ $n_{\varphi\cup\phi}(\chi) = \min\left\{n_{\varphi}(\chi), n_{\phi}(\chi)\right\}$

2) Intersection: $\varphi \cap \phi = \left\{\left\langle \chi, \hbar_{\varphi\cap\phi}(\chi) e^{ia_{\varphi\cap\phi}(\chi)}, ƛ_{\varphi\cap\phi}(\chi) e^{ib_{\varphi\cap\phi}(\chi)}, \xi_{\varphi\cap\phi}(\chi) e^{in_{\varphi\cap\phi}(\chi)} \right\rangle : \chi \in X\right\}$

   where $\hbar_{\varphi\cap\phi}(\chi) = \min\left\{\hbar_{\varphi}(\chi), \hbar_{\phi}(\chi)\right\}, ƛ_{\varphi\cap\phi}(\chi) = \min\left\{ƛ_{\varphi}(\chi), ƛ_{\phi}(\chi)\right\},$ $\xi_{\varphi\cap\phi}(\chi) = \max\left\{\xi_{\varphi}(\chi), \xi_{\phi}(\chi)\right\}$

   and $a_{\varphi\cap\phi}(\chi) = \min\left\{a_{\varphi}(\chi), a_{\phi}(\chi)\right\}, b_{\varphi\cap\phi}(\chi) = \min\left\{b_{\varphi}(\chi), b_{\phi}(\chi)\right\},$ $n_{\varphi\cap\phi}(\chi) = \max\left\{n_{\varphi}(\chi), n_{\phi}(\chi)\right\}$

3) Complement: $\varphi^c = \left\{\left\langle \chi, \xi(\chi) e^{in(\chi)}, ƛ(\chi) e^{ib(\chi)}, \hbar(\chi) e^{ia(\chi)} \right\rangle : \chi \in X\right\}$

Definition 9: Let $\upsilon_1 = \left(\hbar_1 e^{ia_1}, ƛ_1 e^{ib_1}, \xi_1 e^{in_1}\right)$ and $\upsilon_2 = \left(\hbar_2 e^{ia_2}, ƛ_2 e^{ib_2}, \xi_2 e^{in_2}\right)$ are any two CPFNs, then

i) If $\hbar_1 < \hbar_2,\ a_1 < a_2,\ ƛ_1 < ƛ_2,\ b_1 < b_2, \xi_1 > \xi_2, n_1 > n_2$, then $\upsilon_2 > \upsilon_1$

ii) If $\hbar_1 > \hbar_2,\ a_1 > a_2,\ ƛ_1 > ƛ_2,\ b_1 > b_2, \xi_1 < \xi_2, n_1 < n_2$, then $\upsilon_1 > \upsilon_2$

iii) If $\hbar_1 = \hbar_2,\ a_1 = a_2,\ ƛ_1 = ƛ_2,\ b_1 = b_2, \xi_1 = \xi_2, n_1 = n_2$, then $\upsilon_1 = \upsilon_2$

Theorem 1: Let a family of CPF numbers be such that $\upsilon_j = \left(\hbar_j e^{ia_j}, ƛ_j e^{ib_j}, \xi_j e^{in_j}\right) (j = 1, 2, 3)$, then the following laws hold:

1) Commutative laws:

   i) $\upsilon_1 \oplus \upsilon_2 = \upsilon_2 \oplus \upsilon_1$

ii) $\upsilon_1 \otimes \upsilon_2 = \upsilon_2 \otimes \upsilon_1$

2) Associative laws:

i) $(\upsilon_1 \oplus \upsilon_2) \oplus \upsilon_3 = \upsilon_1 \oplus (\upsilon_2 \oplus \upsilon_3)$

ii) $(\upsilon_1 \otimes \upsilon_2) \otimes \upsilon_3 = \upsilon_1 \otimes (\upsilon_2 \otimes \upsilon_3)$

3) Distributive laws:

i) $\upsilon_1 \otimes (\upsilon_2 \oplus \upsilon_3) = \upsilon_1 \otimes \upsilon_2 \oplus \upsilon_1 \otimes \upsilon_3$

ii) $(\upsilon_1 \oplus \upsilon_2) \otimes \upsilon_3 = \upsilon_1 \otimes \upsilon_3 \oplus \upsilon_2 \otimes \upsilon_3$

Proof: Since $\upsilon_1 = \left(\hbar_1 e^{ia_1}, \lambda_1 e^{ib_1}, \xi_1 e^{in_1}\right)$ and $\upsilon_2 = \left(\hbar_2 e^{ia_2}, \lambda_2 e^{ib_2}, \xi_2 e^{in_2}\right)$ are two CPFNs, then, by using Definition 6, w have

$$\upsilon_1 \oplus \upsilon_2 = \begin{pmatrix} \left(\hbar_1^q + \hbar_2^q - \hbar_1^q \hbar_2^q\right)^{\frac{1}{q}} .e^{i2\pi\left(\left(\frac{a_1}{2\pi}\right)^q + \left(\frac{a_2}{2\pi}\right)^q - \left(\frac{a_1}{2\pi}\right)^q \left(\frac{a_2}{2\pi}\right)^q\right)^{\frac{1}{q}}}, \\ (\lambda_1 \lambda_2).e^{i2\pi\left(\frac{b_1}{2\pi}\right)\left(\frac{b_2}{2\pi}\right)}, (\xi_1 \xi_2).e^{i2\pi\left(\frac{n_1}{2\pi}\right)\left(\frac{n_2}{2\pi}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\hbar_2^q + \hbar_1^q - \hbar_2^q \hbar_1^q\right)^{\frac{1}{q}} .e^{i2\pi\left(\left(\frac{a_2}{2\pi}\right)^q + \left(\frac{a_1}{2\pi}\right)^q - \left(\frac{a_2}{2\pi}\right)^q \left(\frac{a_1}{2\pi}\right)^q\right)^{\frac{1}{q}}}, \\ (\lambda_2 \lambda_1).e^{i2\pi\left(\frac{b_2}{2\pi}\right)\left(\frac{b_1}{2\pi}\right)}, (\xi_2 \xi_1).e^{i2\pi\left(\frac{n_2}{2\pi}\right)\left(\frac{n_1}{2\pi}\right)} \end{pmatrix}$$

$$= \upsilon_2 \oplus \upsilon_1$$

Moreover, the remaining parts of this Theorem can be proved easily by using definition 6.

Theorem 2: Let a family of CPF numbers be such that $\upsilon_j = \left(\hbar_j e^{ia_j}, \lambda_j e^{ib_j}, \xi_j e^{in_j}\right) (j = 1, 2, 3),$ with $\gamma, \gamma_1, \gamma_2 > 0,$ then the following laws hold:

1) $\gamma(\upsilon_1 \oplus \upsilon_2) = \gamma(\upsilon_1) \oplus \gamma(\upsilon_2)$

2) $(\upsilon_1 \otimes \upsilon_2)^\gamma = (\upsilon_1)^\gamma \otimes (\upsilon_2)^\gamma$

3) $\gamma_1(\upsilon_1) \oplus \gamma_2(\upsilon_1) = (\gamma_1 \oplus \gamma_2)\upsilon_1$

4) $(\upsilon_1)^{\gamma_1} \otimes (\upsilon_1)^{\gamma_2} = (\upsilon_1)^{\gamma_1 \oplus \gamma_2}$

5) $\left(\upsilon_1^c\right)^\gamma = (\gamma \upsilon_1)^c$

6) $\gamma\left(\upsilon_1^c\right) = \left(\upsilon_1^\gamma\right)^c$

Proof: Let $\upsilon_j = \left(\hbar_j e^{ia_j}, \lambda_j e^{ib_j}, \xi_j e^{in_j}\right) (j = 1, 2, \ldots, n)$ be a family of CPFNs, then, by using Definition 6, we have

$$\gamma\left(\upsilon_1 \oplus \upsilon_2\right)=\gamma\left(\begin{array}{l}\left(1-\prod_{j=1}^{2}\left(1-\hbar_j^q\right)\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\prod_{j=1}^{2}\left(1-\left(\frac{a_j}{2\pi}\right)^q\right)\right)^{\frac{1}{q}}}, \prod_{j=1}^{2}\left(\lambdabar_j\right).e^{i2\pi\left(\prod_{j=1}^{2}\left(\frac{e_j}{2\pi}\right)\right)}, \\ \prod_{j=1}^{2}\left(\xi_j\right).e^{i2\pi\left(\prod_{j=1}^{2}\left(\frac{n_j}{2\pi}\right)\right)}\end{array}\right)$$

$$=\left(\begin{array}{l}\left(1-\prod_{j=1}^{2}\left(1-\hbar_j^q\right)^{\gamma}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\prod_{j=1}^{2}\left(1-\left(\frac{a_j}{2\pi}\right)^q\right)^{\gamma}\right)^{\frac{1}{q}}}, \prod_{j=1}^{2}\left(\lambdabar_j\right)^{\gamma}.e^{i2\pi\left(\prod_{j=1}^{2}\left(\frac{e_j}{2\pi}\right)^{\gamma}\right)}, \\ \prod_{j=1}^{2}\left(\xi_j\right)^{\gamma}.e^{i2\pi\left(\prod_{j=1}^{2}\left(\frac{n_j}{2\pi}\right)^{\gamma}\right)}\end{array}\right)$$

$$=\left(\left(1-\left(1-\hbar_1^q\right)^{\gamma}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\left(1-\left(\frac{a_1}{2\pi}\right)^q\right)^{\gamma}\right)^{\frac{1}{q}}},\left(\lambdabar_1\right)^{\gamma}.e^{i2\pi\left(\frac{e_1}{2\pi}\right)^{\gamma}},\left(\xi_1\right)^{\gamma}.e^{i2\pi\left(\frac{n_1}{2\pi}\right)^{\gamma}}\right)$$

$$+\left(\left(1-\left(1-\hbar_2^q\right)^{\gamma}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\left(1-\left(\frac{a_2}{2\pi}\right)^q\right)^{\gamma}\right)^{\frac{1}{q}}},\left(\lambdabar_2\right)^{\gamma}.e^{i2\pi\left(\frac{e_2}{2\pi}\right)^{\gamma}},\left(\xi_2\right)^{\gamma}.e^{i2\pi\left(\frac{n_2}{2\pi}\right)^{\gamma}}\right)$$

$$=\gamma\left(\upsilon_1\right)\oplus\gamma\left(\upsilon_2\right)$$

Next, we prove 5), whereas, the remaining parts can be proved easily with the help of Definition 6.

$$\left(\upsilon_1^{c}\right)^{\gamma}=\left(\left(\xi_1\right)^{\gamma}.e^{i2\pi\left(\frac{n_1}{2\pi}\right)^{\gamma}},\left(\lambdabar_1\right)^{\gamma}.e^{i2\pi\left(\frac{e_1}{2\pi}\right)^{\gamma}},\left(1-\left(1-\hbar_1^q\right)^{\gamma}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\left(1-\left(\frac{a_1}{2\pi}\right)^q\right)^{\gamma}\right)^{\frac{1}{q}}}\right)$$

$$=\left(\left(1-\left(1-\hbar_1^q\right)^{\gamma}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\left(1-\left(\frac{a_1}{2\pi}\right)^q\right)^{\gamma}\right)^{\frac{1}{q}}},\left(\lambdabar_1\right)^{\gamma}.e^{i2\pi\left(\frac{e_1}{2\pi}\right)^{\gamma}},\left(\xi_1\right)^{\gamma}.e^{i2\pi\left(\frac{n_1}{2\pi}\right)^{\gamma}}\right)^{c}$$

$$=\left(\upsilon_1^{\gamma}\right)^{c}$$

Theorem 3: Let $\upsilon_j=\left(\hbar_j e^{ia_j},\lambda_j e^{ib_j},\xi_j e^{in_j}\right)(j=1,2)$ be a family of CPFNs, and $\gamma>0$, be a real number then the resulting values of $\upsilon_1\oplus\upsilon_2$, $\upsilon_1\otimes\upsilon_2$, υ^{γ} and $\gamma(\upsilon)$ are also CPFNs.

Proof: The required result can be proved easily with a straight forward calculation by using Definition 6. Hence, the proof is omitted here.

Theorem 4: Let $\upsilon_j=\left(\hbar_j e^{ia_j},\lambda_j e^{ib_j},\xi_j e^{in_j}\right)$ be a family of CPFNs, then $\upsilon_1\otimes\upsilon_2\subseteq\upsilon_1\oplus\upsilon_2$.

Proof: Let $\upsilon_j=\left(\hbar_j e^{ia_j},\lambda_j e^{ib_j},\xi_j e^{in_j}\right)(j=1,2)$ are two CPFNs, then by Definition 6, we have:

$$\upsilon_1\oplus\upsilon_2=\left(\begin{array}{l}\left(1-\prod_{j=1}^{2}\left(1-\hbar_j^q\right)\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\prod_{j=1}^{2}\left(1-\left(\frac{a_j}{2\pi}\right)^q\right)\right)^{\frac{1}{q}}},\prod_{j=1}^{2}\left(\lambda_j\right).e^{i2\pi\left(\prod_{j=1}^{2}\left(\frac{e_j}{2\pi}\right)\right)},\\ \prod_{j=1}^{2}\left(\xi_j\right).e^{i2\pi\left(\prod_{j=1}^{2}\left(\frac{n_j}{2\pi}\right)\right)}\end{array}\right)$$

Again, by using Definition 6, we have

$$\upsilon_1\otimes\upsilon_2=\left(\begin{array}{l}\prod_{j=1}^{2}\left(\hbar_j\right).e^{i2\pi\left(\prod_{j=1}^{2}\left(\frac{a_j}{2\pi}\right)\right)},\prod_{j=1}^{2}\left(\lambda_j\right).e^{i2\pi\left(\prod_{j=1}^{2}\left(\frac{b_j}{2\pi}\right)\right)},\\ \left(1-\prod_{j=1}^{2}\left(1-\xi_j^q\right)\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\prod_{j=1}^{2}\left(1-\left(\frac{n_j}{2\pi}\right)^q\right)\right)^{\frac{1}{q}}}\end{array}\right)$$

By using a well-known classical result that arithmetic mean is greater than or equal to geometric mean such that: $\frac{\hbar_1\oplus\hbar_2}{2}\geq\sqrt{\hbar_1\hbar_2}$ hence, we have $\left(1-\prod_{j=1}^{2}\left(1-\hbar_j^q\right)\right)^{\frac{1}{q}}\geq\prod_{j=1}^{2}\hbar_j$. Similarly, $\left(1-\prod_{j=1}^{2}\left(1-\lambda_j^q\right)\right)^{\frac{1}{q}}\geq\prod_{j=1}^{2}\lambda_j$, and $\left(1-\prod_{j=1}^{2}\left(1-\xi_j^q\right)\right)^{\frac{1}{q}}\geq\prod_{j=1}^{2}\xi_j$ $2\pi\left(1-\prod_{j=1}^{2}1-\left(\frac{a_j}{2\pi}\right)^q\right)^{\frac{1}{q}}\geq\prod_{j=1}^{2}\left(\frac{a_j}{2\pi}\right)$ $2\pi\left(1-\prod_{j=1}^{2}1-\left(\frac{b_j}{2\pi}\right)^q\right)^{\frac{1}{q}}\geq\prod_{j=1}^{2}\left(\frac{b_j}{2\pi}\right)$ and $2\pi\left(1-\prod_{j=1}^{2}1-\left(\frac{n_j}{2\pi}\right)^q\right)^{\frac{1}{q}}\geq\prod_{j=1}^{2}\left(\frac{n_j}{2\pi}\right)$. So, we conclude that: $\upsilon_1\otimes\upsilon_2\subseteq\upsilon_1\oplus\upsilon_2$.

Theorem 5: Let $\upsilon=\left(\hbar e^{ia},\lambda e^{ib},\xi e^{in}\right)$ be a CPFN, and a real number $\gamma>0$, then the following conditions hold.

i) $\upsilon^{\gamma}\subseteq\gamma(\upsilon)$ if and only if $\gamma\geq1$

ii) $\gamma(\upsilon)\subseteq(\upsilon)^{\gamma}$ if and only if $0<\gamma\leq1$

Proof: By using Definition 6, we have

$$\gamma \upsilon_1 = \left(\left(1-\left(1-\hbar_1^q\right)^{\gamma}\right)^{\frac{1}{q}} .e^{i2\pi\left(1-\left(1-\left(\frac{a}{2\pi}\right)^q\right)^{\gamma}\right)^{\frac{1}{q}}}, \ (\lambda\!\!\!^{-}_1)^{\gamma} .e^{i2\pi\left(\frac{b}{2\pi}\right)^{\gamma}}, (\xi_1)^{\gamma} .e^{i2\pi\left(\frac{n}{2\pi}\right)^{\gamma}} \right)$$

$$(\upsilon_1)^{\gamma} = \left((\hbar_1)^{\gamma} .e^{i2\pi\left(\frac{a}{2\pi}\right)^{\gamma}}, (\lambda\!\!\!^{-}_1)^{\gamma} .e^{i2\pi\left(\frac{b}{2\pi}\right)^{\gamma}} \left(1-\left(1-\xi_1^q\right)^{\gamma}\right)^{\frac{1}{q}} .e^{i2\pi\left(1-\left(1-\left(\frac{n}{2\pi}\right)^q\right)^{\gamma}\right)^{\frac{1}{q}}} \right)$$

Next, by Theorem 4, we have $\left(1-\left(1-\hbar^q\right)\right)^{\frac{1}{q}} \geq \hbar$, this implies that $\left(1-\left(1-\hbar^q\right)^{\gamma}\right)^{\frac{1}{q}} \geq \hbar^{\gamma}$ for $\gamma \geq 1$. Similarly, we can prove that $2\pi\left(1-\left(1-\left(\frac{a}{2\pi}\right)^q\right)^{\gamma}\right)^{\frac{1}{q}} \geq 2\pi\left(\frac{a}{2\pi}\right)^{\gamma}$, $\left(1-\left(1-\lambda\!\!\!^{-}{}^q\right)^{\gamma}\right)^{\frac{1}{q}} \geq \lambda\!\!\!^{-}{}^{\gamma}$, $\left(1-\left(1-\left(\frac{b}{2\pi}\right)^q\right)^{\gamma}\right)^{\frac{1}{q}} \geq \left(\frac{b}{2\pi}\right)^{\gamma}$, $\left(1-\left(1-\xi^q\right)^{\gamma}\right)^{\frac{1}{q}} \geq \xi^{\gamma}$, $2\pi\left(1-\left(1-\left(\frac{n}{2\pi}\right)^q\right)^{\gamma}\right)^{\frac{1}{q}} \geq 2\pi\left(\frac{n}{2\pi}\right)^{\gamma}$. Thus, we have $\upsilon^{\gamma} \subseteq \gamma(\upsilon)$. Moreover, on similar way, the second part can be proved.

Theorem 6: Let $\upsilon_j = \left(\hbar_j e^{ia_j}, \lambda\!\!\!^{-}_j e^{ib_j}, \xi_j e^{in_j}\right)$ be a family of CPFNs, and a real number $\gamma > 0$, then the following conditions hold.

i. $\upsilon_1 \cup \upsilon_2 = \upsilon_2 \cup \upsilon_1$
ii. $\upsilon_1 \cap \upsilon_2 = \upsilon_2 \cap \upsilon_1$
iii. $\gamma(\upsilon_1 \cap \upsilon_2) = \gamma(\upsilon_1) \cap \gamma(\upsilon_2)$
iv. $\gamma(\upsilon_1 \cup \upsilon_2) = \gamma(\upsilon_1) \cup \gamma(\upsilon_2)$
v. $(\upsilon_1 \cup \upsilon_2)^{\gamma} = (\upsilon_1)^{\gamma} \cup (\upsilon_2)^{\gamma}$
vi. $(\upsilon_1 \cap \upsilon_2)^{\gamma} = (\upsilon_1)^{\gamma} \cap (\upsilon_2)^{\gamma}$

Proof: i. Since $\upsilon_1 = \left(\hbar_1 e^{ia_1}, \lambda\!\!\!^{-}_1 e^{ib_1}, \xi_1 e^{in_1}\right)$ and $\upsilon_2 = \left(\hbar_2 e^{ia_2}, \lambda\!\!\!^{-}_2 e^{ib_2}, \xi_2 e^{in_2}\right)$ are two CPFNs, then by Definition 8, we have:

$$\upsilon_1 \cup \upsilon_2 = \left(\left[\max\{\hbar_1, \hbar_2\} e^{i\left[\max\{a_1,a_2\}\right]}, \max\{\lambda_1, \lambda_2\} e^{i\left[\max\{b_1,b_2\}\right]}, \min\{\xi_1, \xi_2\} e^{i\left[\min\{n_1,n_2\}\right]} \right] \right)$$

$$= \left(\left[\max\{\hbar_2, \hbar_1\} e^{i\left[\max\{a_2,a_1\}\right]}, \max\{\lambda_2, \lambda_1\} e^{i\left[\max\{b_2,b_1\}\right]}, \min\{\xi_2, \xi_1\} e^{i\left[\min\{n_2,n_1\}\right]} \right] \right)$$

$$= \upsilon_2 \cup \upsilon_1$$

Next, we prove result iv. The remaining parts can be prove directly by using Definition 8.

iv. Let $\gamma > 0$, we have:

$$\gamma(\upsilon_1 \cup \upsilon_2) = \gamma\left(\left[\max\{\hbar_1, \hbar_2\} e^{i\left[\max\{a_1,a_2\}\right]}, \max\{\lambda_1, \lambda_2\} e^{i\left[\max\{b_1,b_2\}\right]}, \min\{\xi_1, \xi_2\} e^{i\left[\min\{n_1,n_2\}\right]} \right] \right)$$

$$= \left(\begin{array}{l} \left[\max\left\{ \left(1-\left(1-\hbar_1^q\right)^\gamma\right)^{\frac{1}{q}}, \left(1-\left(1-\hbar_2^q\right)^\gamma\right)^{\frac{1}{q}} \right\} \right] .e^{i\left[\max\left\{ 2\pi\left(1-\left(1-\left(\frac{a_1}{2\pi}\right)^q\right)^\gamma\right)^{\frac{1}{q}},\ 2\pi\left(1-\left(1-\left(\frac{a_2}{2\pi}\right)^q\right)^\gamma\right)^{\frac{1}{q}} \right\}\right]}, \\ \left[\max\left\{ \left(1-\left(1-\lambda_1^q\right)^\gamma\right)^{\frac{1}{q}}, \left(1-\left(1-\lambda_2^q\right)^\gamma\right)^{\frac{1}{q}} \right\} \right] .e^{i\left[\max\left\{ 2\pi\left(1-\left(1-\left(\frac{b_1}{2\pi}\right)^q\right)^\gamma\right)^{\frac{1}{q}},\ 2\pi\left(1-\left(1-\left(\frac{b_2}{2\pi}\right)^q\right)^\gamma\right)^{\frac{1}{q}} \right\}\right]}, \\ \left[\min\left\{ (\xi_1)^\gamma, (\xi_2)^\gamma \right\} \right] .e^{i\left[\min\left\{ 2\pi\left(\frac{n_1}{2\pi}\right)^\gamma, 2\pi\left(\frac{n_2}{2\pi}\right)^\gamma \right\}\right]} \end{array} \right)$$

$$= \left(\begin{array}{l} \left(1-\left(1-\hbar_1^q\right)^\gamma\right)^{\frac{1}{q}} .e^{i2\pi\left(1-\left(1-\left(\frac{a_1}{2\pi}\right)^q\right)^\gamma\right)^{\frac{1}{q}}}, \\ \left(1-\left(1-\lambda_1^q\right)^\gamma\right)^{\frac{1}{q}} .e^{i2\pi\left(1-\left(1-\left(\frac{b_1}{2\pi}\right)^q\right)^\gamma\right)^{\frac{1}{q}}}, \\ (\xi_1)^\gamma .e^{i2\pi\left(\frac{n_1}{2\pi}\right)^\gamma} \end{array} \right) \cup \left(\begin{array}{l} \left(1-\left(1-\hbar_2^q\right)^\gamma\right)^{\frac{1}{q}} .e^{i2\pi\left(1-\left(1-\left(\frac{a_2}{2\pi}\right)^q\right)^\gamma\right)^{\frac{1}{q}}}, \\ \left(1-\left(1-\lambda_2^q\right)^\gamma\right)^{\frac{1}{q}} .e^{i2\pi\left(1-\left(1-\left(\frac{b_2}{2\pi}\right)^q\right)^\gamma\right)^{\frac{1}{q}}}, \\ (\xi_2)^\gamma .e^{i2\pi\left(\frac{n_2}{2\pi}\right)^\gamma} \end{array} \right)$$

$$= \gamma(\upsilon_1) \cup \gamma(\upsilon_2)$$

Theorem 7: Let $\upsilon_j = \left(\hbar_j e^{ia_j}, \lambda_j e^{ib_j}, \xi_j e^{in_j}\right)$ be a family of CPFNs, and a real number $\gamma > 0$, then the following conditions hold.

i. $(\upsilon_1 \cup \upsilon_2)^c = (\upsilon_1)^c \cap (\upsilon_2)^c$

ii. $(\upsilon_1 \cap \upsilon_2)^c = (\upsilon_1)^c \cap (\upsilon_2)^c$

iii. $(\upsilon_1 \oplus \upsilon_2)^c = (\upsilon_1)^c \otimes (\upsilon_2)^c$

iv. $(\upsilon_1 \otimes \upsilon_2)^c = (\upsilon_1)^c \oplus (\upsilon_2)^c$

Proof: We prove part ii, and iv, the remaining parts can be proved by using the same process. i. Since $\upsilon_1 = \left(\hbar_1 e^{ia_1}, \lambda_1 e^{ib_1}, \xi_1 e^{in_1}\right)$, $\upsilon_2 = \left(\hbar_2 e^{ia_2}, \lambda_2 e^{ib_2}, \xi_2 e^{in_2}\right)$ are two CPFNs, then Definition 8, we have

$$(\upsilon_1 \cap \upsilon_2)^c = \left(\left[\max\{\hbar_1, \hbar_2\} e^{i\left[\max\{a_1,a_2\}\right]}, \max\{\lambda_1, \lambda_2\} e^{i\left[\max\{b_1,b_2\}\right]}, \min\{\xi_1, \xi_2\} e^{i\left[\min\{n_1,n_2\}\right]}\right]\right)$$

$$= \left(\hbar_1 e^{ia_1}, \lambda_1 e^{ib_1}, \xi_1 e^{in_1}\right) \cup \left(\hbar_2 e^{ia_2}, \lambda_2 e^{ib_2}, \xi_2 e^{in_2}\right) = (\upsilon_1)^c \cup (\upsilon_2)^c$$

Next, let $\upsilon_1 = \left(\hbar_1 e^{ia_1}, \lambda_1 e^{ib_1}, \xi_1 e^{in_1}\right)$, $\upsilon_2 = \left(\hbar_2 e^{ia_2}, \lambda_2 e^{ib_2}, \xi_2 e^{in_2}\right)$ are two CPFNs, then Definition 6, we have

$$(\upsilon_1 \otimes \upsilon_2)^c = \begin{bmatrix} \left(\hbar_1^q + \hbar_2^q - \hbar_1^q \hbar_2^q\right)^{\frac{1}{q}} . e^{i2\pi\left(\left(\frac{a_1}{2\pi}\right)^q + \left(\frac{a_2}{2\pi}\right)^q - \left(\frac{a_1}{2\pi}\right)^q \left(\frac{a_2}{2\pi}\right)^q\right)^{\frac{1}{q}}}, \\ (\lambda_1 \lambda_2) . e^{i2\pi\left(\frac{b_1}{2\pi}\right)\left(\frac{b_2}{2\pi}\right)}, (\xi_1 \xi_2) . e^{i2\pi\left(\frac{n_1}{2\pi}\right)\left(\frac{n_2}{2\pi}\right)} \end{bmatrix}$$

$$= \begin{bmatrix} \left(1 - \prod_{j=1}^{2}\left(1 - \hbar_j^q\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1 - \prod_{j=1}^{2}\left(1 - \frac{a_j}{2\pi}\right)^q\right)^{\frac{1}{q}}}, (\lambda_j) . e^{i2\pi\left(\prod_{j=1}^{2}\left(\frac{b_j}{2\pi}\right)\right)}, (\xi_j) . e^{i2\pi\left(\prod_{j=1}^{2}\left(\frac{n_j}{2\pi}\right)\right)} \end{bmatrix}$$

$$= \begin{bmatrix} \left(1 - \left(1 - \hbar_1^q\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1 - \left(1 - \frac{a_1}{2\pi}\right)^q\right)^{\frac{1}{q}}}, (\lambda_1) . e^{i2\pi\left(\left(\frac{b_1}{2\pi}\right)\right)}, (\xi_1) . e^{i2\pi\left(\frac{n_1}{2\pi}\right)} \end{bmatrix}$$

$$\oplus \begin{bmatrix} \left(1 - \left(1 - \hbar_2^q\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1 - \left(1 - \frac{a_2}{2\pi}\right)^q\right)^{\frac{1}{q}}}, (\lambda_2) . e^{i2\pi\left(\left(\frac{b_2}{2\pi}\right)\right)}, (\xi_2) . e^{i2\pi\left(\frac{n_2}{2\pi}\right)} \end{bmatrix}$$

$$= (\upsilon_1)^c \oplus (\upsilon_2)^c$$

Theorem 8: Let $\upsilon_j = \left(\hbar_j e^{ia_j}, \lambda_j e^{ib_j}, \xi_j e^{in_j}\right)$ be a family of CPFNs, and a real number, then the following conditions hold:

1) $(\upsilon_1 \cup \upsilon_2) \oplus (\upsilon_1 \cap \upsilon_2) = \upsilon_1 \oplus \upsilon_2$
2) $(\upsilon_1 \cup \upsilon_2) \otimes (\upsilon_1 \cap \upsilon_2) = \upsilon_1 \otimes \upsilon_2$
3) $(\upsilon_1 \cup \upsilon_2) \cap \upsilon_1 = \upsilon_1$
4) $(\upsilon_1 \cap \upsilon_2) \cup \upsilon_1 = \upsilon_1$
5) $(\upsilon_1 \cup \upsilon_2) \cap \upsilon_2 = \upsilon_2$
6) $(\upsilon_1 \cap \upsilon_2) \cup \upsilon_2 = \upsilon_2$

Proof: In the proof, we just prove part 5, the remaining part can be prove easily just by using Definition 8, let $\upsilon_1 = \left(\hbar_1 e^{ia_1}, \lambda_1 e^{ib_1}, \xi_1 e^{in_1}\right)$, $\upsilon_2 = \left(\hbar_2 e^{ia_2}, \lambda_2 e^{ib_2}, \xi_2 e^{in_2}\right)$ are two CPFNs, then we have:

$$(\upsilon_1 \cup \upsilon_2) \cap \upsilon_2 = \begin{pmatrix} \max\{\hbar_1, \hbar_2\} e^{i\left[\max\{a_1, a_2\}\right]}, \max\{\lambda_1, \lambda_2\} e^{i\left[\max\{b_1, b_2\}\right]}, \\ \min\{\xi_1, \xi_2\} e^{i\left[\min\{n_1, n_2\}\right]} \cap \left(\hbar_2 e^{ia_2}, \lambda_2 e^{ib_2}, \xi e^{in_2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \min\{\max\{\hbar_1, \hbar_2\}, \hbar_2\} e^{i\left[\min\{\max\{a_1, a_2\}, a_2\}\right]}, \\ \min\{\max\{\lambda_1, \lambda_2\}, \lambda_2\} e^{i\left[\min\{\max\{b_1, b_2\}, b_2\}\right]}, \\ \max\{\min\{\xi_1, \xi_2\}, \xi_2\} e^{i\left[\max\{\min\{n_1, n_2\}, n_2\}\right]} \end{pmatrix}$$

$$= \left(\hbar_2 e^{ia_2}, \lambda_2 e^{ib_2}, \xi_2 e^{in_2}\right) = \upsilon_2$$

Theorem 9: Let $\upsilon_j = \left(\hbar_j e^{ia_j}, \lambda_j e^{ib_j}, \xi_j e^{in_j}\right)$ be a family of CPFNs, and a real number, then the following conditions hold:

1) $(\upsilon_1 \cup \upsilon_2) \cap \upsilon_3 = (\upsilon_1 \cap \upsilon_3) \cup (\upsilon_2 \cap \upsilon_3)$
2) $(\upsilon_1 \cap \upsilon_2) \cup \upsilon_3 = (\upsilon_1 \cup \upsilon_3) \cap (\upsilon_2 \cup \upsilon_3)$
3) $(\upsilon_1 \cup \upsilon_2) \oplus \upsilon_3 = (\upsilon_1 \oplus \upsilon_3) \cup (\upsilon_2 \oplus \upsilon_3)$
4) $(\upsilon_1 \cap \upsilon_2) \oplus \upsilon_3 = (\upsilon_1 \oplus \upsilon_3) \cap (\upsilon_2 \oplus \upsilon_3)$
5) $(\upsilon_1 \cup \upsilon_2) \otimes \upsilon_3 = (\upsilon_1 \otimes \upsilon_3) \cup (\upsilon_2 \otimes \upsilon_3)$
6) $(\upsilon_1 \cap \upsilon_2) \otimes \upsilon_3 = (\upsilon_1 \otimes \upsilon_3) \cap (\upsilon_2 \otimes \upsilon_3)$

Proof: The proof can easily by using Definition 5 and 8 with a straight forward calculation.

Theorem 10: Let $\upsilon_j = \left(\hbar_j e^{ia_j}, \lambda_j e^{ib_j}, \xi_j e^{in_j}\right)$ be a family of CPFNs, and a real number, then the following hold true:

1) $\upsilon_1 \cup \upsilon_2 \cup \upsilon_3 = \upsilon_1 \cup \upsilon_3 \cup \upsilon_2$
2) $\upsilon_1 \cap \upsilon_2 \cap \upsilon_3 = \upsilon_1 \cap \upsilon_3 \cap \upsilon_2$

3) $\upsilon_1 \oplus \upsilon_2 \oplus \upsilon_3 = \upsilon_1 \oplus \upsilon_3 \oplus \upsilon_2$

4) $\upsilon_1 \otimes \upsilon_2 \otimes \upsilon_3 = \upsilon_1 \otimes \upsilon_3 \otimes \upsilon_2$

Proof: Let three CPFNs be υ_1, υ_2 and υ_3, then according to Definition 8, we get

$$\begin{aligned}
&\upsilon_1 \cup \upsilon_2 \cup \upsilon_3 \\
&= \left(\left[\begin{array}{l} \max\{\hbar_1, \hbar_2, \hbar_3\} e^{i[\max\{a_1,a_2,a_3\}]}, \max\{\lambda_1, \lambda_2, \lambda_3\} e^{i[\max\{b_1,b_2,b_3\}]}, \\ \min\{\xi_1, \xi_2, \xi_3\} e^{i[\min\{n_1,n_2,n_3\}]} \end{array} \right] \right) \\
&= \left(\left[\begin{array}{l} \max\{\hbar_1, \hbar_3, \hbar_2\} e^{i[\max\{a_1,a_3,a_2\}]}, \max\{\lambda_1, \lambda_3, \lambda_2\} e^{i[\max\{b_1,b_3,b_2\}]}, \\ \min\{\xi_1, \xi_3, \xi_2\} e^{i[\min\{n_1,n_3,n_2\}]} \end{array} \right] \right) \\
&= \upsilon_1 \cup \upsilon_3 \cup \upsilon_2
\end{aligned}$$

Next, we prove part 3, the remaining parts can be prove easily by using Definition 5 and 8.

3) By using Definition 5, we have

$$\begin{aligned}
&\upsilon_1 \oplus \upsilon_2 \oplus \upsilon_3 \\
&= \left(\begin{array}{l} \left(1 - \prod_{j=1}^{3}\left(1 - \hbar_j^q\right)\right)^{\frac{1}{q}} . e^{i2\pi\left(1 - \prod_{j=1}^{3}\left(1 - \left(\frac{a_j}{2\pi}\right)^q\right)\right)^{\frac{1}{q}}}, \prod_{j=1}^{3}\left(\lambda_j\right) . e^{i2\pi\left(\prod_{j=1}^{3}\left(\frac{e_j}{2\pi}\right)\right)}, \\ \prod_{j=1}^{3}\left(\xi_j\right) . e^{i2\pi\left(\prod_{j=1}^{3}\left(\frac{n_j}{2\pi}\right)\right)} \end{array} \right) \\
&= \left(\left(1 - \left(1 - \hbar_1^q\right)\right)^{\frac{1}{q}} . e^{i2\pi\left(1 - \left(1 - \left(\frac{a_1}{2\pi}\right)\right)^q\right)^{\frac{1}{q}}}, \left(\lambda_1\right) . e^{i2\pi\left(\left(\frac{e_1}{2\pi}\right)\right)}, \left(\xi_1\right) . e^{i2\pi\left(\left(\frac{n_1}{2\pi}\right)\right)} \right) \oplus \\
&\left(\left(1 - \left(1 - \hbar_3^q\right)\right)^{\frac{1}{q}} . e^{i2\pi\left(1 - \left(1 - \left(\frac{a_3}{2\pi}\right)\right)^q\right)^{\frac{1}{q}}}, \left(\lambda_3\right) . e^{i2\pi\left(\left(\frac{e_3}{2\pi}\right)\right)}, \left(\xi_3\right) . e^{i2\pi\left(\left(\frac{n_3}{2\pi}\right)\right)} \right) \oplus \\
&\left(\left(1 - \left(1 - \hbar_2^q\right)\right)^{\frac{1}{q}} . e^{i2\pi\left(1 - \left(1 - \left(\frac{a_2}{2\pi}\right)\right)^q\right)^{\frac{1}{q}}}, \left(\lambda_2\right) . e^{i2\pi\left(\left(\frac{e_2}{2\pi}\right)\right)}, \left(\xi_2\right) . e^{i2\pi\left(\left(\frac{n_2}{2\pi}\right)\right)} \right) \\
&= \upsilon_1 \oplus \upsilon_3 \oplus \upsilon_2
\end{aligned}$$

4. Complex Polytopic fuzzy geometric operators

This section introduces the concept of several complex aggregation operators, including the I-CPFOWG operator, I-CPFHG operator, CPFWG operator, CPFOWG operator, and CPFHG operator, as well as its structural qualities, namely idempotency, boundedness, and monotonicity.

Definition 10: Let $\upsilon_j = \left(\hbar_j e^{ia_j}, \lambda_j e^{ib_j}, \xi_j e^{in_j}\right)(1 \le j \le n)$, a family of CPF numbers with weighted vector $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$, such that $(1 \le \varpi_j \le n)$ and $\sum_{j=1}^{n} \varpi_j = 1$ .Then the complex Polytopic fuzzy weighted geometric aggregation operator is mathematically given by:

$$\text{CPFWG}_{\varpi}(\upsilon_1, \upsilon_2, ..., \upsilon_n) = \left(\begin{array}{c} \prod_{j=1}^{n} (\hbar_j)^{\varpi_j} . e^{i2\pi \prod_{j=1}^{n} \left(\frac{a_j}{2\pi}\right)^{\varpi_j}}, \prod_{j=1}^{n} (\lambda_j)^{\varpi_j} . e^{i2\pi \prod_{j=1}^{n} \left(\frac{b_j}{2\pi}\right)^{\varpi_j}}, \\ \left(1 - \prod_{j=1}^{n} \left(1 - \xi_j^q\right)^{\varpi_j}\right)^{\frac{1}{q}} . e^{i2\pi \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{n_j}{2\pi}\right)^q\right)^{\varpi_j}\right)^{\frac{1}{q}}} \end{array} \right) \tag{1}$$

Theorem 11: Let $\upsilon_j = \left(\hbar_j e^{ia_j}, \lambda_j e^{ib_j}, \xi_j e^{in_j}\right)(1 \le j \le n)$, be a group of CPFVs, then their resulting value under CPFWG operator remains a CPFV, such that

$$\text{CPFWG}_{\varpi}(\upsilon_1, \upsilon_2, ..., \upsilon_n) = \left(\begin{array}{c} \prod_{j=1}^{n} (\hbar_j)^{\varpi_j} . e^{i2\pi \prod_{j=1}^{n} \left(\frac{a_j}{2\pi}\right)^{\varpi_j}}, \prod_{j=1}^{n} (\lambda_j)^{\varpi_j} . e^{i2\pi \prod_{j=1}^{n} \left(\frac{b_j}{2\pi}\right)^{\varpi_j}}, \\ \left(1 - \prod_{j=1}^{n} \left(1 - \xi_j^q\right)^{\varpi_j}\right)^{\frac{1}{q}} . e^{i2\pi \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{n_j}{2\pi}\right)^q\right)^{\varpi_j}\right)^{\frac{1}{q}}} \end{array} \right) \tag{2}$$

Proof: The required result can be proved by mathematical induction principle, the major steps are as follows:

Step 01: As $\upsilon_j = \left(\hbar_j e^{ia_j}, \lambda_j e^{ib_j}, \xi_j e^{in_j}\right)(1 \le j \le n)$, then for $n = 2$, we have

$$(\upsilon_1)^{\varpi_1} = \left((\hbar_1)^{\varpi_1} . e^{i2\pi \left(\frac{a_1}{2\pi}\right)^{\varpi_1}}, (\lambda_1)^{\varpi_1} . e^{i2\pi \left(\frac{b_1}{2\pi}\right)^{\varpi_1}}, \left(1 - \left(1 - \xi_1^q\right)^{\varpi_1}\right)^{\frac{1}{q}} . e^{i2\pi \left(1 - \left(1 - \left(\frac{n_1}{2\pi}\right)^q\right)^{\varpi_1}\right)^{\frac{1}{q}}} \right)$$

$$\left(\upsilon_2\right)^{\varpi_2}=\left(\left(\hbar_2\right)^{\varpi_2}.e^{i2\pi\left(\frac{a_2}{2\pi}\right)^{\varpi_2}},\left(\lambda_2\right)^{\varpi_2}.e^{i2\pi\left(\frac{b_2}{2\pi}\right)^{\varpi_2}},\left(1-\left(1-\xi_2^q\right)^{\varpi_2}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\left(1-\left(\frac{n_2}{2\pi}\right)^q\right)^{\varpi_2}\right)^{\frac{1}{q}}}\right)$$

Next, by using Definition 10, we have

$$\text{CPFWG}_{\varpi}\left(\upsilon_1,\upsilon_2\right)=\left(\begin{array}{l}\prod_{j=1}^{2}\left(\hbar_j\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{2}\left(\frac{a_j}{2\pi}\right)^{\varpi_j}},\prod_{j=1}^{2}\left(\lambda_j\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{2}\left(\frac{b_j}{2\pi}\right)^{\varpi_j}},\\ \left(1-\prod_{j=1}^{2}\left(1-\xi_j^q\right)^{\varpi_j}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\prod_{j=1}^{2}\left(1-\left(\frac{n_j}{2\pi}\right)^q\right)^{\varpi_j}\right)^{\frac{1}{q}}}\end{array}\right)$$

Step 02: In step 01, it is proved for $n=2$. Next, suppose that Eq. (2) holds for $n=k$, $k>0$, then it follows that

$$\text{CPFWG}_{\varpi}\left(\upsilon_1,\upsilon_2,...,\upsilon_k\right)=\left(\begin{array}{l}\prod_{j=1}^{k}\left(\hbar_j\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{k}\left(\frac{a_j}{2\pi}\right)^{\varpi_j}},\prod_{j=1}^{k}\left(\lambda_j\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{k}\left(\frac{b_j}{2\pi}\right)^{\varpi_j}},\\ \left(1-\prod_{j=1}^{k}\left(1-\xi_j^q\right)^{\varpi_j}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{n_j}{2\pi}\right)^q\right)^{\varpi_j}\right)^{\frac{1}{q}}}\end{array}\right)$$

Step 03: Next, for $n=k+1$

$$\text{CPFWG}_{\varpi}\left(\upsilon_1,\upsilon_2,...,\upsilon_{k+1}\right)$$

$$=\left(\begin{array}{l}\prod_{j=1}^{k}\left(\hbar_j\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{k}\left(\frac{a_j}{2\pi}\right)^{\varpi_j}},\prod_{j=1}^{k}\left(\lambda_j\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{k}\left(\frac{b_j}{2\pi}\right)^{\varpi_j}},\\ \left(1-\prod_{j=1}^{k}\left(1-\xi_j^q\right)^{\varpi_j}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\prod_{j=1}^{k}\left(1-\left(\frac{n_j}{2\pi}\right)^q\right)^{\varpi_j}\right)^{\frac{1}{q}}}\end{array}\right)$$

$$\otimes\left(\begin{array}{l}\left(\hbar_{k+1}\right)^{\varpi_{k+1}}.e^{i2\pi\left(\frac{a_{k+1}}{2\pi}\right)^{\varpi_{k+1}}},\left(\lambda_{k+1}\right)^{\varpi_{k+1}}.e^{i2\pi\left(\frac{b_{k+1}}{2\pi}\right)^{\varpi_{k+1}}},\\ \left(1-\left(1-\xi_{k+1}^q\right)^{\varpi_{k+1}}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\left(1-\left(\frac{n_{k+1}}{2\pi}\right)^q\right)^{\varpi_{k+1}}\right)^{\frac{1}{q}}}\end{array}\right)$$

$$=\left(\begin{array}{c}\prod_{j=1}^{k+1}\left(\hbar_j\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{k+1}\left(\frac{a_j}{2\pi}\right)^{\varpi_j}},\prod_{j=1}^{k+1}\left(\lambda_j\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{k+1}\left(\frac{b_j}{2\pi}\right)^{\varpi_j}},\\ \left(1-\prod_{j=1}^{k+1}\left(1-\xi_j^q\right)^{\varpi_j}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\prod_{j=1}^{k+1}\left(1-\left(\frac{n_j}{2\pi}\right)^q\right)^{\varpi_j}\right)^{\frac{1}{q}}}\end{array}\right)$$

Hence, it holds for $n=k+1$. As a result, Eq. (2) holds for all positive integers according to the principle of mathematical induction. So, the proof is completed.

Property 1 (Idempotency): Let $\upsilon_j=\left(\hbar_j e^{ia_j},\lambda_j e^{ib_j},\xi_j e^{in_j}\right)(1\le j\le n)$, be a family of CPFNs, and let $\upsilon^*=\left(\hbar^* e^{ia^*},\lambda^* e^{ib^*},\xi^* e^{in^*}\right)$ be another CPFN satisfying $\upsilon_j=\upsilon^*$, then

$$\mathrm{CPFWG}_{\varpi}\left(\upsilon_1,\upsilon_2,\ldots,\upsilon_n\right)=\left(\begin{array}{c}\prod_{j=1}^{n}\left(\hbar_j\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{n}\left(\frac{a_j}{2\pi}\right)^{\varpi_j}},\prod_{j=1}^{n}\left(\lambda_j\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{n}\left(\frac{b_j}{2\pi}\right)^{\varpi_j}},\\ \left(1-\prod_{j=1}^{n}\left(1-\xi_j^q\right)^{\varpi_j}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{n_j}{2\pi}\right)^q\right)^{\varpi_j}\right)^{\frac{1}{q}}}\end{array}\right)$$

Property II (Boundedness): Let $\upsilon_j=\left(\hbar_j e^{ia_j},\lambda_j e^{ib_j},\xi_j e^{in_j}\right)(1\le j\le n)$, be a family of CPFNs, and let $\upsilon_{\max}=\left(\hbar_{\max}e^{ia_{\max}},\lambda_{\max}e^{ib_{\max}},\xi_{\max}e^{in_{\max}}\right)$, with $\hbar_{\max}=\max_j\left\{\hbar_j\right\}$, $a_{\max}=\max_j\left\{a_j\right\}$, $\lambda_{\max}=\max_j\left\{\lambda_j\right\}$, $b_{\max}=\max_j\left\{b_j\right\}$ $\xi_{\max}=\max_j\left\{\xi_j\right\}$, $n_{\max}=\max_j\left\{n_j\right\}$, $\upsilon_{\min}=\left(\hbar_{\min}e^{ia_{\min}},\lambda_{\min}e^{ib_{\min}},\xi_{\min}e^{in_{\min}}\right)$, with $\hbar_{\min}=\min_j\left\{\hbar_j\right\}$, $a_{\min}=\min_j\left\{a_j\right\}$, $\lambda_{\min}=\min_j\left\{\lambda_j\right\}$, $b_{\min}=\min_j\left\{b_j\right\}$ $\xi_{\min}=\min_j\left\{\xi_j\right\}$, $n_{\min}=\min_j\left\{n_j\right\}$ then, we have

$$\upsilon_{\min}\le\mathrm{CPFWG}_{\varpi}\left(\upsilon_1,\upsilon_2,\ldots,\upsilon_n\right)\le\upsilon_{\max}\qquad(3)$$

Proof: Let $\upsilon_j=\left(\hbar_j e^{ia_j},\lambda_j e^{ib_j},\xi_j e^{in_j}\right)(1\le j\le n)$, then for υ_j, we have

$$\Leftrightarrow\left(\left(\min_j\left\{\hbar_{\min}\right\}\right)^q\right)^{\frac{1}{q}}\le\left(\hbar_j^q\right)^{\frac{1}{q}}\le\left(\left(\max_j\left\{\hbar_{\max}\right\}\right)^q\right)^{\frac{1}{q}}$$

$$\Leftrightarrow\left(1-\left(\max_j\left\{\hbar_{\max}\right\}\right)^q\right)^{\frac{1}{q}}\le\left(1-\hbar_j^q\right)^{\frac{1}{q}}\le\left(1-\left(\min_j\left\{\hbar_{\min}\right\}\right)^q\right)^{\frac{1}{q}}$$

$$\Leftrightarrow\left(\prod_{j=1}^{n}\left(1-\left(\max_j\left\{\hbar_{\max}\right\}\right)^q\right)^{\varpi_j}\right)^{\frac{1}{q}}\le\left(\prod_{j=1}^{n}\left(1-\hbar_j^q\right)^{\varpi_j}\right)^{\frac{1}{q}}\le\left(\prod_{j=1}^{n}\left(1-\left(\min_j\left\{\hbar_{\min}\right\}\right)^q\right)^{\varpi_j}\right)^{\frac{1}{q}}$$

$$\Leftrightarrow\left(\left(\min_j\{\hbar_{\min}\}\right)^q\right)^{\frac{1}{q}} \le \left(1-\prod_{j=1}^{n}\left(1-\hbar_j^q\right)^{\varpi_j}\right)^{\frac{1}{q}} \le \left(\left(\max_j\{\hbar_{\max}\}\right)^q\right)^{\frac{1}{q}}$$

$$\min_j\{\hbar_{\min}\} \le \left(1-\prod_{j=1}^{n}\left(1-\hbar_j^q\right)^{\varpi_j}\right)^{\frac{1}{q}} \le \max_j\{\hbar_{\max}\}$$

Moreover, we have $\min_j\{a_j\} \le a_j \le \max_j\{a_j\}$. Similarly, for neutral membership, we have

$$\Leftrightarrow \min_j\{\lambda_j\} \le \lambda_j \le \max_j\{\lambda_j\} \Leftrightarrow \prod_{j=1}^{n}\left(\min_j\{\lambda_j\}\right)^{\varpi_j} \le \prod_{j=1}^{n}\left(\lambda_j\right)^{\varpi_j} \le \prod_{j=1}^{n}\left(\max_j\{\lambda_j\}\right)^{\varpi_j}$$

$$\Leftrightarrow\left(\min_j\{\lambda_j\}\right)^{\sum_{j=1}^{n}\varpi_j} \le \prod_{j=1}^{n}\left(\lambda_j\right)^{\varpi_j} \le \left(\max_j\{\lambda_j\}\right)^{\sum_{j=1}^{n}\varpi_j} \Leftrightarrow \min_j\{\lambda_j\} \le \prod_{j=1}^{n}\left(\lambda_j\right)^{\varpi_j} \le \max_j\{\lambda_j\}$$

Thus, on the same way $\min_j\{b_j\} \le b_j \le \max_j\{b_j\}$,

$$\Leftrightarrow \min_j\{\xi_j\} \le \xi_j \le \max_j\{\xi_j\} \Leftrightarrow \prod_{j=1}^{n}\left(\min_j\{\xi_j\}\right)^{\varpi_j} \le \prod_{j=1}^{n}\left(\xi_j\right)^{\varpi_j} \le \prod_{j=1}^{n}\left(\max_j\{\xi_j\}\right)^{\varpi_j}$$

$$\Leftrightarrow\left(\min_j\{\xi_j\}\right)^{\sum_{j=1}^{n}\varpi_j} \le \prod_{j=1}^{n}\left(\xi_j\right)^{\varpi_j} \le \left(\max_j\{\xi_j\}\right)^{\sum_{j=1}^{n}\varpi_j} \Leftrightarrow \min_j\{\xi_j\} \le \prod_{j=1}^{n}\left(\xi_j\right)^{\varpi_j} \le \max_j\{\xi_j\}$$

Also, we have that $\min_j\{n_j\} \le n_j \le \max_j\{n_j\}$. The above expression, yields that $\upsilon_{\min} \le \text{CPFWG}_{\varpi}(\upsilon_1,\upsilon_2,\upsilon_3,\ldots,\upsilon_n) \le \upsilon_{\max}$. Hence, the proof is completed.

Property III (Monotonicity): Let the two CPFNs families be such that $\upsilon_j=\left(\hbar_j e^{ia_j},\lambda_j e^{ib_j},\xi_j e^{in_j}\right)(1\le j\le n)$ and $\upsilon_j^*=\left(\hbar_j^* e^{ia_j^*},\lambda_j^* e^{ib_j^*},\xi_j^* e^{in_j^*}\right)(1\le j\le n)$ be satisfying the conditions, $\hbar_j \le \hbar_j^*$, $a_j \le a_j^*$, $\lambda_j \le \lambda_j^*$, $b_j \le b_j^*$ $\xi_j^* \le \xi_j$, $n_j^* \le n_j$, then the following hold:

$$\text{CPFWG}_{\varpi}(\upsilon_1,\upsilon_2,\upsilon_3,\ldots,\upsilon_n) \le \text{CPFWG}_{\varpi}\left(\upsilon_1^*,\upsilon_2^*,\upsilon_1^*,\ldots,\upsilon_n^*\right) \tag{4}$$

Proof: To avoid the repetition again and again its proof is omitted.

Definition 11: Let $\upsilon_j=\left(\hbar_j e^{ia_j},\lambda_j e^{ib_j},\xi_j e^{in_j}\right)(1\le j\le n)$, be a family of CPFNs with its weighted vector $\varpi=(\varpi_1,\varpi_2,\ldots,\varpi_n)^T$ satisfying $(1\le\varpi_j\le n)$ and $\sum_{j=1}^{n}\varpi_j=1$. Then the complex Polytopic fuzzy ordered weighted geometric aggregation operator is given by:

$$\text{CPFOWG}_{\varpi}(\upsilon_1,\upsilon_2,\upsilon_3,\ldots,\upsilon_n)$$

$$=\begin{pmatrix} \prod_{j=1}^{n}\left(\hbar_{\propto(j)}\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{n}\left(\frac{a_{\propto(j)}}{2\pi}\right)^{\varpi_j}}, \prod_{j=1}^{n}\left(\lambda_{\propto(j)}\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{n}\left(\frac{b_{\propto(j)}}{2\pi}\right)^{\varpi_j}}, \\ \left(1-\prod_{j=1}^{n}\left(1-\left(\xi_{\propto(j)}\right)^q\right)^{\varpi_j}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{n_{\propto(j)}}{2\pi}\right)^q\right)^{\varpi_j}\right)^{\frac{1}{q}}} \end{pmatrix} \quad (5)$$

with $\left(\propto(1),\propto(2),...,\propto(n)\right)$, be arranged such that $\upsilon_{\propto(j-1)}\geq\upsilon_{\propto(j)}$.

Theorem 12: Let $\upsilon_j=\left(\hbar_j e^{ia_j},\lambda_j e^{ib_j},\xi_j e^{in_j}\right)(1\leq j\leq n)$, be a family of CPFVs, for which their output under the CPFOWG operator remains a CPFV.

Proof: Required result can be proved as Theorem 11.

Definition 12: The complex Polytopic fuzzy hybrid geometric aggregation operator is mathematically given by:

$$\text{CPFHG}_{\Xi,\varpi}\left(\upsilon_1,\upsilon_2,\upsilon_3,...,\upsilon_n\right)=$$

$$\begin{pmatrix} \prod_{j=1}^{n}\left(\hbar_{\upsilon_{\propto(j)}}\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{n}\left(\frac{a_{\upsilon_{\propto(j)}}}{2\pi}\right)^{\varpi_j}}, \prod_{j=1}^{n}\left(\lambda_{\upsilon_{\propto(j)}}\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{n}\left(\frac{b_{\upsilon_{\propto(j)}}}{2\pi}\right)^{\varpi_j}} \\ ,\left(1-\prod_{j=1}^{n}\left(1-\left(\xi_{\upsilon_{\propto(j)}}\right)^q\right)^{\varpi_j}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{n_{\upsilon_{\propto(j)}}}{2\pi}\right)^q\right)^{\varpi_j}\right)^{\frac{1}{q}}} \end{pmatrix} \quad (6)$$

Where, $\upsilon_j=\left(\hbar_j e^{ia_j},\lambda_j e^{ib_j},\xi_j e^{in_j}\right)(1\leq j\leq n)$, be a family of CPFNs, with weighted vector $\varpi=(\varpi_1,\varpi_2,...,\varpi_n)^T$ and associated vector $\Xi=(\Xi_1,\Xi_2,...,\Xi_n)^T$ satisfying the additional conditions $\left(1\leq\varpi_j\leq n\right)$, $\sum_{j=1}^{n}\varpi_j=1$, $(0\leq\Xi\leq1)$ and $\sum_{j=1}^{n}\Xi_j=1$. Moreover, $\upsilon_{\propto(j)}=\left(\upsilon_j\right)^{n\Xi_j}$, and n be the balancing coefficient. Let the weighted vector $\varpi=(\varpi_1,\varpi_2,...,\varpi_n)^T$ approaches to $\left(\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right)^T$, then $\left((\upsilon_1)^{n\Xi_1},(\upsilon_2)^{n\Xi_2},...,(\upsilon_n)^{n\Xi_n}\right)^T$ approaches to $(\upsilon_1,\upsilon_2,...,\upsilon_n)^T$.

Definition 13: Let a family $\langle\Im_j,\upsilon_j\rangle(1\leq j\leq n)$ of 2-tuple, along with their weighted vector $\varpi=(\varpi_1,\varpi_2,...,\varpi_n)^T$ such that $\left(1\leq\varpi_j\leq n\right)$ and $\sum_{j=1}^{n}\varpi_j=1$. Then the induced complex Polytopic fuzzy ordered weighted geometric aggregation operator can be stated as:

$$\text{I-CPFOWG}_{\varpi}\left(\langle \Im_1,\upsilon_1\rangle,\langle \Im_2,\upsilon_2\rangle,\ldots,\langle \Im_2,\upsilon_2\rangle\right)=$$

$$\left(\begin{array}{c}\prod_{j=1}^{n}\left(\hbar_{\propto(j)}\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{n}\left(\frac{a_{\propto(j)}}{2\pi}\right)^{\varpi_j}},\prod_{j=1}^{n}\left(\lambda_{\propto(j)}\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{n}\left(\frac{b_{\propto(j)}}{2\pi}\right)^{\varpi_j}},\\ \left(1-\prod_{j=1}^{n}\left(1-\left(\xi_{\propto(j)}\right)^{q}\right)^{\varpi_j}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{n_{\propto(j)}}{2\pi}\right)^{q}\right)^{\varpi_j}\right)^{\frac{1}{q}}}\end{array}\right) \tag{7}$$

where $\langle \Im_j,\upsilon_j\rangle$ be the pair of CPFOWG having the j[th] largest value is known as the order -inducing variable and υ_j as the complex Polytopic fuzzy argument.

Definition 14: Let $\langle \Im_j,\upsilon_j\rangle(1\le j\le n)$ be a 2-tuple family, along with their weighted vector $\varpi=(\varpi_1,\varpi_2,\ldots,\varpi_n)^T$ and $\Xi=(\Xi_1,\Xi_2,\ldots,\Xi_n)^T$ satisfying the conditions $(1\le\varpi_j\le n)$, $\sum_{j=1}^{n}\varpi_j=1$, $(0\le\Xi\le1)$ with $\sum_{j=1}^{n}\Xi_j=1$. Following that, the mathematical expression for the induced complex Polytopic fuzzy hybrid geometric aggregation operator is:

$$\text{I-CPFHG}_{\Xi,\varpi}\left(\langle \Im_1,\upsilon_1\rangle,\langle \Im_2,\upsilon_2\rangle,\ldots,\langle \Im_n,\upsilon_n\rangle\right)=$$

$$\left(\begin{array}{c}\prod_{j=1}^{n}\left(\hbar_{\upsilon_{\propto(j)}}\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{n}\left(\frac{a_{\upsilon_{\propto(j)}}}{2\pi}\right)^{\varpi_j}},\prod_{j=1}^{n}\left(\lambda_{\upsilon_{\propto(j)}}\right)^{\varpi_j}.e^{i2\pi\prod_{j=1}^{n}\left(\frac{b_{\upsilon_{\propto(j)}}}{2\pi}\right)^{\varpi_j}},\\ \left(1-\prod_{j=1}^{n}\left(1-\left(\xi_{\upsilon_{\propto(j)}}\right)^{q}\right)^{\varpi_j}\right)^{\frac{1}{q}}.e^{i2\pi\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{n_{\upsilon_{\propto(j)}}}{2\pi}\right)^{q}\right)^{\varpi_j}\right)^{\frac{1}{q}}}\end{array}\right) \tag{8}$$

where $\upsilon_{\propto(j)}=\left(\upsilon_j\right)^{n\Xi_j}$, with n being the balancing coefficient. If $\varpi=(\varpi_1,\varpi_2,\ldots,\varpi_n)^T$ approaches to $\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^T$, then $\left((\upsilon_1)^{n\Xi_1},(\upsilon_2)^{n\Xi_2},\ldots,(\upsilon_n)^{n\Xi_n}\right)^T$ approaches to $(\upsilon_1,\upsilon_2,\ldots,\upsilon_n)^T$.

5. An application of the proposed aggregation operators

The CPFWG operator, CPFOWG operator, CPFHG operator, I-CPFOWG operator, and I-CPFHG operator are some of the novel techniques we design for decision-making in this section.

Algorithm: Let $A=\{A_1,A_2,\ldots,A_m\}$ be a fixed set of $\boldsymbol{m}$ alternatives and $\aleph=\{\aleph_1,\aleph_2,\ldots,\aleph_n\}$ be a fixed set of $\boldsymbol{n}$ criteria whose weighted vector is $\varpi=(\varpi_1,\varpi_2,\ldots,\varpi_n)^T$ with restriction, such as $(1\le\varpi_j\le n)$ and $\sum_{j=1}^{n}\varpi_j=1$. Let $E=\{E_1,E_2,\ldots,E_k\}$ be a set of k experts/decision makers whose weight is $\exists=(\exists_1,\exists_2,\ldots,\exists_k)^T$ with restriction $(1\le\exists_j\le n)$ and $\sum_{j=1}^{k}\exists_j=1$. To find the suitable option, we develop a MAGDM under the CPF environment. The main steps are as follows:

Step 1: Develop matrices based on the expertise of experts.

Step 2: Make a single matrix out of all the separate matrices by combining them using the specified operators.

Step 3: Again compute all of the preference values using the specified techniques.

Step 4: Calculating the scores uses all preference values.

Step 5: Choose the one with the highest score value.

6. Illustrative Example

Consider a businessman intends to invest his money in a certain business. After a very careful consideration, he choosing the four alternatives, such as, A_1: Medicine Company, A_2: Mobile Company, A_3: Cement Company and A_4: Cloth Company. The businessman will decide based on the following five attributes, namely, $\aleph_1$: Growth Analysis, $\aleph_2$: Environmental analysis, $\aleph_3$: Corporate Reputation, $\aleph_4$: Economic Benefit and $\aleph_5$: Enterprise Management level, whose weighted vector is $\varpi = (0.30, 0.10, 0.30, 0.10, 0.20)^T$. Furthermore, there are four decision makers $E_k\ (k = 1,2,3,4)$ for decision with weighted vector $\exists = (0.20, 0.10, 0.40, 0.30)^T$ and let $q = 4$.

6.1 By Algebraic Aggregation Operators:

Step 1: Decision matrices can be constructed on the expert's ideas as follows:

Table 1: Decision matrix of E_1

	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$
A_1	$\begin{pmatrix} 0.90e^{i2\pi(0.5)}, \\ 0.60e^{i2\pi(0.7)}, \\ 0.70e^{i2\pi(0.8)} \end{pmatrix}$	$\begin{pmatrix} 0.80e^{i2\pi(0.6)}, \\ 0.70e^{i2\pi(0.5)}, \\ 0.60e^{i2\pi(0.7)} \end{pmatrix}$	$\begin{pmatrix} 0.70e^{i2\pi(0.5)}, \\ 0.90e^{i2\pi(0.4)}, \\ 0.50e^{i2\pi(0.8)} \end{pmatrix}$	$\begin{pmatrix} 0.90e^{i2\pi(0.5)}, \\ 0.60e^{i2\pi(0.6)}, \\ 0.80e^{i2\pi(0.9)} \end{pmatrix}$	$\begin{pmatrix} 0.70e^{i2\pi(0.5)}, \\ 0.90e^{i2\pi(0.6)}, \\ 0.60e^{i2\pi(0.7)} \end{pmatrix}$
A_2	$\begin{pmatrix} 0.80e^{i2\pi(0.6)}, \\ 0.70e^{i2\pi(0.4)}, \\ 0.60e^{i2\pi(0.5)} \end{pmatrix}$	$\begin{pmatrix} 0.50e^{i2\pi(0.6)}, \\ 0.90e^{i2\pi(0.7)}, \\ 0.60e^{i2\pi(0.7)} \end{pmatrix}$	$\begin{pmatrix} 0.80e^{i2\pi(0.6)}, \\ 0.60e^{i2\pi(0.7)}, \\ 0.80e^{i2\pi(0.5)} \end{pmatrix}$	$\begin{pmatrix} 0.70e^{i2\pi(0.6)}, \\ 0.50e^{i2\pi(0.4)}, \\ 0.80e^{i2\pi(0.9)} \end{pmatrix}$	$\begin{pmatrix} 0.70e^{i2\pi(0.8)}, \\ 0.80e^{i2\pi(0.5)}, \\ 0.90e^{i2\pi(0.6)} \end{pmatrix}$
A_3	$\begin{pmatrix} 0.70e^{i2\pi(0.5)}, \\ 0.80e^{i2\pi(0.6)}, \\ 0.90e^{i2\pi(0.5)} \end{pmatrix}$	$\begin{pmatrix} 0.90e^{i2\pi(0.7)}, \\ 0.60e^{i2\pi(0.5)}, \\ 0.50e^{i2\pi(0.9)} \end{pmatrix}$	$\begin{pmatrix} 0.70e^{i2\pi(0.4)}, \\ 0.60e^{i2\pi(0.5)}, \\ 0.90e^{i2\pi(0.6)} \end{pmatrix}$	$\begin{pmatrix} 0.70e^{i2\pi(0.6)}, \\ 0.60e^{i2\pi(0.5)}, \\ 0.80e^{i2\pi(0.4)} \end{pmatrix}$	$\begin{pmatrix} 0.70e^{i2\pi(0.9)}, \\ 0.50e^{i2\pi(0.4)}, \\ 0.90e^{i2\pi(0.5)} \end{pmatrix}$
A_4	$\begin{pmatrix} 0.80e^{i2\pi(0.4)}, \\ 0.70e^{i2\pi(0.7)}, \\ 0.80e^{i2\pi(0.9)} \end{pmatrix}$	$\begin{pmatrix} 0.80e^{i2\pi(0.6)}, \\ 0.50e^{i2\pi(0.7)}, \\ 0.60e^{i2\pi(0.7)} \end{pmatrix}$	$\begin{pmatrix} 0.80e^{i2\pi(0.4)}, \\ 0.90e^{i2\pi(0.6)}, \\ 0.40e^{i2\pi(0.8)} \end{pmatrix}$	$\begin{pmatrix} 0.90e^{i2\pi(0.7)}, \\ 0.70e^{i2\pi(0.4)}, \\ 0.60e^{i2\pi(0.5)} \end{pmatrix}$	$\begin{pmatrix} 0.90e^{i2\pi(0.7)}, \\ 0.50e^{i2\pi(0.4)}, \\ 0.80e^{i2\pi(0.5)} \end{pmatrix}$

Table 2: Decision matrix of E_2

	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$
A_1	$\left(0.59e^{i2\pi(0.7)}, 0.71e^{i2\pi(0.5)}, 0.81e^{i2\pi(0.6)}\right)$	$\left(0.59e^{i2\pi(0.8)}, 0.79e^{i2\pi(0.7)}, 0.81e^{i2\pi(0.6)}\right)$	$\left(0.61e^{i2\pi(0.3)}, 0.91e^{i2\pi(0.6)}, 0.81e^{i2\pi(0.7)}\right)$	$\left(0.79e^{i2\pi(0.7)}, 0.81e^{i2\pi(0.5)}, 0.89e^{i2\pi(0.8)}\right)$	$\left(0.89e^{i2\pi(0.7)}, 0.61e^{i2\pi(0.6)}, 0.69e^{i2\pi(0.4)}\right)$
A_2	$\left(0.49e^{i2\pi(0.7)}, 0.91e^{i2\pi(0.6)}, 0.79e^{i2\pi(0.4)}\right)$	$\left(0.51e^{i2\pi(0.9)}, 0.79e^{i2\pi(0.5)}, 0.69e^{i2\pi(0.4)}\right)$	$\left(0.68e^{i2\pi(0.5)}, 0.91e^{i2\pi(0.8)}, 0.49e^{i2\pi(0.8)}\right)$	$\left(0.68e^{i2\pi(0.6)}, 0.51e^{i2\pi(0.4)}, 0.81e^{i2\pi(0.9)}\right)$	$\left(0.91e^{i2\pi(0.6)}, 0.39e^{i2\pi(0.7)}, 0.71e^{i2\pi(0.5)}\right)$
A_3	$\left(0.79e^{i2\pi(0.4)}, 0.71e^{i2\pi(0.7)}, 0.59e^{i2\pi(0.8)}\right)$	$\left(0.81e^{i2\pi(0.6)}, 0.91e^{i2\pi(0.4)}, 0.89e^{i2\pi(0.5)}\right)$	$\left(0.79e^{i2\pi(0.4)}, 0.91e^{i2\pi(0.5)}, 0.59e^{i2\pi(0.9)}\right)$	$\left(0.59e^{i2\pi(0.7)}, 0.49e^{i2\pi(0.6)}, 0.39e^{i2\pi(0.8)}\right)$	$\left(0.89e^{i2\pi(0.7)}, 0.39e^{i2\pi(0.5)}, 0.51e^{i2\pi(0.8)}\right)$
A_4	$\left(0.39e^{i2\pi(0.8)}, 0.71e^{i2\pi(0.7)}, 0.89e^{i2\pi(0.8)}\right)$	$\left(0.59e^{i2\pi(0.8)}, 0.69e^{i2\pi(0.5)}, 0.71e^{i2\pi(0.6)}\right)$	$\left(0.41e^{i2\pi(0.8)}, 0.59e^{i2\pi(0.9)}, 0.79e^{i2\pi(0.4)}\right)$	$\left(0.71e^{i2\pi(0.9)}, 0.39e^{i2\pi(0.7)}, 0.49e^{i2\pi(0.6)}\right)$	$\left(0.69e^{i2\pi(0.9)}, 0.39e^{i2\pi(0.5)}, 0.49e^{i2\pi(0.8)}\right)$

Table 3: Decision matrix of E_3

	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$
A_1	$\left(0.71e^{i2\pi(0.8)}, 0.69e^{i2\pi(0.6)}, 0.89e^{i2\pi(0.5)}\right)$	$\left(0.69e^{i2\pi(0.8)}, 0.49e^{i2\pi(0.7)}, 0.71e^{i2\pi(0.6)}\right)$	$\left(0.79e^{i2\pi(0.5)}, 0.41e^{i2\pi(0.9)}, 0.81e^{i2\pi(0.5)}\right)$	$\left(0.49e^{i2\pi(0.9)}, 0.69e^{i2\pi(0.6)}, 0.91e^{i2\pi(0.8)}\right)$	$\left(0.79e^{i2\pi(0.6)}, 0.91e^{i2\pi(0.8)}, 0.79e^{i2\pi(0.6)}\right)$
A_2	$\left(0.59e^{i2\pi(0.8)}, 0.39e^{i2\pi(0.7)}, 0.51se^{i2\pi(0.6)}\right)$	$\left(0.59e^{i2\pi(0.5)}, 0.69e^{i2\pi(0.9)}, 0.71e^{i2\pi(0.6)}\right)$	$\left(0.59e^{i2\pi(0.8)}, 0.69e^{i2\pi(0.6)}, 0.51e^{i2\pi(0.8)}\right)$	$\left(0.61e^{i2\pi(0.7)}, 0.39e^{i2\pi(0.5)}, 0.89e^{i2\pi(0.8)}\right)$	$\left(0.79e^{i2\pi(0.7)}, 0.49e^{i2\pi(0.8)}, 0.61e^{i2\pi(0.9)}\right)$
A_3	$\left(0.51e^{i2\pi(0.7)}, 0.89e^{i2\pi(0.4)}, 0.79e^{i2\pi(0.6)}\right)$	$\left(0.69e^{i2\pi(0.8)}, 0.49e^{i2\pi(0.6)}, 0.89e^{i2\pi(0.6)}\right)$	$\left(0.79e^{i2\pi(0.4)}, 0.91e^{i2\pi(0.7)}, 0.59e^{i2\pi(0.8)}\right)$	$\left(0.89e^{i2\pi(0.5)}, 0.79e^{i2\pi(0.9)}, 0.39e^{i2\pi(0.6)}\right)$	$\left(0.89e^{i2\pi(0.7)}, 0.61e^{i2\pi(0.5)}, 0.79e^{i2\pi(0.6)}\right)$
A_4	$\left(0.89e^{i2\pi(0.6)}, 0.79e^{i2\pi(0.7)}, 0.91e^{i2\pi(0.5)}\right)$	$\left(0.79e^{i2\pi(0.6)}, 0.49e^{i2\pi(0.7)}, 0.59e^{i2\pi(0.7)}\right)$	$\left(0.89e^{i2\pi(0.7)}, 0.61e^{i2\pi(0.5)}, 0.79e^{i2\pi(0.6)}\right)$	$\left(0.79e^{i2\pi(0.6)}, 0.89e^{i2\pi(0.5)}, 0.71e^{i2\pi(0.4)}\right)$	$\left(0.69e^{i2\pi(0.8)}, 0.55e^{i2\pi(0.3)}, 0.88e^{i2\pi(0.4)}\right)$

Table 4: Decision matrix of E_4

	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$
A_1	$\left(0.68e^{i2\pi(0.4)}, 0.87e^{i2\pi(0.6)}, 0.79e^{i2\pi(0.7)}\right)$	$\left(0.67e^{i2\pi(0.5)}, 0.48e^{i2\pi(0.4)}, 0.89e^{i2\pi(0.6)}\right)$	$\left(0.88e^{i2\pi(0.6)}, 0.69e^{i2\pi(0.3)}, 0.78e^{i2\pi(0.6)}\right)$	$\left(0.77e^{i2\pi(0.7)}, 0.68e^{i2\pi(0.4)}, 0.92e^{i2\pi(0.6)}\right)$	$\left(0.89e^{i2\pi(0.8)}, 0.49e^{i2\pi(0.5)}, 0.68e^{i2\pi(0.9)}\right)$
A_2	$\left(0.87e^{i2\pi(0.6)}, 0.49e^{i2\pi(0.4)}, 0.78e^{i2\pi(0.5)}\right)$	$\left(0.57e^{i2\pi(0.5)}, 0.78e^{i2\pi(0.6)}, 0.68e^{i2\pi(0.8)}\right)$	$\left(0.86e^{i2\pi(0.5)}, 0.72e^{i2\pi(0.6)}, 0.58e^{i2\pi(0.4)}\right)$	$\left(0.77e^{i2\pi(0.7)}, 0.69e^{i2\pi(0.5)}, 0.89e^{i2\pi(0.6)}\right)$	$\left(0.88e^{i2\pi(0.6)}, 0.72e^{i2\pi(0.4)}, 0.58e^{i2\pi(0.8)}\right)$

A_3	$\left(0.88e^{i2\pi(0.7)}, 0.67e^{i2\pi(0.4)}, 0.58e^{i2\pi(0.8)}\right)$	$\left(0.58e^{i2\pi(0.8)}, 0.49e^{i2\pi(0.4)}, 0.88e^{i2\pi(0.6)}\right)$	$\left(0.59e^{i2\pi(0.7)}, 0.47e^{i2\pi(0.6)}, 0.78e^{i2\pi(0.4)}\right)$	$\left(0.77e^{i2\pi(0.5)}, 0.68e^{i2\pi(0.6)}, 0.91e^{i2\pi(0.4)}\right)$	$\left(0.88e^{i2\pi(0.6)}, 0.78e^{i2\pi(0.4)}, 0.68e^{i2\pi(0.3)}\right)$
A_4	$\left(0.89e^{i2\pi(0.5)}, 0.48e^{i2\pi(0.9)}, 0.69e^{i2\pi(0.4)}\right)$	$\left(0.78e^{i2\pi(0.4)}, 0.57e^{i2\pi(0.5)}, 0.69e^{i2\pi(0.6)}\right)$	$\left(0.57e^{i2\pi(0.7)}, 0.68e^{i2\pi(0.5)}, 0.78e^{i2\pi(0.9)}\right)$	$\left(0.59e^{i2\pi(0.4)}, 0.89e^{i2\pi(0.7)}, 0.68e^{i2\pi(0.5)}\right)$	$\left(0.48e^{i2\pi(0.5)}, 0.77e^{i2\pi(0.8)}, 0.86e^{i2\pi(0.6)}\right)$

Step 2: Applying the CPFWG aggregation operator to aggregate all individual matrices into a single matrix, where $\exists = (0.20, 0.10, 0.40, 0.30)^T$ and $q = 4$.

Table 5: Collective decision-matrix E

	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$
A_1	$\left(0.72e^{i2\pi(0.58)}, 0.73e^{i2\pi(0.61)}, 0.84e^{i2\pi(0.67)}\right)$	$\left(0.71e^{i2\pi(0.65)}, 0.56e^{i2\pi(0.55)}, 0.80e^{i2\pi(0.62)}\right)$	$\left(0.94e^{i2\pi(0.63)}, 0.72e^{i2\pi(0.34)}, 0.86e^{i2\pi(0.60)}\right)$	$\left(0.89e^{i2\pi(0.74)}, 0.77e^{i2\pi(0.47)}, 0.94e^{i2\pi(0.64)}\right)$	$\left(0.98e^{i2\pi(0.83)}, 0.56e^{i2\pi(0.56)}, 0.72e^{i2\pi(0.96)}\right)$
A_2	$\left(0.70e^{i2\pi(0.68)}, 0.67e^{i2\pi(0.42)}, 0.69e^{i2\pi(0.54)}\right)$	$\left(0.57e^{i2\pi(0.55)}, 0.68e^{i2\pi(0.61)}, 0.81e^{i2\pi(0.70)}\right)$	$\left(0.96e^{i2\pi(0.58)}, 0.72e^{i2\pi(0.67)}, 0.68e^{i2\pi(0.45)}\right)$	$\left(0.89e^{i2\pi(0.77)}, 0.75e^{i2\pi(0.55)}, 0.96e^{i2\pi(0.68)}\right)$	$\left(0.90e^{i2\pi(0.68)}, 0.72e^{i2\pi(0.42)}, 0.65e^{i2\pi(0.84)}\right)$
A_3	$\left(0.67e^{i2\pi(0.62)}, 0.79e^{i2\pi(0.46)}, 0.78e^{i2\pi(0.70)}\right)$	$\left(0.68e^{i2\pi(0.68)}, 0.55e^{i2\pi(0.49)}, 0.90e^{i2\pi(0.59)}\right)$	$\left(0.68e^{i2\pi(0.75)}, 0.59e^{i2\pi(0.64)}, 0.86e^{i2\pi(0.48)}\right)$	$\left(0.89e^{i2\pi(0.54)}, 0.75e^{i2\pi(0.68)}, 0.96e^{i2\pi(0.45)}\right)$	$\left(0.96e^{i2\pi(0.65)}, 0.83e^{i2\pi(0.42)}, 0.75e^{i2\pi(0.37)}\right)$
A_4	$\left(0.81e^{i2\pi(0.53)}, 0.67e^{i2\pi(0.75)}, 0.85e^{i2\pi(0.71)}\right)$	$\left(0.83e^{i2\pi(0.44)}, 0.65e^{i2\pi(0.56)}, 0.78e^{i2\pi(0.62)}\right)$	$\left(0.65e^{i2\pi(0.74)}, 0.75e^{i2\pi(0.53)}, 0.86e^{i2\pi(0.92)}\right)$	$\left(0.68e^{i2\pi(0.43)}, 0.95e^{i2\pi(0.72)}, 0.73e^{i2\pi(0.56)}\right)$	$\left(0.56e^{i2\pi(0.58)}, 0.83e^{i2\pi(0.82)}, 0.97e^{i2\pi(0.64)}\right)$

Step 3: Next, again applying the CPFWG operator, with $\varpi = (0.30, 0.10, 0.30, 0.10, 0.20)^T$, we have the following preference values:

$$\varsigma_1 = \left(0.85e^{i2\pi(0.66)}, 0.67e^{i2\pi(0.48)}, 0.85e^{i2\pi(0.80)}\right) \varsigma_2 = \left(0.81e^{i2\pi(0.64)}, 0.70e^{i2\pi(0.52)}, 0.77e^{i2\pi(0.67)}\right)$$

$$\varsigma_3 = \left(0.74e^{i2\pi(0.66)}, 0.68e^{i2\pi(0.51)}, 0.81e^{i2\pi(0.58)}\right) \varsigma_4 = \left(0.69e^{i2\pi(0.57)}, 0.70e^{i2\pi(0.66)}, 0.89e^{i2\pi(0.80)}\right).$$

Step 4: By using Definition 6, we get the score functions as below.

$$S(\varsigma_1) = 0.64,\ S(\varsigma_2) = 0.79, S(\varsigma_3) = 0.74,\ S(\varsigma_4) = 0.50.$$

Step 5: Thus the best option is A_2.

6.2 Induced Aggregation Operators:

Step 1: Construct the following matrices based on expert's ideas:

Table 6: Decision matrix of E_1 on induced aggregation operator

	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$
A_1	$\left\langle 0.8, \left(0.65e^{i2\pi(0.4)}, 0.87e^{i2\pi(0.6)}, 0.78e^{i2\pi(0.7)}\right)\right\rangle$	$\left\langle 0.5, \left(0.68e^{i2\pi(0.5)}, 0.47e^{i2\pi(0.4)}, 0.86e^{i2\pi(0.6)}\right)\right\rangle$	$\left\langle 0.6, \left(0.88e^{i2\pi(0.6)}, 0.67e^{i2\pi(0.3)}, 0.78e^{i2\pi(0.6)}\right)\right\rangle$	$\left\langle 0.6, \left(0.78e^{i2\pi(0.7)}, 0.67e^{i2\pi(0.4)}, 0.88e^{i2\pi(0.6)}\right)\right\rangle$	$\left\langle 0.4, \left(0.89e^{i2\pi(0.8)}, 0.48e^{i2\pi(0.5)}, 0.72e^{i2\pi(0.9)}\right)\right\rangle$
A_2	$\left\langle 0.7, \left(0.89e^{i2\pi(0.6)}, 0.48e^{i2\pi(0.4)}, 0.77e^{i2\pi(0.5)}\right)\right\rangle$	$\left\langle 0.4, \left(0.58e^{i2\pi(0.5)}, 0.78e^{i2\pi(0.6)}, 0.67e^{i2\pi(0.8)}\right)\right\rangle$	$\left\langle 0.5, \left(0.89e^{i2\pi(0.5)}, 0.67e^{i2\pi(0.6)}, 0.57e^{i2\pi(0.4)}\right)\right\rangle$	$\left\langle 0.5, \left(0.78e^{i2\pi(0.7)}, 0.67e^{i2\pi(0.5)}, 0.89e^{i2\pi(0.6)}\right)\right\rangle$	$\left\langle 0.3, \left(0.89e^{i2\pi(0.6)}, 0.68e^{i2\pi(0.4)}, 0.57e^{i2\pi(0.8)}\right)\right\rangle$
A_3	$\left\langle 0.6, \left(0.87e^{i2\pi(0.7)}, 0.67e^{i2\pi(0.4)}, 0.58e^{i2\pi(0.8)}\right)\right\rangle$	$\left\langle 0.3, \left(0.57e^{i2\pi(0.8)}, 0.48e^{i2\pi(0.4)}, 0.87e^{i2\pi(0.6)}\right)\right\rangle$	$\left\langle 0.4, \left(0.58e^{i2\pi(0.7)}, 0.47e^{i2\pi(0.6)}, 0.79e^{i2\pi(0.4)}\right)\right\rangle$	$\left\langle 0.4, \left(0.78e^{i2\pi(0.5)}, 0.67e^{i2\pi(0.6)}, 0.88e^{i2\pi(0.4)}\right)\right\rangle$	$\left\langle 0.2, \left(0.87e^{i2\pi(0.6)}, 0.78e^{i2\pi(0.4)}, 0.67e^{i2\pi(0.3)}\right)\right\rangle$
A_4	$\left\langle 0.5, \left(0.89e^{i2\pi(0.5)}, 0.47e^{i2\pi(0.9)}, 0.67e^{i2\pi(0.4)}\right)\right\rangle$	$\left\langle 0.2, \left(0.78e^{i2\pi(0.4)}, 0.56e^{i2\pi(0.5)}, 0.68e^{i2\pi(0.6)}\right)\right\rangle$	$\left\langle 0.3 \left(0.58e^{i2\pi(0.7)}, 0.68e^{i2\pi(0.5)}, 0.78e^{i2\pi(0.9)}\right)\right\rangle$	$\left\langle 0.3, \left(0.58e^{i2\pi(0.4)}, 0.87e^{i2\pi(0.7)}, 0.67e^{i2\pi(0.5)}\right)\right\rangle$	$\left\langle 0.1, \left(0.47e^{i2\pi(0.5)}, 0.78e^{i2\pi(0.8)}, 0.89e^{i2\pi(0.6)}\right)\right\rangle$

Table 7: Decision matrix of E_2 on induced aggregation operator

	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$
A_1	$\left\langle 0.5, \left(0.58e^{i2\pi(0.7)}, 0.68e^{i2\pi(0.5)}, 0.79e^{i2\pi(0.6)}\right)\right\rangle$	$\left\langle 0.6, \left(0.59e^{i2\pi(0.8)}, 0.81e^{i2\pi(0.7)}, 0.79e^{i2\pi(0.6)}\right)\right\rangle$	$\left\langle 0.7, \left(0.59e^{i2\pi(0.3)}, 0.87e^{i2\pi(0.6)}, 0.78e^{i2\pi(0.7)}\right)\right\rangle$	$\left\langle 0.8, \left(0.78e^{i2\pi(0.7)}, 0.79e^{i2\pi(0.5)}, 0.92e^{i2\pi(0.8)}\right)\right\rangle$	$\left\langle 0.9, \left(0.89e^{i2\pi(0.7)}, 0.57e^{i2\pi(0.6)}, 0.72e^{i2\pi(0.4)}\right)\right\rangle$
A_2	$\left\langle 0.4, \left(0.48e^{i2\pi(0.7)}, 0.89e^{i2\pi(0.6)}, 0.78e^{i2\pi(0.4)}\right)\right\rangle$	$\left\langle 0.5, \left(0.47e^{i2\pi(0.9)}, 0.86e^{i2\pi(0.5)}, 0.69e^{i2\pi(0.4)}\right)\right\rangle$	$\left\langle 0.6, \left(0.68e^{i2\pi(0.5)}, 0.89e^{i2\pi(0.8)}, 0.48e^{i2\pi(0.8)}\right)\right\rangle$	$\left\langle 0.7, \left(0.69e^{i2\pi(0.6)}, 0.47e^{i2\pi(0.4)}, 0.78e^{i2\pi(0.9)}\right)\right\rangle$	$\left\langle 0.6, \left(0.87e^{i2\pi(0.6)}, 0.37e^{i2\pi(0.7)}, 0.69e^{i2\pi(0.5)}\right)\right\rangle$
A_3	$\left\langle 0.3, \left(0.78e^{i2\pi(0.4)}, 0.67e^{i2\pi(0.7)}, 0.56e^{i2\pi(0.8)}\right)\right\rangle$	$\left\langle 0.4, \left(0.78e^{i2\pi(0.6)}, 0.86e^{i2\pi(0.4)}, 0.89e^{i2\pi(0.5)}\right)\right\rangle$	$\left\langle 0.5, \left(0.78e^{i2\pi(0.4)}, 0.87e^{i2\pi(0.5)}, 0.56e^{i2\pi(0.9)}\right)\right\rangle$	$\left\langle 0.6, \left(0.56e^{i2\pi(0.7)}, 0.47e^{i2\pi(0.6)}, 0.38e^{i2\pi(0.8)}\right)\right\rangle$	$\left\langle 0.5, \left(0.87e^{i2\pi(0.7)}, 0.39e^{i2\pi(0.5)}, 0.48e^{i2\pi(0.8)}\right)\right\rangle$
A_4	$\left\langle 0.2, \left(0.39e^{i2\pi(0.8)}, 0.67e^{i2\pi(0.7)}, 0.87e^{i2\pi(0.8)}\right)\right\rangle$	$\left\langle 0.3, \left(0.62e^{i2\pi(0.8)}, 0.69e^{i2\pi(0.5)}, 0.72e^{i2\pi(0.6)}\right)\right\rangle$	$\left\langle 0.4, \left(0.39e^{i2\pi(0.8)}, 0.58e^{i2\pi(0.9)}, 0.79e^{i2\pi(0.4)}\right)\right\rangle$	$\left\langle 0.5, \left(0.68e^{i2\pi(0.9)}, 0.32e^{i2\pi(0.7)}, 0.47e^{i2\pi(0.6)}\right)\right\rangle$	$\left\langle 0.4, \left(0.68e^{i2\pi(0.9)}, 0.37e^{i2\pi(0.5)}, 0.51e^{i2\pi(0.8)}\right)\right\rangle$

Table 8: Decision matrix of E_3 on induced aggregation operator

	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$
A_1	$\left\langle 0.9, \left(0.68e^{i2\pi(0.8)}, 0.69e^{i2\pi(0.6)}, 0.87e^{i2\pi(0.5)}\right)\right\rangle$	$\left\langle 0.3, \left(0.68e^{i2\pi(0.8)}, 0.47e^{i2\pi(0.7)}, 0.69e^{i2\pi(0.6)}\right)\right\rangle$	$\left\langle 0.5, \left(0.79e^{i2\pi(0.5)}, 0.38e^{i2\pi(0.9)}, 0.78e^{i2\pi(0.5)}\right)\right\rangle$	$\left\langle 0.7, \left(0.47e^{i2\pi(0.9)}, 0.68e^{i2\pi(0.6)}, 0.87e^{i2\pi(0.8)}\right)\right\rangle$	$\left\langle 0.8, \left(0.78e^{i2\pi(0.6)}, 0.89e^{i2\pi(0.8)}, 0.69e^{i2\pi(0.6)}\right)\right\rangle$

A_2	$\left\langle 0.7, \begin{pmatrix} 0.57e^{i2\pi(0.8)}, \\ 0.39e^{i2\pi(0.7)}, \\ 0.52e^{i2\pi(0.6)} \end{pmatrix} \right\rangle$	$\left\langle 0.2, \begin{pmatrix} 0.62e^{i2\pi(0.5)}, \\ 0.68e^{i2\pi(0.9)}, \\ 0.72e^{i2\pi(0.6)} \end{pmatrix} \right\rangle$	$\left\langle 0.3, \begin{pmatrix} 0.59e^{i2\pi(0.8)}, \\ 0.73e^{i2\pi(0.6)}, \\ 0.49e^{i2\pi(0.8)} \end{pmatrix} \right\rangle$	$\left\langle 0.6, \begin{pmatrix} 0.58e^{i2\pi(0.7)}, \\ 0.39e^{i2\pi(0.5)}, \\ 0.92e^{i2\pi(0.8)} \end{pmatrix} \right\rangle$	$\left\langle 0.5, \begin{pmatrix} 0.82e^{i2\pi(0.7)}, \\ 0.48e^{i2\pi(0.8)}, \\ 0.59e^{i2\pi(0.9)} \end{pmatrix} \right\rangle$
A_3	$\left\langle 0.6, \begin{pmatrix} 0.48e^{i2\pi(0.7)}, \\ 0.87e^{i2\pi(0.4)}, \\ 0.79e^{i2\pi(0.6)} \end{pmatrix} \right\rangle$	$\left\langle 0.2, \begin{pmatrix} 0.67e^{i2\pi(0.8)}, \\ 0.51e^{i2\pi(0.6)}, \\ 0.89e^{i2\pi(0.6)} \end{pmatrix} \right\rangle$	$\left\langle 0.2, \begin{pmatrix} 0.78e^{i2\pi(0.4)}, \\ 0.89e^{i2\pi(0.7)}, \\ 0.58e^{i2\pi(0.8)} \end{pmatrix} \right\rangle$	$\left\langle 0.4, \begin{pmatrix} 0.89e^{i2\pi(0.5)}, \\ 0.79e^{i2\pi(0.9)}, \\ 0.38e^{i2\pi(0.6)} \end{pmatrix} \right\rangle$	$\left\langle 0.3, \begin{pmatrix} 0.87e^{i2\pi(0.7)}, \\ 0.57e^{i2\pi(0.5)}, \\ 0.78e^{i2\pi(0.6)} \end{pmatrix} \right\rangle$
A_4	$\left\langle 0.2, \begin{pmatrix} 0.87e^{i2\pi(0.6)}, \\ 0.82e^{i2\pi(0.7)}, \\ 0.89e^{i2\pi(0.5)} \end{pmatrix} \right\rangle$	$\left\langle 0.1, \begin{pmatrix} 0.76e^{i2\pi(0.6)}, \\ 0.47e^{i2\pi(0.7)}, \\ 0.57e^{i2\pi(0.7)} \end{pmatrix} \right\rangle$	$\left\langle 0.1, \begin{pmatrix} 0.89e^{i2\pi(0.7)}, \\ 0.61e^{i2\pi(0.5)}, \\ 0.78e^{i2\pi(0.6)} \end{pmatrix} \right\rangle$	$\left\langle 0.4, \begin{pmatrix} 0.79e^{i2\pi(0.6)}, \\ 0.87e^{i2\pi(0.5)}, \\ 0.67e^{i2\pi(0.4)} \end{pmatrix} \right\rangle$	$\left\langle 0.1, \begin{pmatrix} 0.67e^{i2\pi(0.8)}, \\ 0.51e^{i2\pi(0.3)}, \\ 0.89e^{i2\pi(0.4)} \end{pmatrix} \right\rangle$

Table 9: Decision matrix of E_4 on induced aggregation operator

	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$
A_1	$\left\langle 0.8, \begin{pmatrix} 0.89e^{i2\pi(0.5)}, \\ 0.62e^{i2\pi(0.7)}, \\ 0.68e^{i2\pi(0.8)} \end{pmatrix} \right\rangle$	$\left\langle 0.5, \begin{pmatrix} 0.78e^{i2\pi(0.6)}, \\ 0.69e^{i2\pi(0.5)}, \\ 0.58e^{i2\pi(0.7)} \end{pmatrix} \right\rangle$	$\left\langle 0.6, \begin{pmatrix} 0.67e^{i2\pi(0.5)}, \\ 0.89e^{i2\pi(0.4)}, \\ 0.48e^{i2\pi(0.8)} \end{pmatrix} \right\rangle$	$\left\langle 0.8, \begin{pmatrix} 0.87e^{i2\pi(0.5)}, \\ 0.56e^{i2\pi(0.6)}, \\ 0.79e^{i2\pi(0.9)} \end{pmatrix} \right\rangle$	$\left\langle 0.7, \begin{pmatrix} 0.72e^{i2\pi(0.5)}, \\ 0.91e^{i2\pi(0.6)}, \\ 0.58e^{i2\pi(0.7)} \end{pmatrix} \right\rangle$
A_2	$\left\langle 0.7, \begin{pmatrix} 0.78e^{i2\pi(0.6)}, \\ 0.69e^{i2\pi(0.4)}, \\ 0.59e^{i2\pi(0.5)} \end{pmatrix} \right\rangle$	$\left\langle 0.4, \begin{pmatrix} 0.47e^{i2\pi(0.6)}, \\ 0.89e^{i2\pi(0.7)}, \\ 0.48e^{i2\pi(0.7)} \end{pmatrix} \right\rangle$	$\left\langle 0.3, \begin{pmatrix} 0.79e^{i2\pi(0.6)}, \\ 0.61e^{i2\pi(0.7)}, \\ 0.78e^{i2\pi(0.5)} \end{pmatrix} \right\rangle$	$\left\langle 0.6, \begin{pmatrix} 0.67e^{i2\pi(0.6)}, \\ 0.49e^{i2\pi(0.4)}, \\ 0.78e^{i2\pi(0.9)} \end{pmatrix} \right\rangle$	$\left\langle 0.4, \begin{pmatrix} 0.68e^{i2\pi(0.8)}, \\ 0.78e^{i2\pi(0.5)}, \\ 0.89e^{i2\pi(0.6)} \end{pmatrix} \right\rangle$
A_3	$\left\langle 0.5, \begin{pmatrix} 0.68e^{i2\pi(0.5)}, \\ 0.78e^{i2\pi(0.6)}, \\ 0.92e^{i2\pi(0.5)} \end{pmatrix} \right\rangle$	$\left\langle 0.3, \begin{pmatrix} 0.89e^{i2\pi(0.7)}, \\ 0.57e^{i2\pi(0.5)}, \\ 0.49e^{i2\pi(0.9)} \end{pmatrix} \right\rangle$	$\left\langle 0.2, \begin{pmatrix} 0.68e^{i2\pi(0.4)}, \\ 0.57e^{i2\pi(0.5)}, \\ 0.86e^{i2\pi(0.6)} \end{pmatrix} \right\rangle$	$\left\langle 0.4, \begin{pmatrix} 0.68e^{i2\pi(0.6)}, \\ 0.57e^{i2\pi(0.5)}, \\ 0.79e^{i2\pi(0.4)} \end{pmatrix} \right\rangle$	$\left\langle 0.3, \begin{pmatrix} 0.67e^{i2\pi(0.9)}, \\ 0.48e^{i2\pi(0.4)}, \\ 0.92e^{i2\pi(0.5)} \end{pmatrix} \right\rangle$
A_4	$\left\langle 0.3, \begin{pmatrix} 0.78e^{i2\pi(0.4)}, \\ 0.69e^{i2\pi(0.7)}, \\ 0.77e^{i2\pi(0.9)} \end{pmatrix} \right\rangle$	$\left\langle 0.2, \begin{pmatrix} 0.76e^{i2\pi(0.6)}, \\ 0.51e^{i2\pi(0.7)}, \\ 0.59e^{i2\pi(0.7)} \end{pmatrix} \right\rangle$	$\left\langle 0.1, \begin{pmatrix} 0.78e^{i2\pi(0.4)}, \\ 0.89e^{i2\pi(0.6)}, \\ 0.37e^{i2\pi(0.8)} \end{pmatrix} \right\rangle$	$\left\langle 0.2, \begin{pmatrix} 0.89e^{i2\pi(0.7)}, \\ 0.68e^{i2\pi(0.4)}, \\ 0.57e^{i2\pi(0.5)} \end{pmatrix} \right\rangle$	$\left\langle 0.3, \begin{pmatrix} 0.87e^{i2\pi(0.7)}, \\ 0.46e^{i2\pi(0.4)}, \\ 0.78e^{i2\pi(0.5)} \end{pmatrix} \right\rangle$

Step 2: Applying the I-CPFOWG aggregation operator to aggregate all individual matrices into a single matrix, where $\exists = (0.30, 0.10, 0.40, 0.20)^T$ and $q = 4$, we have:

Table 10: Combined Decision-matrix E on induced aggregation operator

	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$
A_1	$\begin{pmatrix} 0.71e^{i2\pi(0.57)}, \\ 0.72e^{i2\pi(0.60)}, \\ 0.85e^{i2\pi(0.68)} \end{pmatrix}$	$\begin{pmatrix} 0.72e^{i2\pi(0.64)}, \\ 0.57e^{i2\pi(0.56)}, \\ 0.79e^{i2\pi(0.61)} \end{pmatrix}$	$\begin{pmatrix} 0.93e^{i2\pi(0.62)}, \\ 0.71e^{i2\pi(0.35)}, \\ 0.87e^{i2\pi(0.59)} \end{pmatrix}$	$\begin{pmatrix} 0.88e^{i2\pi(0.73)}, \\ 0.78e^{i2\pi(0.45)}, \\ 0.93e^{i2\pi(0.65)} \end{pmatrix}$	$\begin{pmatrix} 0.99e^{i2\pi(0.84)}, \\ 0.57e^{i2\pi(0.55)}, \\ 0.73e^{i2\pi(0.95)} \end{pmatrix}$
A_2	$\begin{pmatrix} 0.69e^{i2\pi(0.70)}, \\ 0.68e^{i2\pi(0.40)}, \\ 0.68e^{i2\pi(0.55)} \end{pmatrix}$	$\begin{pmatrix} 0.56e^{i2\pi(0.55)}, \\ 0.69e^{i2\pi(0.61)}, \\ 0.79e^{i2\pi(0.71)} \end{pmatrix}$	$\begin{pmatrix} 0.95e^{i2\pi(0.57)}, \\ 0.71e^{i2\pi(0.66)}, \\ 0.69e^{i2\pi(0.46)} \end{pmatrix}$	$\begin{pmatrix} 0.89e^{i2\pi(0.77)}, \\ 0.75e^{i2\pi(0.55)}, \\ 0.96e^{i2\pi(0.68)} \end{pmatrix}$	$\begin{pmatrix} 0.90e^{i2\pi(0.68)}, \\ 0.72e^{i2\pi(0.42)}, \\ 0.65e^{i2\pi(0.84)} \end{pmatrix}$

A_3	$\begin{pmatrix}0.66e^{i2\pi(0.68)},\\0.80e^{i2\pi(0.46)},\\0.78e^{i2\pi(0.69)}\end{pmatrix}$	$\begin{pmatrix}0.68e^{i2\pi(0.69)},\\0.56e^{i2\pi(0.50)},\\0.89e^{i2\pi(0.60)}\end{pmatrix}$	$\begin{pmatrix}0.68e^{i2\pi(0.75)},\\0.59e^{i2\pi(0.64)},\\0.86e^{i2\pi(0.48)}\end{pmatrix}$	$\begin{pmatrix}0.89e^{i2\pi(0.54)},\\0.75e^{i2\pi(0.68)},\\0.96e^{i2\pi(0.45)}\end{pmatrix}$	$\begin{pmatrix}0.96e^{i2\pi(0.65)},\\0.83e^{i2\pi(0.42)},\\0.75e^{i2\pi(0.37)}\end{pmatrix}$
A_4	$\begin{pmatrix}0.80e^{i2\pi(0.52)},\\0.66e^{i2\pi(0.74)},\\0.86e^{i2\pi(0.70)}\end{pmatrix}$	$\begin{pmatrix}0.84e^{i2\pi(0.44)},\\0.66e^{i2\pi(0.57)},\\0.80e^{i2\pi(0.63)}\end{pmatrix}$	$\begin{pmatrix}0.65e^{i2\pi(0.74)},\\0.75e^{i2\pi(0.53)},\\0.86e^{i2\pi(0.92)}\end{pmatrix}$	$\begin{pmatrix}0.68e^{i2\pi(0.43)},\\0.95e^{i2\pi(0.72)},\\0.73e^{i2\pi(0.56)}\end{pmatrix}$	$\begin{pmatrix}0.56e^{i2\pi(0.58)},\\0.82e^{i2\pi(0.80)},\\0.98e^{i2\pi(0.65)}\end{pmatrix}$

Step 3: Next, again applying the I-CPFOWG operator, with $\varpi = (0.30, 0.10, 0.30, 0.10, 0.20)^T$, we have the following preference values:

$$\varsigma_1 = \left(0.84e^{i2\pi(0.67)}, 0.68e^{i2\pi(0.49)}, 0.84e^{i2\pi(0.79)}\right),\ \varsigma_2 = \left(0.80e^{i2\pi(0.63)}, 0.69e^{i2\pi(0.51)}, 0.76e^{i2\pi(0.67)}\right),$$
$$\varsigma_3 = \left(0.73e^{i2\pi(0.65)}, 0.68e^{i2\pi(0.49)}, 0.80e^{i2\pi(0.58)}\right),\ \varsigma_4 = \left(0.68e^{i2\pi(0.57)}, 0.70e^{i2\pi(0.66)}, 0.90e^{i2\pi(0.79)}\right).$$

Step 4: Using Definition 6, and get the score functions as below.

$$S(\varsigma_1) = 0.63,\ S(\varsigma_2) = 0.80, S(\varsigma_3) = 0.73,\ S(\varsigma_4) = 0.49.$$

Step 5: Thus the best option is A_2.

Table11: Score functions of different operators

Operators	Score functions	Ranking
CPFWG	$S(\varsigma_2) \succ S(\varsigma_3) \succ S(\varsigma_1) \succ S(\varsigma_4)$	$A_2 \succ A_3 \succ A_1 \succ A_4$
CPFOWG	$S(\varsigma_2) \succ S(\varsigma_3) \succ S(\varsigma_1) \succ S(\varsigma_4)$	$A_2 \succ A_3 \succ A_1 \succ A_4$
CPFHG	$S(\varsigma_2) \succ S(\varsigma_3) \succ S(\varsigma_1) \succ S(\varsigma_4)$	$A_2 \succ A_3 \succ A_1 \succ A_4$
I-CPFOWG	$S(\varsigma_2) \succ S(\varsigma_3) \succ S(\varsigma_1) \succ S(\varsigma_4)$	$A_2 \succ A_3 \succ A_1 \succ A_4$
I-CPFHG	$S(\varsigma_2) \succ S(\varsigma_3) \succ S(\varsigma_1) \succ S(\varsigma_4)$	$A_2 \succ A_3 \succ A_1 \succ A_4$

7. Comparative Analysis

Complex Polytopic fuzzy set is a refinement of earlier work, including: FSs, IFSs, PFSs, FFSs, CFSs, CIFSs, and CPFSs by taking into account a lot more details about an object when processing it and managing two-dimensional data as a single set. For instance, CFSs (only complex-valued membership degrees), IFSs, PFSs, FFSs (consisting of both real-valued membership and real-valued non-membership degrees), FSs (consisting of just membership degrees), and CIFSs and CPFSs (contain complex-valued membership and complex-valued non-membership degrees with conditions such as their sum and square sum less than or equal to one respectively). Additionally, information contained in the complex Polytopic fuzzy set (membership, neutral and non-membership degrees with condition that sum of their q-th power less than or equal to one). As a result, the suggested model is more flexible than their prior studies.

8. Sensitivity Analysis

The proposed technique is not only applicable to complex Polytopic fuzzy data, as it may be used to Polytopic fuzzy data by setting the phase terms to zero. Moreover, it can also be applied to complex q-Rung orthopair fuzzy information without taking of neutral and similarly, it can be applied to q-Rung orthopair fuzzy information by setting their neutral and phase terms zero. The suggested operators are therefore more adaptable and elastic to get around the constraints and limits of their present aggregate operators.

Table 12: Sensitivity analysis

Model	Uncertainty	Falsity	Hesitation	Periodicity	2-D information	q in Power
FSs	✓	×	×	×	×	×
IFSs	✓	✓	✓	×	×	×
PFSs	✓	✓	✓	×	×	×
FFSs	✓	✓	✓	×	×	×
CFSs	✓	×	×	✓	✓	×
CIFSs	✓	✓	✓	✓	✓	×
CPFSs	✓	✓	✓	✓	✓	✓

9. Limitations

Compared with previous studies, complex Polytopic fuzzy sets are more powerful tool for decision-making problems, such as, FSs, IFSs, PFSs, FFSs, CFSs, CIFSs, and CPFSs, which consider more information about objects in the process, and in Work with two-dimensional data in a collection.

However, there are some drawbacks to the suggested model, including: let $\upsilon=\left(\hbar e^{ia},\lambda e^{ib},\xi e^{in}\right)$, be complex Polytopic fuzzy number, where $i=\sqrt{-1}$, $\hbar,\lambda,\xi\in[0,1]$, $a,b,n\in[0,2\pi]$ with conditions $0\le\hbar^{q}+\lambda^{q}+\xi^{q}\le 1$ and $0\le\left(\frac{a}{2\pi}\right)^{q}+\left(\frac{b}{2\pi}\right)^{q}+\left(\frac{n}{2\pi}\right)^{q}\le 1$. Therefore, in this study, we only take into account complex Polytopic fuzzy numbers that meet the aforementioned requirements.

10. Conclusion

This paper is concerned to the study of complex Polytopic fuzzy set, complex Polytopic fuzzy numbers and some of their essential operational laws. We have developed the score function and accuracy degree for the novel model. Using the CPFNs and developed various new techniques, namely, CPFWG operator, CPFOWG operator, I-CPFOWG operator, CPFHG operator, and I-CPFHG operator with their structure characteristics such as idempotency, monotonicity and boundedness. This novel model is explained with an illustrative example related to the selection of the more suitable alternative among the existing alternatives. Finally, a comparison and sensitivity analysis of the innovative model is given, demonstrating the potency of the strategy being offered.

Furthermore, this study can be expanded to complex Logarithmic operators, complex linguistic terms, complex inducing variables, complex Hamacher operators, complex confidence level, complex interval-valued approach, complex symmetric operators, complex Dombi approach, complex power operators, complex Hamacher interval approach, complex Dombi interval approach, complex Einstein interval, complex Einstein approach, etc.

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Declarations

Conflict of interest

The authors declare no conflict of interest.

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