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Research Article

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An Improved Conditional Inference on General Lifetime Model Parameters

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Abstract

It is widely known that the conditional inference is usually efficient as the Bayesian inference based on the non-informative prior. However, it is less efficient than the Bayesian inference based on the informative prior. Therefore, the main objective of this paper is to introduce an improvement to the conditional inference by using the kernel prior distribution. The improved conditional inference has been used for estimating the general lifetime model parameters, based on the generalized progressive hybrid-censoring scheme, and compared with the Bayesian estimates, via the Monte Carlo simulations. The simulation results have been shown that the improved conditional inference is highly efficient and provides better estimates than the Bayesian estimates based on different loss functions. Finally, real data sets have been given to demonstrate the efficiencies of the proposed methods.

Keywords: Bayes Inference; Conditional Inference; Informative Prior; Kernel Prior; Loss Functions.

AMS Subject Classification: 62F15; 62F40; 62F86; 62N01; 62N02

Introduction

In statistical inference, it is known that the conditional inference is highly efficient as the Bayesian inference especially when the sample sizes are small or when the data are heavily censored based on the non-informative prior. However, unfortunately the conditional inference is less efficient than the Bayesian inference based on the informative prior. Thus, in this paper we introduce an improvement to the conditional inference by using the kernel prior of the pivotal quantities. We employed the proposed method on general lifetime model that belongs to the shape-scale family, which has cumulative distribution function (CDF) and probability density function (PDF) as given respectively by:

$$F(x) = 1 - \exp(-\beta g^{\alpha}(x)) \ , \alpha, \beta, x > 0 \ , \quad (1)$$

$$f(x) = \alpha\beta g^{\alpha-1}(x) g'(x) \exp(-\beta g^{\alpha}(x)) \ , \alpha, \beta, x > 0 \ . \quad (2)$$

For convenience, we assume $g(x)$ to be differentiable as well as strictly increasing function of x such that, $g(0^{+}) = 0$ and $g(x) \to \infty$ as $x \to \infty$,with α and β are shape and scale parameters respectively.

This family includes among others, more common parametric distributions such as Weibull, Weibull extension, modified Weibull, Burr-type-XII, Lomax, Generalized Pareto and Pareto distributions with different values of $g^{\alpha}(x)$. Some important members of this family are shown in Table 1.

Table (1): Important members form the general lifetime model

No.	$g^{\alpha}(x)$	F(x)	Distribution
1	x^{α}	$1-\exp(-\beta x^{\alpha})$	Weibull
2	$\exp(x^{\alpha})-1$	$1-\exp(-\beta(\exp(x^{\alpha})-1))$	Weibull Extension
3	$x^{\alpha}\exp(\lambda x)$	$1-\exp(-x^{\alpha}\beta\exp(\lambda x))$	Modified Weibull
4	$\ln(1+x^{\alpha})$	$1-(1+x^{\alpha})^{-\beta}$	Burr-type XII
5	$\ln(1+x/\alpha)$	$1-(1+x/\alpha)^{-\beta}$	Lomax
6	$-\ln(1-x/\alpha)$	$1-(1-x/\alpha)^{\beta}$	Generalized Pareto
7	$\ln(x/\alpha)$	$1-(x/\alpha)^{-\beta}$	Pareto-type I

This family includes the most common parametric models in lifetime distributions. Thus, the improved conditional (IMPC) and the Bayesian methods were applied to estimate the Weibull model parameters as an application based on the generalized progressive hybrid censoring scheme that guarantee the proposed method can be applied to all the members of this family. Many authors used the informative prior for Weibull model parameters, among of them [30] derived an informative conjugate prior by assuming each of the parameters has a gamma distribution [1]. Suggested a different prior based on the prior information about the reliability level or the hazard rate at a given time and converting it into information about the model parameters [2, 3]. Derived the confidence intervals, via some pivotal quantities based on the Type-II progressive censored samples [4]. Derived the parameter estimates based on the classical and Bayesian approaches [5]. Presented the analysis of the reliability and quantile of the Weibull distribution [6]. Applied some methods for estimating Weibull parameters [7]. Applied the Maximum Likelihood (ML) and Bayes methods for estimating Weibull parameters based on censored samples and derived the empirical Bayes inference of Weibull Model parameters [8, 9]. For the continuation of these efforts, the purpose of this paper is deriving the point estimates for the general lifetime model parameters with application to the Weibull model using the improved conditional inference and Bayesian methods based on the generalized progressive hybrid-censoring scheme.

Recently, proposed the generalized progressive hybrid-censoring scheme (GPHCS), which is a generalization of the type-II progressively hybrid censored data [10, 11]. The GPHCS has been described in [12, 13].

Thus, given a generalized progressive hybrid censored sample, the likelihood functions for the three different cases can be written in a unified form as follows:

$$L(\underline{x};\theta) = C\prod_{i=1}^{D} f(x_{i,n,n})\,[1-F(x_{i,n,n})]^{R_i^*}\,[1-F(T)]^{R_T^*\delta}. \quad (3)$$

where $C=\prod_{i=1}^{D}\sum_{j=i}^{m}(R_j+1)$ and R_T^* is the number of surviving units that are removed at the termination time T.

$$D=\begin{cases} J, & \delta=1 \quad if\ X_k\le T< X_m \\ m, & \delta=0 \quad if\ X_k< X_m\le T \\ k, & \delta=0 \quad if\ T< X_k< X_m \end{cases} \text{ and } \underline{X}=\begin{cases} (X_1,X_2,\dots,X_J) & if\ X_k\le T< X_m \\ (X_1,X_2,\dots,X_m) & if\ X_k< X_m\le T \\ (X_1,X_2,\dots,X_J,X_{J+1}\dots,X_k) & if\ T< X_k< X_m \end{cases}$$

$$R_i^*=\begin{cases} (R_1,R_2,\dots,R_J,R_T^*), \quad R_T^*=D-J-\sum_{i=1}^{J}R_i & if\ X_k\le T< X_m \\ (R_1,R_2,\dots,R_m), \quad R_m=D-m-\sum_{i=1}^{m-1}R_i & if\ X_k< X_m\le T \\ (R_1,R_2,\dots,R_J,0,0,\dots0,R_k), \quad R_k=D-k-\sum_{i=1}^{k-1}R_i & if\ T< X_k< X_m \end{cases}$$

The GPHCS has been applied for some distributions such as Weibull distribution, inverse Weibull distribution, Exponential distribution, see and Rayleigh distribution, the inverse Weibull distribution, the generalized shape scale family [10, 12-16].

The improved conditional inference and the Bayesian inference will be derived using the GPHCS based on the following loss functions:

Firstly, the squared error loss function (SLF), $L(\theta,\theta^*)=(\theta-\theta^*)^2$, which is classified as a symmetric loss function and that penalize overestimation and underestimation equally on $(-\infty,\infty)$. For this loss function the Bayes estimator that minimizes the risk function is given by **$\theta^*=E(\theta|x)$.**

Secondly, the Stein's loss function (STLF), $L(\theta,\theta^*)=\theta^*/\theta-\log(\theta^*/\theta)-1$, which penalize gross overestimation and gross underestimation equally, are recommended for the positive restricted parameter space $(0,\infty)$[17, 18]. The Bayes estimator that minimizes the risk function is derived as θ^*=1/E(1/θ|x).

Thirdly, in real applications the underestimation of a parameter value very often implies different results from overestimation, both in quality and quantity. Thus, losses resulting from this can be described by a linear function with different coefficients characterizing positive and negative errors. This function, called LINEX loss function (LLF) that has been introduced in and it can be defined as asymmetric loss function with the following form [19]
$L(\theta,\theta^*)=exp[\delta(\theta^*-\theta)]-\delta(\theta^*-\theta)-1,\ \delta\neq 0.$

The sign and magnitude of the shape parameter δ represents the direction and degree of symmetry respectively, where positive values mean overestimation is more serious than underestimation and vice versa for negative values.

The unique Bayes estimator θ_L^* of θ under the LINEX loss function, the value that minimizes the risk function, is given by
$\theta_L^* = -\frac{1}{\delta} ln[E_\theta(e^{-\delta\theta}|x)]$, provided the expected $E\theta\ (e^{-\delta\theta})|x)$ exists and is finite.

Several authors have used this function [20, 21].

Bayesian estimation based on the informative prior

We suggest using independent priors for both parameters α and β which has a gamma distribution each as given by:

$$g_1(\alpha) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha},\ a,b \geq 0 \text{ and } g_2(\beta) = \frac{d^c}{\Gamma(c)}\beta^{c-1}e^{-d\beta},\ c,d \geq 0.$$

Thus, the joint prior density is given by

$$g(\alpha,\beta) \propto \alpha^{a-1}\beta^{c-1}e^{-d\beta-b\alpha}. \quad (4)$$

Using the informative prior (4) and the likelihood function (3) the joint posterior density is given by

$$f(\alpha,\beta|X) = K\alpha^{D+a-1}\beta^{D+c-1}\, exp[-\alpha(b-\sum_{i=1}^{D} ln(g(x_i)) - \sum_{i=1}^{D} ln(g(x_i)]$$
$$\times\, exp[-\beta(d+\textstyle\sum_{i=1}^{D}(R_i^*+1)g^\alpha(x_i)+\delta R_T^* g^\alpha(t))]. \quad (5)$$

K is the normalizing constant and can be derived as

$$K^{-1} = \Gamma(D+c)\int_0^\infty \alpha^{D+a-1}\,[d+\sum_{i=1}^{D}(R_i^*+1)g^\alpha(x_i)+\delta R_T^* g^\alpha(t)]^{-(D+c)}$$
$$\times\, exp[-\alpha(b-\textstyle\sum_{i=1}^{D} ln(g(x_i)) - \sum_{i=1}^{D} ln(g(x_i)]\, d\alpha.$$

The expected value for any function h(θ) can be derived from (5) as:

$$E(h^\nu(\theta)|X) = K\int_0^\infty\int_0^\infty h^\nu(\theta)\,\alpha^{D+a-1}\beta^{D+c-1}\, exp[-\alpha(b-\sum_{i=1}^{D} ln(g(x_i)) - \sum_{i=1}^{D} ln(g(x_i)]$$
$$\times\, exp[-\beta(d+\textstyle\sum_{i=1}^{D}(R_i^*+1)g^\alpha(x_i)+\delta R_T^* g^\alpha(t))]d\alpha d\beta. \quad (6)$$

An Improved Conditional inference

This section provides an outline for converting the standard likelihood function to function depends on pivotal quantities and ancillary statistics for the general lifetime model (2) based on the generalized progressive hybrid-censoring scheme. For more details, who used the distribution of pivotal quantities for the parameters given ancillary statistics as tools for estimating the parameters based on complete and censored samples [22-26]. Thus, based on the GPHCS the likelihood function (3) can be written using (2) as follows:

$$L(\underline{x};\theta) = C\prod_{i=1}^{D}\alpha\beta g^{\alpha-1}(x_i)g'(x_i)\, exp[-\beta(\textstyle\sum_{i=1}^{D}(1+R_i^*)\,g^\alpha(x_i)+\delta R_T^* g^\alpha(T))]\ . \quad (7)$$

Since, $\beta g^\alpha(x) = \beta\hat{\beta}^{\alpha/\hat{\alpha}}(g^{\hat{\alpha}}(x)/\hat{\beta})^{\alpha/\hat{\alpha}} = z_2 {a_i}^{z_1}$.

Therefore, $Z_1 = \alpha/\hat{\alpha}$, $Z_2 = \beta\hat{\beta}^{Z_1}$ are pivotal quantities as their joint density function does not depend on either α or β, where $a_i = g^{\hat{\alpha}}(x_i)/\hat{\beta}$ being the ancillary statistics $i=1,2,...,D$.

Let $A=(a_1,a_2...,a_{D-2})$ forms a set of ancillary statistics satisfies the maximum likelihood equations, thus only D-2 of which are functionally independent.

Theorem:

Let $\hat{\alpha}$ and $\hat{\beta}$ be the maximum likelihood estimators of α and β based on the generalized progressive hybrid censored sample. Thus, the joint PDF of $Z_1=\alpha/\hat{\alpha}$, $Z_2=\beta\hat{\beta}^{Z_1}$ and $A=(a_1,a_2...,a_{D-2})$ is of the form

$$f(z_1,z_2:A) = KZ_1^{D-1}Z_2^{D-1}\prod_{i=1}^{D}a_i^{Z_1-1}\, exp[-Z_2[\textstyle\sum_{i=1}^{D}(1+R_i^*)\,a_i^{Z_1}+\delta R_T^* a_T^{Z_1}]], \quad (8)$$

where K is the normalizing constant.

Proof

Make the change of variables from $(x_1,x_2,x_3,...,x_D)$ that has joint density function (7) to . $(\hat{\alpha},\hat{\beta},a_1,a_2...,a_{D-2})$. This transformation can be written as follows: $g(x_i)=(a_i\hat{\beta})^{1/\hat{\alpha}}$, $i=1,2,....,D-2$, $g(x_{D-1})=(a_{D-1}\hat{\beta})^{1/\hat{\alpha}}$ and $g(x_D)=(a_D\hat{\beta})^{1/\hat{\alpha}}$, where $\mathbf{a}_D$ and $\mathbf{a}_{D-1}$ can be expressed in terms of . The Jacobin of this transformation is independent of Z_1 and Z_2, therefore the joint PDF of $\hat{\alpha}$, $\hat{\beta}$, $a_1,a_2,....,a_{D-2}$ can be derived as where $\alpha\beta g^{\alpha-1}(x)g'(x)=z_1 z_2 a_i z_1^{-1}$.

Making further change of variables from ($\hat{\alpha},\hat{\beta}, a_1,a_2...,a_{D-2}$) to ($Z_1,Z_2,a_1,a_2...,a_{D-2}$), where the Jacobin $1/Z_1 Z_2$ of this transformation is proportional to . Finally, the joint distribution function of ($Z_1,Z_2,a_1,a_2...,a_{D-2}$) can be derived as in (8)

From (10), we can derive the Bayes estimators for Z_1 and Z_2 , and converting to the conditional estimators for α and β fiducially from the pivotal quantities $Z_1=\alpha/\hat{\alpha}$ and $Z_2=\beta\hat{\beta} Z_1$) as:

$\alpha^*=\hat{\alpha} E(Z_1)$ and $\beta^*= exp\ [(E(log\ (z_2)\text{-}log\ (\hat{\beta})E(z_1)]$.

Kernel Prior Estimation

For deriving the kernel prior, we introduce the bivariate kernel density estimator for the unknown probability density function $\boldsymbol{g}(z_1, z_2)$ with support on ($\mathbf{0}$,∞), which is defined as

$$\hat{\boldsymbol{g}}(\boldsymbol{z_1},\boldsymbol{z_2}) = \frac{\mathbf{1}}{\boldsymbol{D h_1 h_2}}\sum_{\boldsymbol{i=1}}^{\boldsymbol{D}} \boldsymbol{K}\left(\frac{\boldsymbol{z_1 - z_{1i}}}{\boldsymbol{h_1}},\frac{\boldsymbol{z_2-z_{2i}}}{\boldsymbol{h_2}}\right), (11)$$

$\boldsymbol{h_i, i=1,2}$ are called the bandwidths or smoothing parameters, which chosen such that $h_i\to 0$ and $\boldsymbol{D h_i}\to\infty$ as $\boldsymbol{D}\to\infty$, where $\boldsymbol{D}$ is the sample size. The influence of the smoothing parameters h_i are critical because they determine the amount of smoothing. Too small value of h_i may cause the estimator to show insignificant details while too large value of h_i cause over smoothing of the information contained in the sample, which in consequence, may mask some of the important characteristics. Thus, a certain compromise is needed. However, the optimal choice for h_i which minimize the mean squared errors are $\mathbf{hi=1.06\ S_i\ D^{-0.2}}$ and $\boldsymbol{S_i}$ the sample standard deviations. The optimal choice for the kernel function K (.,.) can be used as the bivariate standard normal distribution. The basic elements associated with the kernel density estimation function have been studied extensively in [27, 28]. Also, a good discussion of the kernel estimation techniques can be found in [29].

Based on the pivotal quantities, which are functions of the MLEs and whose distributions are free of the unknown parameters, the kernel prior estimate can be derived using the following algorithm:

1. Generate a random sample $\underline{\mathbf{X}}\mathbf{=(x_1,x_2,x_3,...,x_D)}$ from the parent distribution $\mathbf{f(x;\alpha,\beta)}$ with a given specified values for the unknown parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.
2. Bootstrapping with replacement D samples $\mathbf{x_1^*,x_2^*,x_3^*,...,x_D^*}$, with size D each, where $\mathbf{x_i^*=(x_{i1}^*,x_{i2}^*,x_{i3}^*,...,x_{iD}^*)}$ for $\mathbf{i=1,2,....,D}$ from the given random sample in step 1.
3. For each sample in step 2, calculate the MLEs for the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, thus the pivotal quantities $\boldsymbol{Z_1}\mathbf{=(z_{11},z_{12},...,z_{1D})}$ and $\boldsymbol{Z_2}\mathbf{=(}z_{21}, z_{22}\mathbf{,...,z_{2D})}$ can be derived.
4. Finally, based on the random variables $\boldsymbol{Z_1}$ and $\boldsymbol{Z_2}$, the kernel density estimation (11) can be used to derive the kernel density estimator for the density function of the pivotal quantities $\boldsymbol{Z_1}$ and $\boldsymbol{Z_2}$, say $\hat{\boldsymbol{g}}\boldsymbol{(z_1, z_2)}$**,** which is the kernel prior estimate.

It is worthwhile to mention that this kernel prior estimate has been used for some distributions, see [9, 30-32].

Simulation Study

For studying the performance of the improved conditional and Bayes methods, through the root mean squared error (RMSE):

$$\mathrm{RMSE}(\theta^*) = \sqrt{\textstyle\sum_{i=1}^{M}(\theta-\theta^*)^2\,/M}.$$

Here θ^* is the estimator for the unknown parameter θ and M is number of replications.

In our simulation study we choose different combinations for the prior hyperparameters of α and β say: (a,b,c.d)=[(5,3,8,2);(7,8,9,5)]. The true values of the parameter α=(2,3) and for the parameter β=(2,4) respectively. Using the above values of the parameters for generating different samples from the Weibull distribution with sizes n=20,40 and 60 to represent small, moderate and large sizes. To assess the performance of these estimates, the RMSEs for each one were calculated using 1000 replications.
The generation of the generalized progressive hybrid censored order statistics has been described in [12, 13].

From the simulation results in Tables 4,5,6 and 7, it is seen that, some of the points are quite clear based on these estimates and the others have been summarized in the following main points:

i. It is clear that, generally for both parameters the estimated RMSE values based on the improved conditional (IMPC) method are often less than the corresponding values based on the Bayes method for the different loss functions.
ii. The estimated RMSE values increase as the values of α and β increase for the IMPC, while decreasing as the value of β increases for the Bayes method.
iii. It is evident that, the estimated RMSE values decrease with increasing the hyperparameters, the termination time of the experiment T and the sample sizes as expected for both method.
iv. The estimated RMSE values based on the squared error loss function are less than those based on the Linex and Stein loss functions.
v. The estimated RMSE values based on LINEX loss function for negative shape parameter are greter than for positive shape parameter, which ensures overestimation are not serious.
vi. Generally, the estimated RMSE values based on LINEX loss function are less than for Stein's loss function for positive shape parameter.

As a conclusion, it appears that the point estimates based on the IMPC method compete and outperform the Bayes method.

Real Data Analysis

In this section, we studied two sets of real data to study the performance of the proposed methods on Weibull model, which is the most widely used and desirable lifetime distribution. Thus, we have fitted these data sets using some goodness of fit tests such as the Kolmogorov-Smirnov (K-S), Anderson-darling (A-D) and Chi-Square (CH2) tests for significance level test equals 0.05. Presented a comprehensive study of these tests [11, 33].

Vinyl Chloride Data Application

As vinyl chloride is a known human carcinogen, exposure to this compound should be avoided as far as practicable, and levels should be kept as low as technically feasible. Where it is known that a concentration of vinyl chloride in drinking-water of 0.5 mg/liter was calculated as being associated with an excess risk of liver and Brain tumors for exposure beginning at adulthood and it would double cancer risk for continuous exposure from birth. Therefore, we consider dataset used by which represents 34 data points in mg/L from the vinyl chloride that obtained from clean upgrade monitoring wells as [34].

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3,
3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

We found the Weibull model is a good fit for this dataset as shown in Table 2 and the Figure (1 a). For studying the concentration of the vinyl chloride in the water of these wells based on this dataset we find the estimates for the parameters which represent the scale and the shape of the concentration using our model to determine the average concentration in the water. We noticed that the IMPC and Bayes estimates for α lie in the interval [0.0018, 0.22], which indicates that the above dataset is moderately right skewed and that means the concentration decreases with increasing time, see Figure (1 b). Also, the IMPC and Bayes estimates for β lie in the interval [0.015, 0.15] which ensure the dataset is right-skewed and the vinyl chloride concentration will decrease with increasing time and therefore monitoring these wells is very significant.

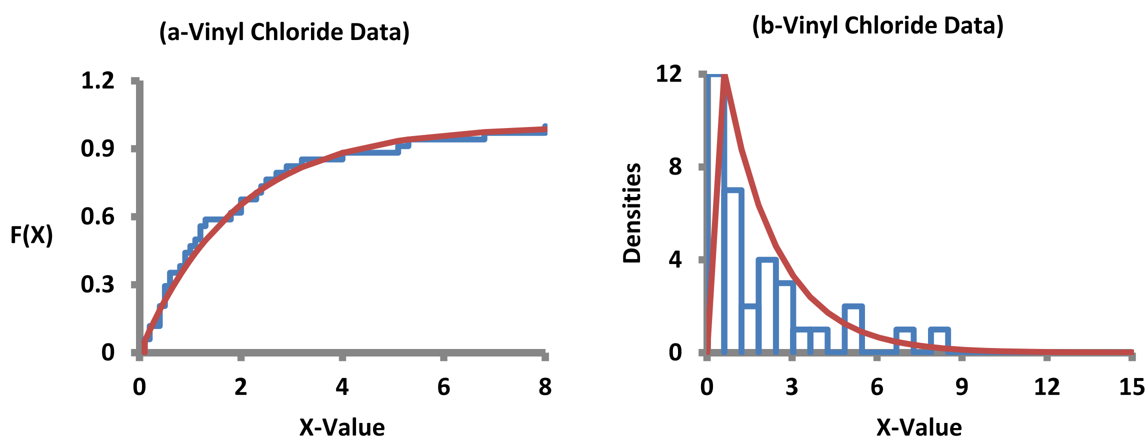

Figure 1: (a) The Empirical and the estimated CDF for the Vinyl Chloride Data.
(b) The Histogram and the estimated densities PDF for the Vinyl Chloride Data.

Leukemia Data Application

In the area of health care, leukemia affects blood status and can be discovered by using the Blood Cell Counter (BCC). Mostly, leukemia patients undergo chemotherapy treatment. Therefore, we study the effect of this treatment on the leukaemia patients based on a dataset collected by the Ministry of Health Hospital in Saudi Arabia and used by which indicates the lifetimes in days for forty-three blood patients with leukemia after they are given chemotherapy treatment [5]:

115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1025, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852, 1899, 1925, 1965.

We found the Weibull model is a good fit for this dataset as shown in Table 2 and the Figure (2 a). For studying the effect of the chemotherapy treatment on the patients based on this dataset we find the estimates for the distribution parameters, which represent the scale and the shape of the lifetime. We noticed that the IMPC and Bayes estimates for α lie in the interval [0.135, 1.07], which are greater than one. However, the IMPC and Bayes estimates for β lie in the interval [0.0011, 0.08], which are approximately zero that means the curve that represent this dataset is approximately symmetric, see Figure (2b). Therefore in general, based on this model, this dataset indicates the patient's lifetimes are more stable and they live longer due to the dose of the chemotherapy treatment, which is very efficient for giving the patients more antibodies against cancer.

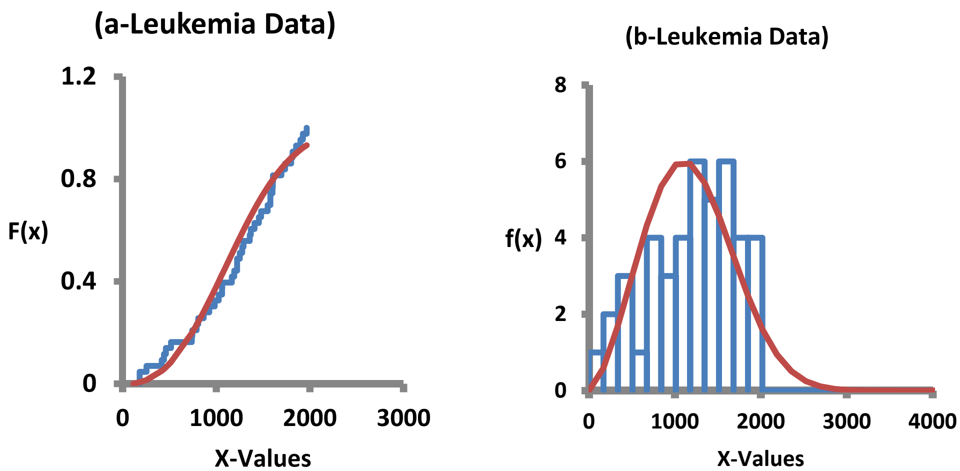

Figure 2: a) The Empirical and the estimated CDF for the Leukemia Data.
b) The Histogram and the estimated densities PDF for the Leukemia Data.

Table 2: The critical and calculated values for the K-S, A-D and CH2 tests and the powers (p-values) for Weibull model. The MLE's for the parameters for these data sets have been calculated.

Data	The Tests	Critical value	Calculated value	The P-values	$\hat{\alpha}$	$\hat{\beta}$
Chloride data N=34	K-S	0.8624	0.5355	0.6525	1.0102	0.5263
	A-D	0.7504	0.2826	0.6708		
	CH2	15.428	4.9912	0.4474		
Leukemia data N=43	K-S	0.8699	0.7285	0.1915	2.5533	1.04E-08
	A-D	0.7598	0.9159	0.0206		
	CH2	15.399	12.409	0.0528		

Table 3: The estimated parameters and the root mean squared errors (RMSEs) for the parameters α and based on the IMPC and Bayes methods at the hyper parameters (A=5, B=3, C=8, D=2) for the GPHCS: basd on $m = n/2$, $k = m/2$

Data	T	Parameters	Improved Conditional Method				Bayes Method			
			Squard Loss	Stein' Loss	Linex Loss		Squard Loss	Stein' Loss	Linex Loss	
					$\delta = -2$	$\delta = 2$			$\delta = -2$	$\delta = 2$
Vinyl Chl. Data N=34	0.75	α	0.0018	0.0123	0.0180	0.0143	0.0023	0.0823	0.1285	0.0992
		β	0.0156	0.0139	0.0135	0.0177	0.2014	0.1739	0.2145	0.1892
	1.5	α	0.0057	0.0049	0.0179	0.0064	0.0753	0.1438	0.0083	0.1454
		β	0.0082	0.0087	0.0063	0.0099	0.1104	0.0952	01144	0.1067
	3.5	α	0.0092	0.0177	0.0009	0.0193	0.1991	0.2324	01673	0.2286
		β	0.0155	0.0157	0.0136	0.0017	0.1593	0.1359	0.1686	0.1506
Leuk. Data N=43	850	α	0.1358	0.1225	0.1534	0.1182	0.2464	0.3847	0.3198	0.2705
		β	0.0017	0.0019	0.0012	0.0021	0.1171	0.0993	0.1194	0.1148
	1250	α	0.1433	0.1338	0.1562	0.1304	0.8068	0.8372	0.786	0.8261
		β	0.0010	0.0013	0.0068	0.0014	0,0813	0.0687	0.0825	0.0802
	1700	α	0.0851	0.0737	0.0999	0.0704	0.0467	0.5746	0.4208	0.2709
		β	0.0011	0.0013	0.0083	0.0013	0.0900	0.0747	0.0916	0.0885

The results in Table 2, refer that the Weibull model is a good fit for these data sets where the calculated value are less than the critical values for the goodness of fit tests and the power of the tests are greater than the significance level of the tests. Also, the results in Table 3 indicate that the estimated RMSE values based on the improved conditional method are less than those based on the Bayes method for large values of T with considering the MLEs are the true values of the parameters. The estimated values of RMSE based on LINEX loss function are less than those under Stein's loss function, while the results under the squared error loss function are less than the other loss functions. Thus, the results of these data sets ensure the simulation results.

Conclusions

It is known that, the Bayesian inference based on the informative prior is more efficient than the conditional inference. However, using the kernel prior leads to the improvement of the conditional inference. It becomes strongly unbiased and much more efficient than the Bayesian inference even when using informative priors based on different loss functions.

Conflicts of Interest

The author declares no conflict of interest.

Table 4: The root mean square errors (RMSEs) for the Weibull parameter using the IMPC and Bayes estimations at T=0.75 with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) for different values of α and .

N	M	K	α	β	Improved Conditional Estimations				Bayes Estimations			
					Squared Loss	Stein's Loss	LINEX Loss δ= -2	LINEX Loss δ= 2	Squared Loss	Stein's Loss	LINEX Loss δ= -2	LINEX Loss δ= 2
20	10	5	2	*2*	0.1298	0.1475	0.1219	0.1432	0.4123	0.3861	0.6131	0.3535
				4	0.4378	0.4752	0.4111	0.4644	0.4583	0.5330	0.3833	0.5616
			3	*2*	0.1830	0.1848	0.1981	0.1803	0.5079	0.5571	0.8503	0.6694
				4	0.3705	0.4185	0.3329	0.4095	0.7251	0.8443	0.5311	0.9641
		8	2	*2*	0.1213	0.1365	0.1158	0.1327	0.4403	0.4019	0.6617	0.3533
				4	0.3359	0.3666	0.3124	0.3594	0.4291	0.4902	0.3587	0.5165
			3	*2*	0.1672	0.1782	0.1711	0.1738	0.5032	0.5185	0.8377	0.5833
				4	0.3004	0.3411	0.2674	0.3345	0.6467	0.7446	0.4769	0.8546
	15	8	2	*2*	0.1229	0.1357	0.1211	0.1314	0.5025	0.4349	0.7285	0.3447
				4	0.1231	0.1371	0.1195	0.1330	0.2596	0.2709	0.3010	0.2786
			3	*2*	0.1854	0.2072	0.1788	0.2013	0.4411	0.4375	0.7485	0.4819
				4	0.1799	0.2005	0.1742	0.1951	0.4581	0.5253	0.3864	0.6234
		11	2	*2*	0.1105	0.1201	0.1101	0.1168	0.4559	0.4034	0.6487	0.3346
				4	0.1082	0.1167	0.1086	0.1138	0.2591	0.2653	0.3073	0.2691
			3	*2*	0.1630	0.1745	0.1644	0.1705	0.4672	0.4564	0.7516	0.4731
				4	0.1634	0.1783	0.1608	0.1743	0.4471	0.5074	0.3854	0.5969
40	20	10	2	*2*	0.1255	0.1364	0.1199	0.1341	0.3521	0.3853	0.3549	0.4010
				4	0.4467	0.4703	0.4296	0.4638	0.6378	0.6897	0.5667	0.7037
			3	*2*	0.1766	0.1759	0.1855	0.1734	0.5269	0.5895	0.5465	0.6796
				4	0.3725	0.4036	0.3472	0.3982	0.8831	0.9639	0.7127	1.0387
		15	2	*2*	0.1158	0.1261	0.1104	0.1239	0.3099	0.3156	0.3654	0.3150
				4	0.3191	0.3389	0.3035	0.3346	0.4867	0.5279	0.4267	0.5453
			3	*2*	0.1551	0.1591	0.1587	0.1569	0.4404	0.4549	0.5803	0.4935
				4	0.2899	0.3167	0.2679	0.3126	0.6608	0.7241	0.5254	0.7983

	30	15	2	*2*	0.1226	0.1330	0.1172	0.1308	0.3313	0.3438	0.3756	0.3463
				4	0.3487	0.3692	0.3328	0.3646	0.5208	0.5645	0.4577	0.5814
			3	*2*	0.1631	0.1713	0.1629	0.1686	0.4351	0.4365	0.6031	0.4561
				4	0.1703	0.1823	0.1663	0.1793	0.3756	0.4171	0.3252	0.4840
		23	2	*2*	0.1046	0.1037	0.1094	0.1024	0.3966	0.3647	0.4972	0.3203
				4	0.1037	0.1068	0.1047	0.1054	0.2241	0.2268	0.2456	0.2272
			3	*2*	0.1727	0.1646	0.1857	0.163	0.5593	0.5153	0.7832	0.4193
				4	0.1484	0.1505	0.1521	0.1488	0.3385	0.3633	0.3312	0.4080
60	30	15	2	*2*	0.1133	0.1209	0.1088	0.1195	0.3241	0.3514	0.3001	0.3665
				4	0.3974	0.4133	0.3853	0.4095	0.6234	0.6557	0.5785	0.6659
			3	*2*	0.1767	0.1853	0.1743	0.1828	0.4376	0.4362	0.5569	0.4409
				4	0.1867	0.1983	0.1812	0.1956	0.3328	0.3691	0.2833	0.4304
		23	2	*2*	0.1124	0.1211	0.1080	0.1185	0.2947	0.3166	0.2836	0.3292
				4	0.3693	0.3848	0.3573	0.3813	0.5841	0.6155	0.5394	0.6266
			3	*2*	0.1538	0.1571	0.1559	0.1552	0.4243	0.4262	0.5251	0.4346
				4	0.1682	0.1753	0.1663	0.1733	0.3270	0.3540	0.3005	0.4013
	45	23	2	*2*	0.1052	0.1104	0.1038	0.1089	0.3635	0.3351	0.4517	0.2968
				4	0.1074	0.1115	0.1070	0.1101	0.1961	0.1996	0.2123	0.2016
			3	*2*	0.1548	0.1636	0.1515	0.1615	0.4054	0.3863	0.5547	0.3602
				4	0.1609	0.1680	0.1591	0.1660	0.3221	0.3497	0.2943	0.3982
		34	2	*2*	0.0999	0.0968	0.1056	0.0959	0.3724	0.3469	0.4468	0.3110
				4	0.0997	0.1012	0.1009	0.1003	0.2041	0.2043	0.2209	0.2027
			3	*2*	0.1833	0.1735	0.1959	0.1724	0.6063	0.5653	0.7887	0.4645
				4	0.1416	0.1437	0.1435	0.1424	0.2944	0.3107	0.2897	0.3419

Table 5: The root mean square errors (RMSEs) for the Weibull parameter α using the IMPC and Bayes estimations at T=1.5 with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) for different values of α and β.

N	M	K	α	β	Improved Conditional Estimations				Bayes Estimations			
					Squared Loss	Stein's Loss	LINEX Loss		Squared Loss	Stein's Loss	LINEX Loss	
							δ= -2	δ= 2			δ= -2	δ= 2
20	10	5	2	*2*	0.1133	0.1250	0.1113	0.1214	0.4661	0.4110	0.6704	0.3363
				4	0.1093	0.1207	0.1073	0.1175	0.2492	0.2606	0.2893	0.2693
			3	*2*	0.1662	0.1842	0.1619	0.1795	0.4426	0.4458	0.7073	0.4890
				4	0.1762	0.1940	0.1713	0.1893	0.4706	0.5316	0.4105	0.6204
		8	2	*2*	0.1144	0.1251	0.1135	0.1214	0.4733	0.4158	0.6771	0.3399
				4	0.1159	0.1273	0.1137	0.1239	0.2627	0.2731	0.3031	0.2795
			3	*2*	0.1667	0.1842	0.1630	0.1795	0.4597	0.4559	0.7502	0.4875
				4	0.1667	0.1841	0.1624	0.1797	0.4526	0.5163	0.3823	0.6095
	15	8	2	*2*	0.1054	0.1080	0.1091	0.1062	0.3871	0.3569	0.5136	0.3163
				4	0.1033	0.1070	0.1061	0.1050	0.2706	0.2741	0.3139	0.2735
			3	*2*	0.1571	0.1583	0.1649	0.1561	0.4564	0.4639	0.6208	0.4905
				4	0.1605	0.1658	0.1641	0.1631	0.4354	0.4780	0.4178	0.5454
		11	2	*2*	0.1051	0.1075	0.1091	0.1056	0.3914	0.3591	0.5226	0.3160
				4	0.1094	0.1126	0.1124	0.1106	0.2774	0.2780	0.3261	0.2750
			3	*2*	0.1607	0.1644	0.1663	0.1619	0.4585	0.4702	0.6085	0.5021
				4	0.1539	0.1587	0.1582	0.1562	0.4271	0.4710	0.4043	0.5406

40	20	10	2	*2*	0.1033	0.1075	0.1038	0.1059	0.3786	0.3469	0.4833	0.3041
				4	0.1044	0.1095	0.1039	0.1078	0.2181	0.2242	0.2357	0.2281
			3	*2*	0.1555	0.1649	0.1530	0.1625	0.4132	0.4061	0.5553	0.4045
				4	0.1552	0.1625	0.1548	0.1602	0.3609	0.3934	0.3318	0.4481
		15	2	*2*	0.1040	0.1076	0.1052	0.1060	0.3974	0.3645	0.5052	0.3189
				4	0.1038	0.1079	0.1046	0.1062	0.2152	0.2194	0.2368	0.2217
			3	*2*	0.1587	0.1669	0.1573	0.1645	0.4247	0.4144	0.5746	0.4066
				4	0.1607	0.1670	0.1611	0.1647	0.3626	0.3931	0.3417	0.4451
	15	15	2	*2*	0.2073	0.2187	0.1978	0.2171	0.3612	0.3815	0.3362	0.3916
				4	0.4551	0.4668	0.4461	0.4638	0.5907	0.6131	0.5591	0.6212
			3	*2*	0.5164	0.5345	0.5011	0.5315	0.8774	0.9136	0.7996	0.9509
				4	0.8789	0.8944	0.8680	0.8899	0.1395	0.1689	0.2841	0.1924
		23	2	*2*	0.2240	0.2371	0.2127	0.2353	0.3740	0.4003	0.3331	0.4149
				4	0.4511	0.4629	0.4422	0.4599	0.5831	0.6054	0.5511	0.6138
			3	*2*	0.5130	0.5311	0.4978	0.5282	0.8706	0.9069	0.7922	0.9446
				4	0.8823	0.8978	0.8714	0.8933	0.1433	0.1727	0.2882	0.1960
60	30	15	2	*2*	0.2397	0.2523	0.2291	0.2503	0.4009	0.4255	0.3624	0.4388
				4	0.4602	0.4714	0.4519	0.4685	0.6175	0.6374	0.5901	0.6442
			3	*2*	0.5338	0.5513	0.5196	0.5480	0.8964	0.9297	0.8260	0.9637
				4	0.8656	0.8803	0.8552	0.8761	0.1451	0.1717	0.2957	0.1929
		23	2	*2*	0.2428	0.2555	0.2322	0.2535	0.4067	0.4314	0.3682	0.4445
				4	0.4595	0.4707	0.4512	0.4678	0.6161	0.6360	0.5886	0.6429
			3	*2*	0.5375	0.5549	0.5234	0.5517	0.9019	0.9350	0.8321	0.9686
				4	0.8667	0.8814	0.8563	0.8770	0.1478	0.1744	0.2987	0.1954
	45	23	2	*2*	0.2680	0.2781	0.2594	0.2767	0.4085	0.4275	0.3786	0.4378
				4	0.5217	0.5305	0.5153	0.5280	0.6358	0.6514	0.6144	0.6567
			3	*2*	0.6026	0.6163	0.5913	0.6138	0.9051	0.9305	0.8513	0.9567
				4	0.9885	1.0011	0.9806	0.9963	0.1944	0.2148	0.1573	0.2303
		34	2	*2*	0.2605	0.2702	0.2521	0.2688	0.3949	0.4133	0.3658	0.4235
				4	0.5791	0.5879	0.5728	0.5852	0.6865	0.7022	0.6658	0.7067
			3	*2*	0.6733	0.6872	0.6622	0.6843	0.9784	0.8040	0.9262	0.8284
				4	0.7781	0.6895	0.5707	0.5855	0.2704	0.2907	0.2349	0.3047

Table 6: The root mean square errors (RMSEs) for the Weibull parameter β using the IMPC and Bayes estimations at T=0.75 with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) for different values of α and β .

N	M	K	α	β	Improved Conditional Estimations				Bayes Estimations			
					Squared Loss	Stein's Loss	LINEX Loss		Squared Loss	Stein's Loss	LINEX Loss	
							δ= -2	δ= 2			δ= -2	δ= 2
20	10	5	2	2	0.2226	0.2424	0.1994	0.2460	0.4891	0.6125	0.4366	0.6501
				4	0.7446	0.7874	0.6888	0.8001	2.8105	0.9337	0.6112	0.9386
			3	2	0.2849	0.3085	0.2561	0.3138	0.6293	0.7956	0.3505	0.8202
				4	0.8626	0.9093	0.8046	0.9196	0.6503	0.8063	0.3150	0.8281
		8	2	2	0.2144	0.2339	0.1915	0.2375	0.4271	0.5570	0.3510	0.6041
				4	0.5457	0.5819	0.4997	0.5917	0.3673	0.5097	0.0349	0.5645
			3	2	0.1902	0.2088	0.1673	0.2133	0.3624	0.4905	0.6018	0.5472
				4	0.5574	0.5939	0.5106	0.6040	0.1859	0.3607	0.6526	0.4453
	15	8	2	2	0.0762	0.0902	0.0619	0.0945	0.3351	0.2387	0.9684	0.6182
				4	0.1397	0.1623	0.1135	0.1735	0.4509	0.5714	0.5296	0.6553
			3	2	0.0864	0.1008	0.0696	0.1060	0.9968	0.7422	0.9020	0.4360
				4	0.1831	0.2083	0.1496	0.2208	0.5849	0.8463	0.0987	0.2913
		11	2	2	0.0665	0.0777	0.0570	0.0807	1.1533	0.9372	0.7420	0.6094
				4	0.1316	0.1521	0.1096	0.1607	0.5474	0.5899	0.3813	0.9537
			3	2	0.0553	0.0660	0.0461	0.0697	0.9474	0.7463	0.9004	0.4759
				4	0.1360	0.1566	0.1115	0.1661	0.6049	0.7757	0.6104	0.1528
40	20	10	2	2	0.3123	0.3285	0.2936	0.3310	0.9237	0.9018	0.8232	0.8008
				4	0.8453	0.8781	0.8044	0.8859	0.9455	0.1168	0.9640	0.9088
			3	2	0.3066	0.3238	0.2859	0.3273	0.8557	0.9606	0.6893	0.9642
				4	0.9665	1.0006	0.9256	0.9068	0.8854	0.9810	0.7433	0.9822
		15	2	2	0.2039	0.2178	0.1874	0.2205	0.5549	0.6427	0.4151	0.6699
				4	0.5302	0.5560	0.4907	0.5634	0.4282	0.5281	0.2271	0.5677
			3	2	0.1743	0.1876	0.1577	0.1911	0.4321	0.5242	0.3996	0.5630
				4	0.5302	0.5559	0.4966	0.5637	0.2105	0.3384	0.8788	0.4074
	15	15	2	2	0.2341	0.2486	0.2170	0.2512	0.6734	0.7598	0.5358	0.7776
				4	0.6167	0.6441	0.5821	0.6512	0.6203	0.7099	0.4654	0.7319
			3	2	0.1514	0.1632	0.1369	0.1665	0.7465	0.6686	0.1956	0.5599
				4	0.1715	0.1905	0.1451	0.2003	0.6289	0.7965	0.8891	0.1661
		23	2	2	0.0466	0.0502	0.0466	0.0513	0.8674	0.7440	0.5582	0.5493
				4	0.1046	0.1158	0.0947	0.1206	0.6604	0.6338	0.7070	0.7800
			3	2	0.0420	0.0366	0.0522	0.0351	0.8323	0.8818	0.8482	0.6465
				4	0.0906	0.0994	0.0843	0.1041	0.6121	0.6557	0.4376	0.8804
60	30	15	2	2	0.3049	0.3172	0.2909	0.3189	0.9070	0.9609	0.8397	0.9626
				4	0.8019	0.8267	0.7724	0.8314	0.8771	0.9311	0.8075	0.9343
			3	2	0.1958	0.2071	0.1820	0.2103	0.9768	0.8273	0.7881	0.6552
				4	0.2583	0.2811	0.2253	0.2922	0.6106	0.8346	6.0489	1.2523
		23	2	2	0.2583	0.2701	0.2446	0.2720	0.7804	0.8378	0.7063	0.8453
				4	0.7053	0.7285	0.6769	0.7336	0.7337	0.7940	0.6447	0.8051
			3	2	0.1747	0.1846	0.1627	0.1871	0.7027	0.6710	0.3654	0.6088
				4	0.1703	0.1865	0.1473	0.1946	0.6104	0.7218	0.5894	0.5366

	45	23	2	2	0.0801	0.0881	0.0712	0.0905	0.9509	0.7903	0.3945	0.5554
				4	0.1287	0.1424	0.1109	0.1493	0.6240	0.6248	0.6626	0.8788
			3	2	0.0793	0.0881	0.0683	0.0914	0.7683	0.6248	0.9298	0.4364
				4	0.1754	0.1916	0.1524	0.1997	0.6193	0.7331	0.5488	0.6466
		34	2	2	0.0427	0.0442	0.0443	0.0448	0.7437	0.6503	0.1397	0.5074
				4	0.1067	0.1156	0.0980	0.1194	0.6862	0.6481	0.2781	0.7088
			3	2	0.0480	0.0415	0.0581	0.0392	0.8594	0.9349	0.6598	0.7284
				4	0.0945	0.1020	0.0879	0.1058	0.6072	0.6198	0.8431	0.7703

Table 7: The root mean square errors (RMSEs) for the Weibull parameter using the IMPC and Bayes estimations at T=1.5 with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) for different values of α and β .

N	M	K	α	β	Improved Conditional Estimations				Bayes Estimations			
					Squared Loss	Stein's Loss	LINEX Loss		Squared Loss	Stein's Loss	LINEX Loss	
							δ= -2	δ= 2			δ= -2	δ= 2
20	10	5	2	2	0.0747	0.0870	0.0629	0.0903	0.1852	0.9491	0.4316	0.6027
				4	0.1349	0.1561	0.1117	0.1653	0.5231	0.5093	0.3249	0.9959
			3	2	0.0694	0.0816	0.0567	0.0857	0.9531	0.7419	0.4498	0.4668
				4	0.1550	0.1771	0.1272	0.1875	0.6143	0.8101	0.3663	0.2057
		8	2	2	0.0679	0.0801	0.0571	0.0834	0.2175	0.9742	0.5556	0.6154
				4	0.1363	0.1568	0.1145	0.1658	0.5431	0.5942	0.5769	0.9855
			3	2	0.0711	0.0833	0.0584	0.0873	0.9371	0.7280	0.3232	0.4581
				4	0.1573	0.1791	0.1301	0.1894	0.6194	0.8111	0.4087	0.2048
	15	8	2	2	0.0651	0.0727	0.0608	0.0742	0.9215	0.7884	0.8024	0.5707
				4	0.1204	0.1361	0.1081	0.1416	0.6378	0.6101	0.4858	0.8052
			3	2	0.0601	0.0669	0.0563	0.0688	0.7414	0.6238	0.4157	0.4514
				4	0.1014	0.1131	0.0950	0.1182	0.5889	0.6663	0.6144	0.9485
		11	2	2	0.0679	0.0757	0.0633	0.0772	0.9056	0.7739	0.7286	0.5609
				4	0.1197	0.1365	0.1057	0.1423	0.6255	0.6041	0.4471	0.8095
			3	2	0.0611	0.0677	0.0578	0.0694	0.7393	0.6221	0.4729	0.4500
				4	0.1039	0.1150	0.0982	0.1199	0.6078	0.6711	0.7795	0.9401
40	20	10	2	2	0.0571	0.0642	0.0514	0.0663	0.9033	0.7573	0.9087	0.5380
				4	0.1122	0.1259	0.0976	0.1321	0.6092	0.6107	0.8121	0.8481
			3	2	0.0559	0.0627	0.0501	0.0651	0.7486	0.6222	0.5901	0.4462
				4	0.1171	0.1305	0.1014	0.1373	0.6221	0.7093	0.0609	0.9934
		15	2	2	0.0590	0.0659	0.0535	0.0679	0.9390	0.7907	0.0382	0.5612
				4	0.1079	0.1212	0.0941	0.1273	0.6292	0.6064	0.0677	0.8215
			3	2	0.0553	0.0623	0.0491	0.0649	0.7360	0.6102	0.5939	0.4360
				4	0.1143	0.1280	0.0983	0.1349	0.6119	0.6996	0.0724	0.9872
	15	15	2	2	0.1324	0.1431	0.1207	0.1446	0.5150	0.5325	0.5474	0.5275
				4	0.2965	0.3196	0.2681	0.3252	0.9741	0.8365	0.8345	0.0912
			3	2	0.1557	0.1680	0.1413	0.1702	0.6235	0.6614	0.5690	0.6733
				4	0.3102	0.3351	0.2796	0.3409	0.2317	0.2838	0.1319	0.3181
		23	2	2	0.1462	0.1582	0.1327	0.1599	0.4895	0.5304	0.4262	0.5481
				4	0.3036	0.3267	0.2751	0.3323	0.9805	0.8426	0.8420	0.9967
			3	2	0.1599	0.1723	0.1455	0.1744	0.6316	0.6692	0.5777	0.6806
				4	0.3102	0.3351	0.2797	0.3409	0.2321	0.2841	0.1323	0.3184

60	30	15	2	*2*	0.1505	0.1611	0.1378	0.1633	0.5589	0.6013	0.4946	0.6168
				4	0.3380	0.3577	0.3121	0.3639	0.1225	0.1848	0.9923	0.2308
			3	*2*	0.1886	0.1998	0.1750	0.2022	0.7378	0.7735	0.6904	0.7810
				4	0.4361	0.4586	0.4075	0.4647	0.3897	0.4393	0.3030	0.4649
		23	2	*2*	0.1519	0.1625	0.1392	0.1647	0.5652	0.6075	0.5014	0.6228
				4	0.3416	0.3614	0.3157	0.3675	0.1284	0.1905	0.9992	0.2360
			3	*2*	0.1849	0.1962	0.1713	0.1985	0.7311	0.7671	0.6829	0.7748
				4	0.4279	0.4504	0.3993	0.4565	0.3814	0.4313	0.2937	0.4573
	45	23	2	*2*	0.1710	0.1813	0.1595	0.1826	0.5928	0.6209	0.5518	0.6311
				4	0.3567	0.3768	0.3322	0.3813	0.1517	0.1927	0.0709	0.2237
			3	*2*	0.1945	0.2053	0.1822	0.2069	0.7374	0.7626	0.7043	0.7683
				4	0.3774	0.3993	0.3510	0.4039	0.3776	0.4119	0.3189	0.4311
		34	2	*2*	0.1677	0.1777	0.1564	0.1790	0.5697	0.5967	0.5298	0.6070
				4	0.4039	0.4251	0.3784	0.4295	0.3139	0.3515	0.2467	0.3744
			3	*2*	0.2262	0.2376	0.2133	0.2391	0.8275	0.8514	0.7984	0.8547
				4	0.4302	0.4535	0.4024	0.4579	0.5180	0.5497	0.4688	0.5631

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