ISSN: 2769-545X

Research Article

Advances in Machine Learning & Artificial Intelligence

About Structure of a connected Quaternion-JUllA-Set and Symmetries of u related JULIA-Network

Udo E. Steinemann*

Findeisen-Str, 5/7, 71665 Vaihingen an der Enz, Germany.

*Corresponding Author
Udo E. Steinemann, Findeisen-Str, 5/7, 71665 Vaihingen an der Enz, Germany.

Submitted: 2024, Mar 12; **Accepted**: 2024, Apr 23; **Published**: 2024, Apr 29

Citation: Udo E. Steinemann. (2024). About Structure of a connected Quaternion-JUllA-Set and Symmetries of u related JULIA-Network. *Adv Mach Lear Art Inte, 5*(2), 01-18.

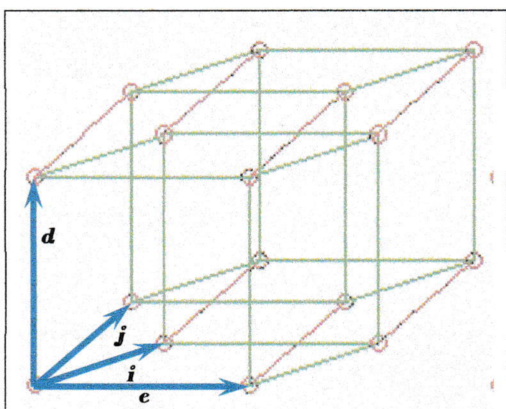
Abstract

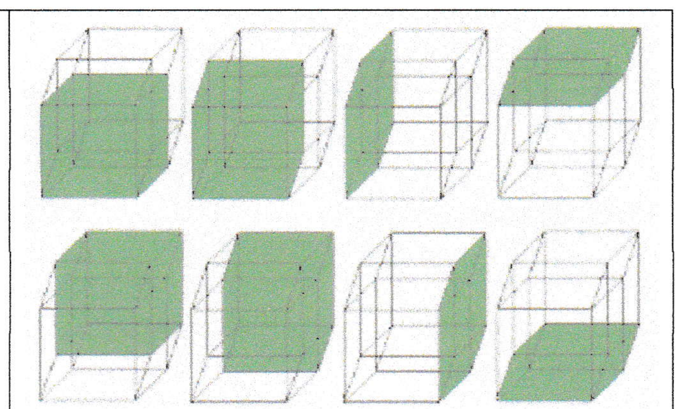
If a variable is replace by its square and subsequently enlarged by a constant during a number of iteration-steps in quaternion-space, a network of (3) sets will be built gradually. As long as for the iteration-constant certain conditions are fulfilled, the network will consist of: an rrnbounded set (escape-set) with trajectories escaping to infinity during course of the iteration, a bounded set (prisoner-set) with trajectories tending to a sink-point and a further bounded one (JULIA-set) with a fixed-point as repeller having a repulsive effect on all points of both the other sets. The iteration will continue until the attracting sink-point of prisoner-set and the repeliing fixedpoint on JULIA-set have been found. This situation is reached if predecessor- and successor-state of the iteration became equal. The fixed-point-condition provisionally formulated in general terms of quaternions, can be separated into (3) sub-conditions. When heeding the HAMILTONian-rules for interactions of the imaginary sub-spaces of the quaternion-space, each sub-condition will be appropriate for one imaginary subspaces and independently debatable. Knowledge of fixed-points from this fundamental network will one enable to study the structure of a connected JULIA-set.

The Iteration will start from (1) on real-axis, this is not a restriction on generaiity because an appropriate scaling on real-axis can always be archived this way. It will become obvious, that the fixed-points in prisonerand JIILIA-set will depend on the iteration-constant only. Thus (16) different constants chosen appropriately will enable to arrange (16) fixed-points of JLTLIA-sets in the square-points of a hyper-cube and thereby together with the JULIA-sets to built a related JULIA-network. The symmetry-properties of this related .IULIA-network can be studied on base of a hyper-cube's symmetry-group extended by some additional considerations.

1. Introduction.

In the following attention is appiied to the results of an iteration, which takes place in **quaternion-space** (a **space of hyper-cubes** with its space-elements) a layout of this is given next:





Each hyper-cube:

- Is surrounded by **(8) cubes** each one with **(6) surfaces**. Thus all together, cubes will have **(48) surfaces.**
- Because the cubes wiil slmre surfaces, onlv **(24) surfaces** will have to be counted **effectively**.

The quaternion-space is spanned by a real **unit-vector (e)** vertical to a tripod of imaginary unit-vectors **{*i*^*j*^*d*,}**. Among these reference-vectors t}re **HAMILTONian rules** must hold:

$$1\hat{}1.\quad \boldsymbol{e}^2 = (-\boldsymbol{i}^2) = (-\boldsymbol{j}^2) = (-\boldsymbol{d}^2) = 1$$
$$[\![\boldsymbol{ij} = (-\boldsymbol{ji}) = \boldsymbol{d}]\!] \wedge [\![\boldsymbol{jd} = (-\boldsymbol{dj}) = \boldsymbol{i}]\!] \wedge [\![\boldsymbol{di} = (-\boldsymbol{id}) = \boldsymbol{j}]\!].$$

Any point in the space is given by:

- $\mathbb{Q} = \boldsymbol{e}\mathrm{Q}_0 + \boldsymbol{i}\mathrm{Q}_1 + \boldsymbol{j}\mathrm{Q}_2 + \boldsymbol{d}\mathrm{Q}_3 \quad \Rightarrow \quad \langle \mathbb{Q} = \text{quaternion}-\text{variable}\rangle \wedge \langle\, [\mathrm{Q}_0\hat{}\mathrm{Q}_1\hat{}\mathrm{Q}_2\hat{}\mathrm{Q}_3] = \text{real components}\rangle.$

A sequence:

$$1\hat{}2.\quad [\mathbb{Q} \to \mathbb{Q}^2 + (\mathbb{N} = \mathrm{N}_0 + \boldsymbol{i}\mathrm{N}_1 + \boldsymbol{j}\mathrm{N}_2 + \boldsymbol{d}\mathrm{N}_3\,)]^2 + \mathbb{N} \to \ldots \quad \Rightarrow \quad \langle \mathbb{N} = \text{constant}\rangle \wedge \langle [\mathrm{N}_0\hat{}\mathrm{N}_1\hat{}\mathrm{N}_2\hat{}\mathrm{N}_3] = \text{real components}\rangle$$

iteratively executed is to considered next, where by observing the HAMILTONian rules **(1^1.)** the following relations between $\mathbb{Q}$ and $\mathbb{Q}^2$ must hold:

Derivation 1^1.			
$\mathbb{Q} = \boldsymbol{e}\mathrm{Q}_0+\boldsymbol{i}\mathrm{Q}_1+\boldsymbol{j}\mathrm{Q}_2+\boldsymbol{d}\mathrm{Q}_3$	●		
leads to	↓		
$\mathbb{Q}^2 = (\boldsymbol{e}\mathrm{Q}_0+\boldsymbol{i}\mathrm{Q}_1+\boldsymbol{j}\mathrm{Q}_2+\boldsymbol{d}\mathrm{Q}_3)^2$	●		
$\mathbb{Q}^2 = \boldsymbol{e}^2\mathrm{Q}_0{}^2+\boldsymbol{i}^2\mathrm{Q}_1{}^2+\boldsymbol{j}^2\mathrm{Q}_2{}^2+\boldsymbol{d}^2\mathrm{Q}_3{}^2+$ $\boldsymbol{i}2\mathrm{Q}_0\mathrm{Q}_1+\boldsymbol{j}2\mathrm{Q}_0\mathrm{Q}_2+\boldsymbol{d}2\mathrm{Q}_0\mathrm{Q}_3+$ $\boldsymbol{i}(\boldsymbol{j}\mathrm{Q}_1\mathrm{Q}_2+\boldsymbol{d}\mathrm{Q}_1\mathrm{Q}_3)+$ $\boldsymbol{j}(\boldsymbol{i}\mathrm{Q}_2\mathrm{Q}_1+\boldsymbol{d}\mathrm{Q}_2\mathrm{Q}_3)+$ $\boldsymbol{d}(\boldsymbol{i}\mathrm{Q}_3\mathrm{Q}_1+\boldsymbol{j}\mathrm{Q}_3\mathrm{Q}_2)$	●	●	
leads to ♦ with	↓	↓	
$\boldsymbol{e}^2 = (-\boldsymbol{i}^2) = (-\boldsymbol{j}^2) = (-\boldsymbol{d}^2) = 1$ $\boldsymbol{i}\cdot\boldsymbol{j} = (-\boldsymbol{j}\cdot\boldsymbol{i}) = \boldsymbol{d}$ $\boldsymbol{j}\cdot\boldsymbol{d} = (-\boldsymbol{d}\cdot\boldsymbol{j}) = \boldsymbol{i}$ $\boldsymbol{d}\cdot\boldsymbol{i} = (-\boldsymbol{i}\cdot\boldsymbol{d}) = \boldsymbol{j}$		●	
$\mathbb{Q}^2 = \mathrm{Q}_0{}^2-\mathrm{Q}_1{}^2-\mathrm{Q}_2{}^2-\mathrm{Q}_3{}^2+$ $\boldsymbol{i}2\mathrm{Q}_1\mathrm{Q}_0+\boldsymbol{j}2\mathrm{Q}_2\mathrm{Q}_0+\boldsymbol{d}2\mathrm{Q}_3\mathrm{Q}_0+$ $\boldsymbol{d}\mathrm{Q}_1\mathrm{Q}_2-\boldsymbol{j}\mathrm{Q}_1\mathrm{Q}_3-\boldsymbol{d}\mathrm{Q}_2\mathrm{Q}_1+\boldsymbol{i}\mathrm{Q}_2\mathrm{Q}_3+\boldsymbol{j}\mathrm{Q}_3\mathrm{Q}-\boldsymbol{i}\mathrm{Q}_3\mathrm{Q}_2$	●		
leads to	↓		
$\mathbb{Q}^2 = \mathrm{Q}_0{}^2+\boldsymbol{i}2\mathrm{Q}_1\mathrm{Q}_0-\mathrm{Q}_1{}^2+$ $\mathrm{Q}_0{}^2+\boldsymbol{j}2\mathrm{Q}_2\mathrm{Q}_0-\mathrm{Q}_2{}^2+$ $\mathrm{Q}_0{}^2+\boldsymbol{d}2\mathrm{Q}_3\mathrm{Q}_0-\mathrm{Q}_3{}^2-2\mathrm{Q}_0{}^2$	●		
leads to	↓		
$\mathbb{Q}^2 = (\mathrm{Q}_0+\boldsymbol{i}\mathrm{Q}_1)^2+(\mathrm{Q}_0+\boldsymbol{j}\mathrm{Q}_2)^2+(\mathrm{Q}_0+\boldsymbol{d}\mathrm{Q}_3)^2-2\mathrm{Q}_0{}^2$	●		●
leads to ♦ with	↓		↓
$[\![\mathrm{Q}_i = \mathrm{Q}_0+\boldsymbol{i}\mathrm{Q}_1]\!] \wedge [\![\mathrm{Q}_j = \mathrm{Q}_0+\boldsymbol{j}\mathrm{Q}_2]\!] \wedge [\![\mathrm{Q}_d = \mathrm{Q}_0+\boldsymbol{d}\mathrm{Q}_3]\!]$			●
$\mathbb{Q} = (\mathrm{Q}_0+\boldsymbol{i}\mathrm{Q}_1)+(\mathrm{Q}_0+\boldsymbol{j}\mathrm{Q}_2)+(\mathrm{Q}_0+\boldsymbol{d}\mathrm{Q}_3)-2\mathrm{Q}_0$	●		

Without restriction on generality due to a free choice of an appropriate scaling on the e–axis, **(Q_0 =1)** can be assumed in **(1^2.)** and thus one may further write:

$$1\hat{}3.\quad [(\mathbb{P} = \mathrm{Q}_i+\mathrm{Q}_j+\mathrm{Q}_d-2) \to (\mathbb{P}^2 = \mathrm{Q}_i{}^2+\mathrm{Q}_j{}^2+\mathrm{Q}_d{}^2-2)+\mathbb{N}]^2+\mathbb{N} \to \ldots \quad \Rightarrow \quad \mathrm{N}_0 = \mathrm{N}_{i0}+\mathrm{N}_{j0}+\mathrm{N}_{d0}$$

This iteration will run until its predecessor– and successor–state become equal. When certain restrictions on **($\mathbb{N}$)** are observed, a network of **(3) connected sets** will be generated:

- An unbounded **escape–set** with trajectories escaping to infinity in execution–time of the iteration,
- A bounded **prisoner–set** with trajectories tending to a sink–point while the iteration is going on and
- A bounded **JULIA–set** with a fractal structure formed by points acting as repellers against all points of both the other sets.

At the moment iteration stops, **(2) fixed–points** have been generated:

- A **repeller**–point **($\mathbb{H}_{[1]}$)** on JULIA–**set** and
- A attractive **sink**–point **($\mathbb{H}_{[2]}$)** in **prisoner–set.**

From sequence **(1^3.)** the following condition for the fixed–points must hold:

- $\mathbf{Q}_i^2+\mathbf{Q}_j^2+\mathbf{Q}_d^2-\mathbf{Q}_i-\mathbf{Q}_j-\mathbf{Q}_d+\mathbf{N}_0+\boldsymbol{i}\mathbf{N}_1+\boldsymbol{j}\mathbf{N}_2+\boldsymbol{d}\mathbf{N}_3=0.$

This will result in the **(2) fixed–point–solutions ($\mathbb{H}_{[1\&2]}$)** with their components:

- $[\mathbb{H}_i \leftarrow \mathbf{Q}_i] \wedge [\mathbb{H}_j \leftarrow \mathbf{Q}_j] \wedge [\mathbb{H}_d \leftarrow \mathbf{Q}_d]$.

Thus equation **(1^3.)** can now be re–written as:

- $\mathbb{H}_i^2+\mathbb{H}_j^2+\mathbb{H}_d^2-\mathbb{H}_i-\mathbb{H}_j-\mathbb{H}_d+\mathbf{N}_0+\boldsymbol{i}\mathbf{N}_1+\boldsymbol{j}\mathbf{N}_2+\boldsymbol{d}\mathbf{N}_3=0,$

under ($\mathrm{N}_0=\mathrm{N}_{i0}+\mathrm{N}_{j0}+\mathrm{N}_{d0}$) can be separated into:

1^4. $\mathbb{H}_i^2-\mathbb{H}_i+\mathbf{N}_{i0}+\boldsymbol{i}\mathbf{N}_1=0$
1^5. $\mathbb{H}_j^2-\mathbb{H}_j+\mathbf{N}_{j0}+\boldsymbol{j}\mathbf{N}_2=0$
1^6. $\mathbb{H}_d^2-\mathbb{H}_d+\mathbf{N}_{d0}+\boldsymbol{d}\mathbf{N}_3=0.$

2. About the Structure of a connected Quaternion-JULIA-Set.

Searching for the **fixed–points** of an appropriate **network** (escape–, prisoner– and JULIA–set) seems to be a good way to enter the discussion on the structure of a connected JULIA–set. For further discussions an invariance of forward– and backward–iterations relative to the repelling fixed–point is of major interest.

Instead trying to find the fixed–points directly their projections in **complex planes ($[\boldsymbol{e}$^$\boldsymbol{i}] \wedge [\boldsymbol{e}$^$\boldsymbol{i}] \wedge [\boldsymbol{e}$^$\boldsymbol{d}]$)** (obtained via solutions of equations **(1^4.–1^6.)**) are used preliminary in order to specify them indirectly.

2.1. Fixed-Points from Interation (1^3,) of Sequence (1^1,) .

From e.g. **[1 ∧ 2]** it is known, that a network with **complex escape– prisoner–** and **JULIA–set** can be obtained, when a sequence like:

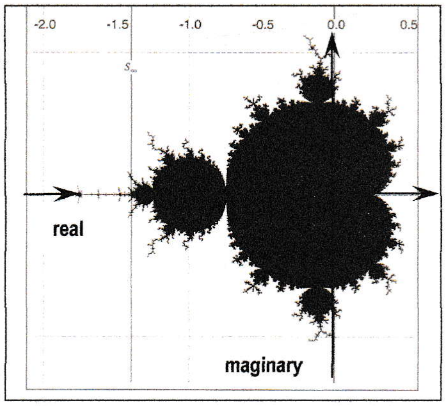
2.1^1. $([h=\mathrm{e}\mathrm{h}_0+\boldsymbol{i}\mathrm{h}_1]\rightarrow h^2+[\ell=\mathrm{el}_0+\boldsymbol{i}\mathrm{l}_1])^2+\ell\rightarrow((h^2+\ell)^2+\ell)^2+\ell\rightarrow\ldots \quad \Leftarrow \quad ([h=\text{variable}] \wedge [\ell=\text{constant}]).$

is executed recursively and the iteration finally stops due to equality of its predecessor– and successor–state. This complex network will have properties comparable with the network specified from **(1^3.)** with the exception, it only exists in complex plane. For this complex network it ihas become obvious, there is a structural dichotomy. Depending on the **constant (ℓ)** both prisoner– and JULIA–set may behave differently:

- For a **specific ℓ–set**, the complex prisoner– and JULIA–set are connected (each on consists of one piece only) and the prisoner–set possesses a fixed–point as sink, while the JULIA–set has a fixed–point as a repeller for the prisoner– and escape–set as well.

- In case of an **alternate ℓ–set**, prisoner– and JULIA–set will become **CANTOR–sets**, which means, they appear completely disconnected.

B. B. MANDELBROT **[3]** had the idea of picturing this dichotomy in a set of **parameters (ℓ)** varying in the complex plane. This leads directly to the **MANDELBROT–set:**



He coloured each point in the plane of ℓ–values black or white depending on whether the associated **JULIA–sets** respectively turned out to be **one piece** or **dust.**

What now a question about the characters of the complex solutions from equations **(1^4.–1^6.)** is concerned, it must be identified, that they are subjected to the same dichotomy as those in case of **(2.1^1.)**. Solutions of **(1^4.–1^6.)** only will become fixed–points, if the complex components $(N_{i0}+iN_1) \wedge (N_{j0}+jN_2) \wedge (N_{d0}+dN_3)$ within **(1^3.)** are extracted from the black part of the MANDELBROT–set.

2.1.1. Conditions to find Components of Fixed-Points .

Under these **conditions (1^4.)** leads to the preliminary solutions:

- $\mathbb{H}_{i[1\&2]} = \tfrac{1}{2} \pm \tfrac{1}{2}\langle\langle 1-4N_{i0}-i4N_1]\rangle\rangle^{1/2}$.

This can be further evaluated by settings:

- $1-4N_{i0}-i4N_1 = (u-ix)^2 = u^2-i2ux+x^2$

leads via a fourth–degree–equation for **(u)**, to the following solutions of **(u)** and **(x)**:

- $u = \pm\langle\langle \tfrac{1}{2}-2N_{i0}+\langle\langle(\tfrac{1}{2}-2N_{i0})^2-4N_1^{\,2}\rangle\rangle^{1/2}\rangle\rangle^{1/2}$
- $x = \pm 2N_1/\langle\langle \tfrac{1}{2}-2N_{i0}+\langle\langle(\tfrac{1}{2}-2N_{i0})^2-4N_1^{\,2}\rangle\rangle^{1/2}\rangle\rangle^{1/2}$

finally to:

2.1.1^1. $\mathbb{H}_{i[1\&2]} = \tfrac{1}{2} \pm \langle\langle \tfrac{1}{8}-\tfrac{1}{2}N_{i0}+\langle\langle(\tfrac{1}{8}-\tfrac{1}{2}N_{i0})^2-\tfrac{1}{4}N_1^{\,2}\rangle\rangle^{1/2}\rangle\rangle^{1/2} \mp iN_1/\langle\langle \tfrac{1}{2}-2N_{i0}+\langle\langle(\tfrac{1}{2}-2N_{i0})^2-4N_1^{\,2}\rangle\rangle^{1/2}\rangle\rangle^{1/2}$.

The attracting or repelling property of the fixed–points is in essence the derivation of the sequence for **($\mathbb{P}$)** at the locations of $\mathbb{H}_{i[1\&2]}$. This derivation can be calculated in the same way as for the real case. A fixed–point is attractive, if the absolute value of the derivation at fixed–point location is **($<$1)**, it is repelling if **($>$1).** Therefore one obtains:

- $|2\mathbb{H}_{i[1]}| > 1 \rightarrow \mathbb{H}_{i[1]}$ is **repeller** and thus a point **on** corresponding **JULIA–set.**
- $|2\mathbb{H}_{i[2]}| < 1 \rightarrow \mathbb{H}_{i[2]}$ is **attractor** and thus a sink **in** the corresponding **prisoner–set.**

More details about the derivations can be found in the **scheme (2.1.1^1.):**

Derivation 2.1.1^1.									
$\mathbb{H}_i^{\,2}-\mathbb{H}_i+N_{i0}+iN_1=0$	●								
leads to	↓								
$\mathbb{H}_{i[1\&2]}=½\pm½\langle\langle 1-4N_{i0}-i4N_1\rangle\rangle^{½}$	●	●							
leads to ♦ *with*	↓	↓							
$1-4N_{i0}-i4N_1=(u-ix)^2=u^2-i2ux+x^2$		●	●	●					
$\mathbb{H}_{i[1\&2]}=½\pm½u\mp i½x$	●						●		
where	↓			↓					
leads to			↓						
$\lvert 2\mathbb{H}_{i[1]}\rvert=[(1+u)^2+x^2]^{½}$	●							●	
	∧						∧		
$\lvert 2\mathbb{H}_{i[2]}\rvert=[(1-u)^2+x^2]^{½}$	●								●
$1-4N_{i0}=u^2+x^2$				●					
				∧					
$(4N_1=2ux)\rightarrow(2N_1/u=x)$				●		●			
$1-4N_{i0}=u^2+4N_1^{\,2}/u^2$			●						
			∨			∧			
$u^4-(1-4N_{i0})u^2+4N_1^{\,2}=0$			●		●				
leads to					↓			↓	↓
$u^2=½-2N_{i0}+\langle\langle(½-2N_{i0})^2-4N_1^{\,2}\rangle\rangle^{½}$					●				
leads to					↓				
$u=\pm\langle\langle ½-2N_{i0}+\langle\langle(½-2N_{i0})^2-4N_1^{\,2}\rangle\rangle^{½}\rangle\rangle^{½}$					●	●	●		
leads to						↓	∧		
$x=\pm 2N_1/\langle\langle ½-2N_{i0}+\langle\langle(½-2N_{i0})^2-4N_1^{\,2}\rangle\rangle^{½}\rangle\rangle^{½}$						●	●		
leads to							↓		
$\mathbb{H}_{i[1\&2]}=½$ $\pm$ $\langle\langle ⅛-½N_{i0}+\langle\langle(⅛-½N_{i0})^2-¼N_1^{\,2}\rangle\rangle^{½}\rangle\rangle^{½}$ $\mp$ $iN_1/\langle\langle ½-2N_{i0}+\langle\langle(½-2N_{i0})^2-4N_1^{\,2}\rangle\rangle^{½}\rangle\rangle^{½}$							●		
$[\![u>0]\!]\rightarrow[\![0<\lvert 2\mathbb{H}_{i[1]}\rvert=(1+\lvert u\rvert)[1+4N_1^{\,2}/u^2(1+\lvert u\rvert)^2]^{½}>1]\!]$								●	
$[\![u>0]\!]\rightarrow[\![0<\lvert 2\mathbb{H}_{i[2]}\rvert=(1-\lvert u\rvert)[1+4N_1^{\,2}/u^2(1-\lvert u\rvert)^2]^{½}<1]\!]$									●
								∧	∧
$[\![u<0]\!]\rightarrow[\![0>\langle\langle-\lvert 2\mathbb{H}_{i[1]}\rvert=-(\lvert u\rvert-1)[1+4N_1^{\,2}/u^2(1+\lvert u\rvert)^2]^{½}\rangle\rangle<-1]\!]$								●	
$[\![u<0]\!]\rightarrow[\![0>\langle\langle-\lvert 2\mathbb{H}_{i[2]}\rvert=-(\lvert u\rvert+1)[1+4N_1^{\,2}/u^2(1-\lvert u\rvert)^2]^{½}\rangle\rangle>-1]\!]$									●
leads to								↓	↓
$0<\lvert 2\mathbb{H}_{i[1]}\rvert>1$								●	
$0<\lvert 2\mathbb{H}_{i[2]}\rvert<1$									●
leads to								↓	↓
$\mathbb{H}_{i[1]}$: *Component associated with repeller-point on quaternion-JULIA-set*								●	
$\mathbb{H}_{i[2]}$: *Component associated with sink-point in quaternion-prisoner-set*									●

Similarly **(1^5.)** will lead to the preliminary solutions:

- $\mathbb{H}_{j[1\&2]}=½\pm½\langle\langle 1-4N_{j0}-j4N_2\rangle\rangle^{½}$.

This can be further evaluated by settings:

- $1-4N_{j0}-j4N_2=(v-jy)^2=v^2-j2vy+y^2$

leads via a fourth−degree−equation for **(v)**, to the following solutions for **(v)** and **(y)**:

- $v = \pm\langle\!\langle \tfrac{1}{2} - 2N_{j0} + \langle\!\langle (\tfrac{1}{2} - 2N_{j0})^2 - 4N_2^2 \rangle\!\rangle^{1/2} \rangle\!\rangle^{1/2}$
- $y = \pm 2N_2 / \langle\!\langle \tfrac{1}{2} - 2N_{j0} + \langle\!\langle (\tfrac{1}{2} - 2N_{j0})^2 - 4N_2^2 \rangle\!\rangle^{1/2} \rangle\!\rangle^{1/2}$

finally to:

2.1.1^2. $\mathbb{H}_{j[1\&2]} = \tfrac{1}{2} \pm \langle\!\langle \tfrac{1}{8} - \tfrac{1}{2}N_{j0} + \langle\!\langle (\tfrac{1}{8} - \tfrac{1}{2}N_{j0})^2 - \tfrac{1}{4}N_2^2 \rangle\!\rangle^{1/2} \rangle\!\rangle^{1/2} \mp j N_2 / \langle\!\langle \tfrac{1}{2} - 2N_{j0} + \langle\!\langle (\tfrac{1}{2} - 2N_{j0})^2 - 4N_2^2 \rangle\!\rangle^{1/2} \rangle\!\rangle^{1/2}$.

The attracting or repelling property of the fixed–points is in essence the derivation of the sequence for **($\mathbb{P}$)** at the locations of $\mathbb{H}_{j[1\&2]}$. This derivation can be calculated in the same way as for the real case. A fixed point is attractive, if the absolute value of the derivation at fixed–point location is **(<1)**, it is repelling, if it is **(>1)**. This leads in the actual cases to:

- $|2\mathbb{H}_{j[1]}| > 1 \rightarrow \mathbb{H}_{j[1]}$ is **repeller** and thus a point **on** corresponding **JULIA–set.**
- $|2\mathbb{H}_{j[2]}| < 1 \rightarrow \mathbb{H}_{j[2]}$ is **attractor** and thus a sink **in** the corresponding **prisoner–set.**

More details about the derivations can be found in the following **scheme (2.1.1^2.):**

Derivation 2.1.1^2.									
$\mathbb{H}_j^2 - \mathbb{H}_j + N_{j0} + jN_2 = 0$	●								
leads to	↓								
$\mathbb{H}_{j[1\&2]} = \tfrac{1}{2} \pm \tfrac{1}{2}\langle\!\langle 1 - 4N_{j0} - j4N_2 \rangle\!\rangle^{1/2}$	●	●							
leads to ♦ *with*	↓	↓							
$1 - 4N_{j0} - j4N_2 = (v - jy)^2 = v^2 - j2vy + y^2$		●	●	●					
$\mathbb{H}_{j[1\&2]} = \tfrac{1}{2} \pm \tfrac{1}{2}v \mp j\tfrac{1}{2}y$	●						●		
where	↓			↓					
leads to			↓						
$\|2\mathbb{H}_{j[1]}\| = [(1+v)^2 + y^2]^{1/2}$	●							●	
	^						^		
$\|2\mathbb{H}_{j[2]}\| = [(1-v)^2 + y^2]^{1/2}$	●								●
$1 - 4N_{j0} = v^2 + y^2$				●					
				^					
$(4N_2 = 2vy) \rightarrow (2N_2/v = y)$				●		●			
$1 - 4N_{j0} = v^2 + 4N_2^2/v^2$			●						
			v			^			
$v^4 - (1 - 4N_{j0})v^2 + 4N_2^2 = 0$			●		●				
leads to					↓			↓	↓
$v^2 = \tfrac{1}{2} - 2N_{j0} + \langle\!\langle (\tfrac{1}{2} - 2N_{j0})^2 - 4N_2^2 \rangle\!\rangle^{1/2}$					●				
leads to					↓				
$v = \pm\langle\!\langle \tfrac{1}{2} - 2N_{j0} + \langle\!\langle (\tfrac{1}{2} - 2N_{j0})^2 - 4N_2^2 \rangle\!\rangle^{1/2} \rangle\!\rangle^{1/2}$					●	●	●		
leads to						↓	^		
$y = \pm 2N_2 / \langle\!\langle \tfrac{1}{2} - 2N_{j0} + \langle\!\langle (\tfrac{1}{2} - 2N_{j0})^2 - 4N_2^2 \rangle\!\rangle^{1/2} \rangle\!\rangle^{1/2}$						●	●		
leads to							↓		
$\mathbb{H}_{j[1\&2]} = \tfrac{1}{2}$ $\pm$ $\langle\!\langle \tfrac{1}{8} - \tfrac{1}{2}N_{j0} + \langle\!\langle (\tfrac{1}{8} - \tfrac{1}{2}N_{j0})^2 - \tfrac{1}{4}N_2^2 \rangle\!\rangle^{1/2} \rangle\!\rangle^{1/2}$ $\mp$ $jN_2 / \langle\!\langle \tfrac{1}{2} - 2N_{j0} + \langle\!\langle (\tfrac{1}{2} - 2N_{j0})^2 - 4N_2^2 \rangle\!\rangle^{1/2} \rangle\!\rangle^{1/2}$							●		
$[\![v > 0]\!] \rightarrow [\![0 < \|2\mathbb{H}_{j[1]}\| = (1+\|v\|)[1 + 4N_2^2/v^2(1+\|v\|)^2]^{1/2} > 1]\!]$								●	
$[\![v > 0]\!] \rightarrow [\![0 < \|2\mathbb{H}_{j[2]}\| = (1-\|v\|)[1 + 4N_2^2/v^2(1-\|v\|)^2]^{1/2} < 1]\!]$									●
								^	^
$[\![v < 0]\!] \rightarrow [\![0 > \langle\!\langle -\|2\mathbb{H}_{j[1]}\| = -(\|v\|-1)[1 + 4N_2^2/v^2(1+\|v\|)^2]^{1/2} \rangle\!\rangle < -1]\!]$								●	
$[\![v < 0]\!] \rightarrow [\![0 > \langle\!\langle -\|2\mathbb{H}_{j[2]}\| = -(\|v\|+1)[1 + 4N_2^2/v^2(1-\|v\|)^2]^{1/2} \rangle\!\rangle > -1]\!]$									●
leads to								↓	↓
$0 < \|2\mathbb{H}_{j[1]}\| > 1$								●	
$0 < \|2\mathbb{H}_{j2}\| < 1$									●
leads to								↓	↓
$\mathbb{H}_{j[1]}$: *Component associated with repeller-point on quaternion-JULIA-set*								●	
$\mathbb{H}_{j[2]}$: *Component associated with sink-point in quaternion-prisoner-set*									●

And last not least **condition (1^6.)** will lead to the preliminary solutions:

- $\mathbb{H}_{d[1\&2]} = ½±½⟪1−4N_{d0}−d4N_3⟫^{½}$.

This can be further ev aluated by settings:

- $1−4N_{d0}−d4N_3 = (w−dz)^2 = w^2−d2wz+z^2$

leads via a fourth−degree−equation for **(w)**, to the following solutions for **(w)** and **(z)**:

- $w = ±⟪½−2N_{d0}+⟪(½−2N_{d0})^2−4N_3{}^2⟫^{½}⟫^{½}$
- $z = ±2N_3/⟪½−2N_{d0}+⟪(½−2N_{d0})^2−4N_3{}^2⟫^{½}⟫^{½}$

finally to:

2.1.1^3. $\mathbb{H}_{d[1\&2]} = ½±⟪⅛−½N_{d0}+⟪(⅛−½N_{d0})^2−¼N_3{}^2⟫^{½}⟫^{½}∓dN_3/⟪½−2N_{d0}+⟪(½−2N_{d0})^2−4N_3{}^2⟫^{½}⟫^{½}$.

The attracting or repelling property of the fixed−points is in essence the derivation of the sequence for **(ℙ)** at the locations of $\mathbb{H}_{d[1\&2]}$. This derivation can be calculated in the same way as for the real case. A fixed− point is attractive, if the absolute value of the derivation at fixed−point location is **(<1)**, it is repelling, if it is **(>1)**. This leads in the actual cases to:

- $|2\mathbb{H}_{d[1]}| > 1 \rightarrow \mathbb{H}_{d[1]}$ is **repeller** and thus a point **on** corresponding **JULIA−set.**
- $|2\mathbb{H}_{d[2]}| < 1 \rightarrow \mathbb{H}_{d[2]}$ is **attractor** and thus a sink **in** the corresponding **prisoner−set.**

More details about the derivation can be found in the following **scheme (2.1.1^3.):**

Derivation 2.1.1^3.									
$\mathbb{H}_d{}^2-\mathbb{H}_d+\mathrm{N}_{d0}+\boldsymbol{d}\mathrm{N}_3=0$	●								
leads to	↓								
$\mathbb{H}_{d[1\&2]}=½\pm½\langle\!\langle 1-4\mathrm{N}_{d0}-\boldsymbol{d}4\mathrm{N}_3\rangle\!\rangle^{½}$	●	●							
leads to ♦ *with*	↓	↓							
$1-4\mathrm{N}_{d0}-\boldsymbol{d}4\mathrm{N}_3=(\mathrm{w}-\boldsymbol{d}\mathrm{z})^2=\mathrm{w}^2-\boldsymbol{d}2\mathrm{wz}+\mathrm{z}^2$		●	●	●					
$\mathbb{H}_{d[1\&2]}=½\pm½\mathrm{w}\mp\boldsymbol{d}½\mathrm{z}$	●						●		
where	↓			↓					
leads to			↓						
$\lvert 2\mathbb{H}_{d[1]}\rvert=[(1+\mathrm{w})^2+\mathrm{z}^2]^{½}$	●							●	
	∧						∧		
$\lvert 2\mathbb{H}_{d[2]}\rvert=[(1-\mathrm{w})^2+\mathrm{z}^2]^{½}$	●								●
$1-4\mathrm{N}_{d0}=\mathrm{w}^2+\mathrm{z}^2$				●					
				∧					
$(4\mathrm{N}_3=2\mathrm{wz})\rightarrow(2\mathrm{N}_3/\mathrm{w}=\mathrm{z})$				●		●			
$1-4\mathrm{N}_{d0}=\mathrm{w}^2+4\mathrm{N}_3{}^2/\mathrm{w}^2$			●						
			∨			∧			
$\mathrm{w}^4-(1-4\mathrm{N}_{d0})\mathrm{w}^2+4\mathrm{N}_3{}^2=0$			●		●				
leads to					↓			↓	↓
$\mathrm{w}^2=½-2\mathrm{N}_{d0}+\langle\!\langle(½-2\mathrm{N}_{d0})^2-4\mathrm{N}_3{}^2\rangle\!\rangle^{½}$					●				
leads to					↓				
$\mathrm{w}\pm\langle\!\langle ½-2\mathrm{N}_{d0}+\langle\!\langle(½-2\mathrm{N}_{d0})^2-4\mathrm{N}_3{}^2\rangle\!\rangle^{½}\rangle\!\rangle^{½}$					●	●	●		
leads to						↓	∧		
$\mathrm{z}=\pm2\mathrm{N}_3/\langle\!\langle ½-2\mathrm{N}_{d0}+\langle\!\langle(½-2\mathrm{N}_{d0})^2-4\mathrm{N}_3{}^2\rangle\!\rangle^{½}\rangle\!\rangle^{½}$						●	●		
leads to							↓		
$\mathbb{H}_{d[1\&2]}=½$ $\pm$ $\langle\!\langle ⅛-½\mathrm{N}_{d0}+\langle\!\langle(⅛-½\mathrm{N}_{d0})^2-¼\mathrm{N}_3{}^2\rangle\!\rangle^{½}\rangle\!\rangle^{½}$ $\mp$ $\boldsymbol{d}\mathrm{N}_3/\langle\!\langle ½-2\mathrm{N}_{d0}+\langle\!\langle(½-2\mathrm{N}_{d0})^2-4\mathrm{N}_3{}^2\rangle\!\rangle^{½}\rangle\!\rangle^{½}$							●		
$[\![\mathrm{w}>0]\!]\rightarrow[\![0<\lvert 2\mathbb{H}_{d[1]}\rvert=(1+\lvert\mathrm{w}\rvert)[1+4\mathrm{N}_3{}^2/\mathrm{w}^2(1+\lvert\mathrm{w}\rvert)^2]^{½}>1]\!]$								●	
$[\![\mathrm{w}>0]\!]\rightarrow[\![0<\lvert 2\mathbb{H}_{d[2]}\rvert=(1-\lvert\mathrm{w}\rvert)[1+4\mathrm{N}_3{}^2/\mathrm{w}^2(1-\lvert\mathrm{w}\rvert)^2]^{½}<1]\!]$									●
								∧	∧
$[\![\mathrm{w}<0]\!]\rightarrow[\![0>\langle\!\langle-\lvert 2\mathbb{H}_{d[1]}\rvert=-(\lvert\mathrm{w}\rvert-1)[1+4\mathrm{N}_3{}^2/\mathrm{w}^2(1+\lvert\mathrm{w}\rvert)^2]^{½}\rangle\!\rangle<-1]\!]$								●	
$[\![\mathrm{w}<0]\!]\rightarrow[\![0>\langle\!\langle-\lvert 2\mathbb{H}_{d[2]}\rvert=-(\lvert\mathrm{w}\rvert+1)[1+4\mathrm{N}_3{}^2/\mathrm{w}^2(1-\lvert\mathrm{w}\rvert)^2]^{½}\rangle\!\rangle>-1]\!]$									●
leads to								↓	↓
$0<\lvert 2\mathbb{H}_{d[1]}\rvert>1$								●	
$0<\lvert 2\mathbb{H}_{d[2]}\rvert<1$									●
leads to								↓	↓
$\mathbb{H}_{d[1]}$: *Component associated with repeller-point on quaternion-JULIA-set*								●	
$\mathbb{H}_{d[2]}$: *Component associated with sink-point in quaternion-prisoner-set*									●

2.1.2. Fixed-Points as Quaternion-Points.

$(\mathbb{H})$ as a quaternion can generally be written in a form like:

- $\mathbb{H}=[(\mathrm{a}_0{}^2+\mathrm{a}_1{}^2+\mathrm{a}_2{}^2+\mathrm{a}_3{}^2)^{½}]\cdot\exp\{\Theta(\boldsymbol{i}\mathrm{a}_1+\boldsymbol{j}\mathrm{a}_2+\boldsymbol{d}\mathrm{a}_3)/(\mathrm{a}_1{}^2+\mathrm{a}_2{}^2+\mathrm{a}_3{}^2)^{½})\}$
  $=\mathrm{T}\cdot\exp\{\underline{\mathrm{n}}\Theta\}$
  $=\mathrm{T}\cdot\exp\{\boldsymbol{i}\Psi_1+\boldsymbol{j}\Psi_2+\boldsymbol{d}\Psi_3\}$
  $=(\mathrm{t}_1\cdot\exp\{\boldsymbol{i}\Psi_1\})\cdot(\mathrm{t}_2\cdot\exp\{\boldsymbol{j}\Psi_2\})\cdot(\mathrm{t}_3\cdot\exp\{\boldsymbol{d}\Psi_3\})$
  $=\mathrm{t}_1(\cos\{\Psi_1\}+\boldsymbol{i}\sin\{\Psi_1\})\cdot\mathrm{t}_2(\cos\{\Psi_2\}+\boldsymbol{j}\sin\{\Psi_2\})\cdot\mathrm{t}_3(\cos\{\Psi_3\}+\boldsymbol{d}\sin\{\Psi_3\})$.

Because $(\mathbb{H}_{i[1\&2]}\wedge\mathbb{H}_{j[1\&2]}\wedge\mathbb{H}_{d[1\&2]})$ may be expressed as (2.1.1^1. − 2.1.1^3.), this will further lead to:

- $\mathrm{t}_1(\cos\{\Psi_1\}+\boldsymbol{i}\sin\{\Psi_1\})\Rightarrow$
  $\mathbb{H}_{i[1\&2]}=\{½\pm\langle\!\langle ⅛-½\mathrm{N}_{i0}+\langle\!\langle(⅛-½\mathrm{N}_{i0})^2-¼\mathrm{N}_1{}^2\rangle\!\rangle^{½}\rangle\!\rangle^{½}\}\mp\boldsymbol{i}\{\mathrm{N}_1/\langle\!\langle ½-2\mathrm{N}_{i0}+\langle\!\langle(½-2\mathrm{N}_{i0})^2-4\mathrm{N}_1{}^2\rangle\!\rangle^{½}\rangle\!\rangle^{½}\}$

- $t_2(\cos\{\Psi_2\}+\boldsymbol{j}\sin\{\Psi_2\}) \Rightarrow$
  $\mathbb{H}_{j[1\&2]}=\{½\pm⟪⅛-½N_{j0}+⟪(⅛-½N_{j0})^2-¼N_2^2⟫^{½}⟫^{½}\}\mp\boldsymbol{j}\{N_2/⟪½-2N_{j0}+⟪(½-2N_{j0})^2-4N_2^2⟫^{½}⟫^{½}\}$
- $t_3(\cos\{\Psi_3\}+\boldsymbol{d}\sin\{\Psi_3\}) \Rightarrow$
  $\mathbb{H}_{d[1\&2]}=\{½\pm⟪⅛-½N_{d0}+⟪(⅛-½N_{d0})^2-¼N_3^2⟫^{½}⟫^{½}\}\mp\boldsymbol{d}\{N_3/⟪½-2N_{d0}+⟪(½-2N_{d0})^2-4N_3^2⟫^{½}⟫^{½}\}.$

Thus the fixed–points for **JULIA–** and **prisoner–set** will become:

2.1.2^1. $\mathbb{H}_{[1]}=\mathbb{H}_{i[1]}\cdot\mathbb{H}_{j[1]}\cdot\mathbb{H}_{d[1]}-2$
$=\{½+⟪⅛-½N_{i0}+⟪(⅛-½N_{i0})^2-¼N_1^2⟫^{½}⟫^{½}\}-\boldsymbol{i}\{N_1/⟪½-2N_{i0}+⟪(½-2N_{i0})^2-4N_1^2⟫^{½}⟫^{½}\}\cdot$
$\{½+⟪⅛-½N_{j0}+⟪(⅛-½N_{j0})^2-¼N_2^2⟫^{½}⟫^{½}\}-\boldsymbol{j}\{N_2/⟪½-2N_{j0}+⟪(½-2N_{j0})^2-4N_2^2⟫^{½}⟫^{½}\}\cdot$
$\{½+⟪⅛-½N_{d0}+⟪(⅛-½N_{d0})^2-¼N_3^2⟫^{½}⟫^{½}\}-\boldsymbol{d}\{N_3/⟪½-2N_{d0}+⟪(½-2N_{d0})^2-4N_3^2⟫^{½}⟫^{½}\}-2$

2.1.2^2. $\mathbb{H}_{[2]}=\mathbb{H}_{i[2]}\cdot\mathbb{H}_{j[2]}\cdot\mathbb{H}_{d[2]}-2$
$=\{½-⟪⅛-½N_{i0}+⟪(⅛-½N_{i0})^2-¼N_1^2⟫^{½}⟫^{½}\}+\boldsymbol{i}\{N_1/⟪½-2N_{i0}+⟪(½-2N_{i0})^2-4N_1^2⟫^{½}⟫^{½}\}\cdot$
$\{½-⟪⅛-½N_{j0}+⟪(⅛-½N_{j0})^2-¼N_2^2⟫^{½}⟫^{½}\}+\boldsymbol{j}\{N_2/⟪½-2N_{j0}+⟪(½-2N_{j0})^2-4N_2^2⟫^{½}⟫^{½}\}\cdot$
$\{½-⟪⅛-½N_{d0}+⟪(⅛-½N_{d0})^2-¼N_3^2⟫^{½}⟫^{½}\}+\boldsymbol{d}\{N_3/⟪½-2N_{d0}+⟪(½-2N_{d0})^2-4N_3^2⟫^{½}⟫^{½}\}-2.$

2.3. The fractal Structure of the JULIA-Set.

A JULIA–set is a complete invariant fractal with respect to forward– and backward–iteration. A **j–th pre–image** (in a **backward–iteration**) and a **k–th image** (in a **forward–iteration**) starting from the repeller **(**$\mathbb{H}_{[1]}$ given by equation **2.1.2^1.)** are to be obtained by:

2.3^1. **Images:** $\mathbb{R}^{(+1)}=\mathbb{H}_{[1]}{}^2+\mathbb{N}$, $\mathbb{R}^{(+2)}=[\mathbb{R}^{(+1)}]^2+\mathbb{N}$,...., $\mathbb{R}^{(+K)}=[\mathbb{R}^{(+K-1)}]^2+\mathbb{N}$,.....

2.3^2. **Pre-images:** $\mathbb{R}_{1\text{-}2}{}^{(-1)}=\pm(\mathbb{H}_{[1]}-\mathbb{N})^{½}$, $\mathbb{R}_{1\text{-}2}{}^{(-2)}=\pm(\mathbb{R}_{1\text{-}2}{}^{(-1)}-\mathbb{N})^{½}$,....., $\mathbb{R}_{1\text{-}2}{}^{(-J)}=\pm(\mathbb{R}_{1\text{-}2}{}^{(-J+1)}-\mathbb{N})^{½}$,..... .

Because **(**$\mathbb{H}_{[1]}$**)** is a point of the JULIA–set, $\mathbb{R}^{(+K)}$ and $\mathbb{R}^{(-J)}$ cannot in the basin of attraction of infinity otherwise the initial point **(**$\mathbb{H}_{[1]}$**)** would have to be **part of** the **escape–set** too. On the other hand, both kinds of images cannot be in the interior (the **prisoner–set**), because then **(**$\mathbb{H}_{[1]}$**)** would then have to be from **prisoner–set** too, what again is not the case. Thus $\mathbb{R}^{(+K)}$ and $\mathbb{R}^{(-J)}$ must be from the boundary (the **JULIA–set**). The reason for all this can also be found in the continuity of the quadratic transformation. Arbitrarily close to the images and pre–images there are escaping– and prisoner–points and the continuity of iteration implies, neighbourhood relation must hold for the whole set of transformation points. This finally leads to a JULIA–set being invariant with respect to forward– and backward–transformation as well.

The total, unlimited set of images and pre–images from the repellers on JULIA–set determines the fractal structure of the JULIA–set.

3. Symmetries of a related JULIA-Network.

It is obvious from **equations (2.1.2^1.)** and **(2.1.2^2.)**, the fixed–points **(**$\mathbb{H}_{[1\&2]}$**)** of the network (escape–prisoner– and JULIA–set) obtained from **iteration (1^3.)** depend on selection of **(**$\mathbb{N}$**)** only. Thus **(16) different choices of (**$\mathbb{N}$**'s)** chosen appropriately from the black part of the MANDELBROT–set will define **(16) different fixed–points (**$\mathbb{H}_{[1]}$**)** for **JULIA–sets** as square–points of a **hyper–cube.** This hyper–cube together with the JULIA–sets belonging to each of the square–points will represent a **related JULIA–network.** The symmetry–properties of this JULIA–network is to be obtained on base of a **hyper–cube's symmetry–group** extended by some additional considerations.

The symmetry–group of a cube can be derived from the symmetry–group of a square. With this knowledge in mind all hints are provided to further obtain the symmetry–group of a hyper–cube. The symmetry–group of a hyper–cube with additional considerations will then finally lead to the symmetry– properties of the related JULIA–network.

3.1. The Symmetries of a Square.

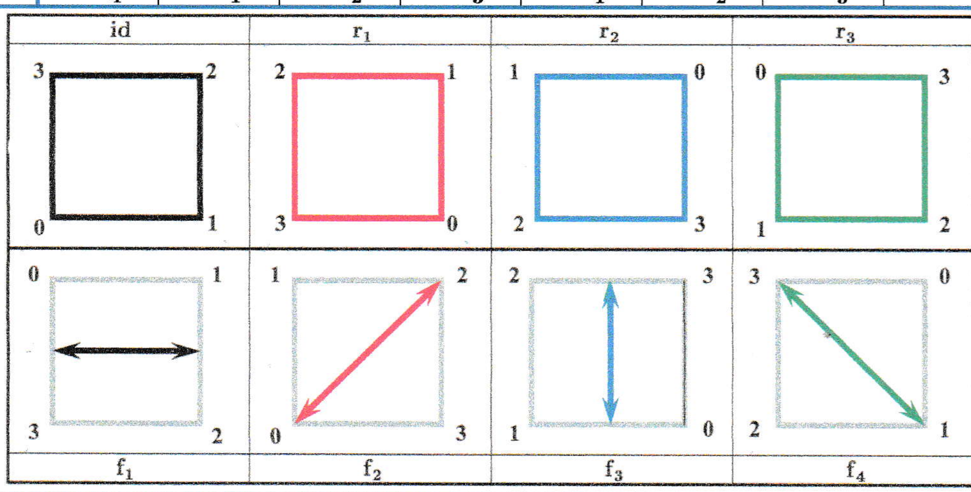
The symmetry–group of a square can best be described by the group–table below, consisting of **(64) permutations** of the square–points (contained in the entries of the table) obtained when **(8) operations** act on the square. The (8) operations consist of:

- The **identity–operation (id)** to reinstall the starting configuration,
- **(3) right–turning rotations** ($[r_1 = \pi/2] \wedge [r_2 = \pi] \wedge [r_3 = 3\pi/2]$) around the centre of the square,
- **(4) flip–operations** (f_1^f_2^f_3^f_4) with respect to indicated directions.

The permutations within **entries (1 → 64)** of the group–table have the meaning:

- Positions of edge–points after an **operation of column(0)** having acted on the square
- Positions of edge–points after **operation of row(0)** being performed on top of operation in **column(0).**

★	id	r_1	r_2	r_3	f_1	f_2	f_3	f_4
id	0 1 2 3 0 1 2 3 = id	0 1 2 3 3 0 1 2 $= r_1$	0 1 2 3 2 3 0 1 $= r_2$	0 1 2 3 1 2 3 0 $= r_3$	0 1 2 3 3 2 1 0 $= f_1$	0 1 2 3 0 3 2 1 $= f_2$	0 1 2 3 1 0 3 2 $= f_3$	0 1 2 3 2 1 0 3 $= f_4$
r_1	3 0 1 2 3 0 1 2 $= r_1$	3 0 1 2 2 3 0 1 $= r_2$	3 0 1 2 1 2 3 0 $= r_3$	3 0 1 2 0 1 2 3 = id	3 0 1 2 2 1 0 3 $= f_4$	3 0 1 2 3 2 1 0 $= f_1$	3 0 1 2 0 3 2 1 $= f_2$	3 0 1 2 1 0 3 2 $= f_3$
r_2	2 3 0 1 2 3 0 1 $= r_2$	2 3 0 1 1 2 3 0 $= r_3$	2 3 0 1 0 1 2 3 = id	2 3 0 1 3 0 1 2 $= r_1$	2 3 0 1 1 0 3 2 $= f_3$	2 3 0 1 2 1 0 3 $= f_4$	2 3 0 1 3 2 1 0 $= f_1$	2 3 0 1 0 3 2 1 $= f_2$
r_3	1 2 3 0 1 2 3 0 $= r_3$	1 2 3 0 0 1 2 3 = id	1 2 3 0 3 0 1 2 $= r_1$	1 2 3 0 2 3 0 1 $= r_2$	1 2 3 0 0 3 2 1 $= f_2$	1 2 3 0 1 0 3 2 $= f_3$	1 2 3 0 2 1 0 3 $= f_4$	1 2 3 0 3 2 1 0 $= f_1$
f_1	3 2 1 0 3 2 1 0 $= f_1$	3 2 1 0 0 3 2 1 $= f_2$	3 2 1 0 1 0 3 2 $= f_3$	3 2 1 0 2 1 0 3 $= f_4$	3 2 1 0 0 1 2 3 = id	3 2 1 0 3 0 1 2 $= r_1$	3 2 1 0 2 3 0 1 $= r_2$	3 2 1 0 1 2 3 0 $= r_3$
f_2	0 3 2 1 0 3 2 1 $= f_2$	0 3 2 1 1 0 3 2 $= f_3$	0 3 2 1 2 1 0 3 $= f_4$	0 3 2 1 3 2 1 0 $= f_1$	0 3 2 1 1 2 3 0 $= r_3$	0 3 2 1 0 1 2 3 = id	0 3 2 1 3 0 1 2 $= r_1$	0 3 2 1 2 3 0 1 $= r_2$
f_3	1 0 3 2 1 0 3 2 $= f_3$	1 0 3 2 2 1 0 3 $= f_4$	1 0 3 2 3 2 1 0 $= f_1$	1 0 3 2 0 3 2 1 $= f_2$	1 0 3 2 2 3 0 1 $= r_2$	1 0 3 2 1 2 3 0 $= r_3$	1 0 3 2 0 1 2 3 = id	1 0 3 2 3 0 1 2 $= r_1$
f_4	2 1 0 3 2 1 0 3 $= f_4$	2 1 0 3 3 2 1 0 $= f_1$	2 1 0 3 0 3 2 1 $= f_2$	2 1 0 3 1 0 3 2 $= f_3$	2 1 0 3 3 0 1 2 $= r_1$	2 1 0 3 2 3 0 1 $= r_2$	2 1 0 3 1 2 3 0 $= r_3$	2 1 0 3 0 1 2 3 = id



The yellow–marked sub–group is the **cyclic group of the square.**

3.2. Symmetries of a Cube.

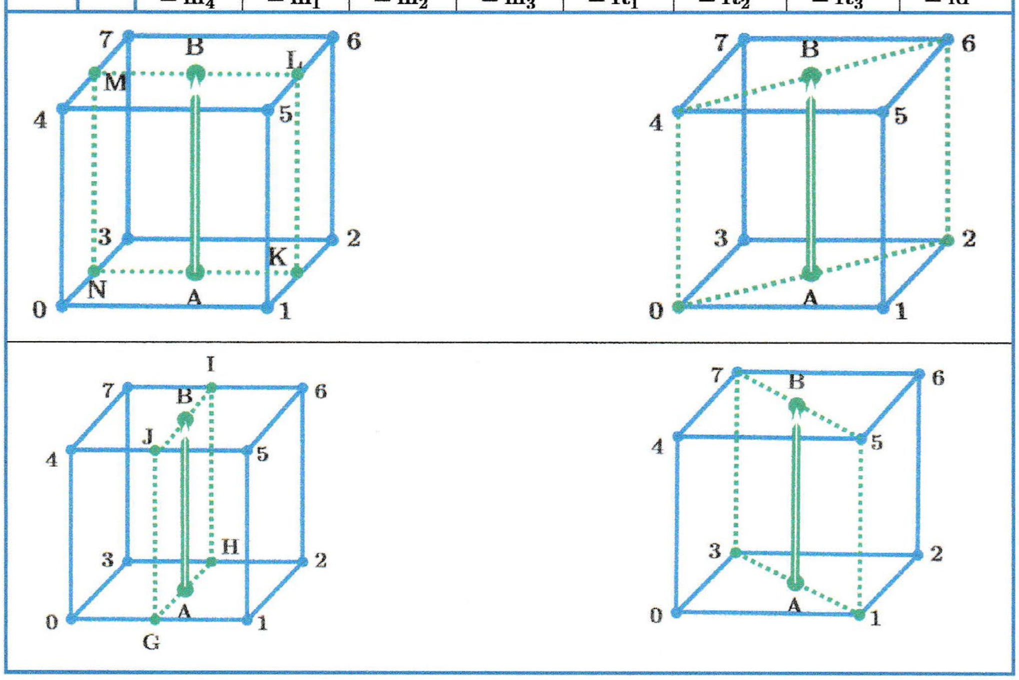
From symmetry−group of a square, **(3) symmetry−sub−groups** of a cube can be derived by replacing:

- Rotations around centre of square by right−turning rotations **(R_1^R_2^R_3)** around each of the axes **(AB^CD^EF)**:
  - ▶ $([AB \perp \langle {}_0{}^4{}_1{}^5{}_2{}^6{}_3{}^7 \rangle] \wedge [CD \perp \langle {}_0{}^1{}_3{}^2{}_7{}^6{}_4{}^5 \rangle] \wedge [EF \perp \langle {}_0{}^3{}_1{}^2{}_5{}^6{}_4{}^7 \rangle])$
- Flip−operations **(f_1^f_2^f_3^f_4)** with respect to directions (**black^red^blue^green**) respectively replaced by mirror−operations **(m_1^m_2^m_3^m_4)** with respect to appropriate mirror−planes:
  - ▶ **⟨NKLM⟩−m_1−, ⟨0264⟩−m_2−, ⟨GHIJ⟩−m_3−** and **⟨1573⟩−m_4−plane** for rotation in **AB−direction**
  - ▶ **⟨OPQR⟩−m_1−, ⟨0167⟩−m_2−, ⟨NKLM⟩−m_3−** and **⟨2543⟩−m_4−plane** for rotation in **CD−direction**
  - ▶ **⟨OPQR⟩−m_1−, ⟨0563⟩−m_2−, ⟨GHIJ⟩−m_3−** and **⟨1274⟩−m_4−plane** for rotation in **EF−direction.**

Under these conditions one will obtain **(3) symmetry−sub−groups** of a cube with respect to the directions **(AB^CD ^EF)**, each one is isomorphic with the symmetry−group of a square.

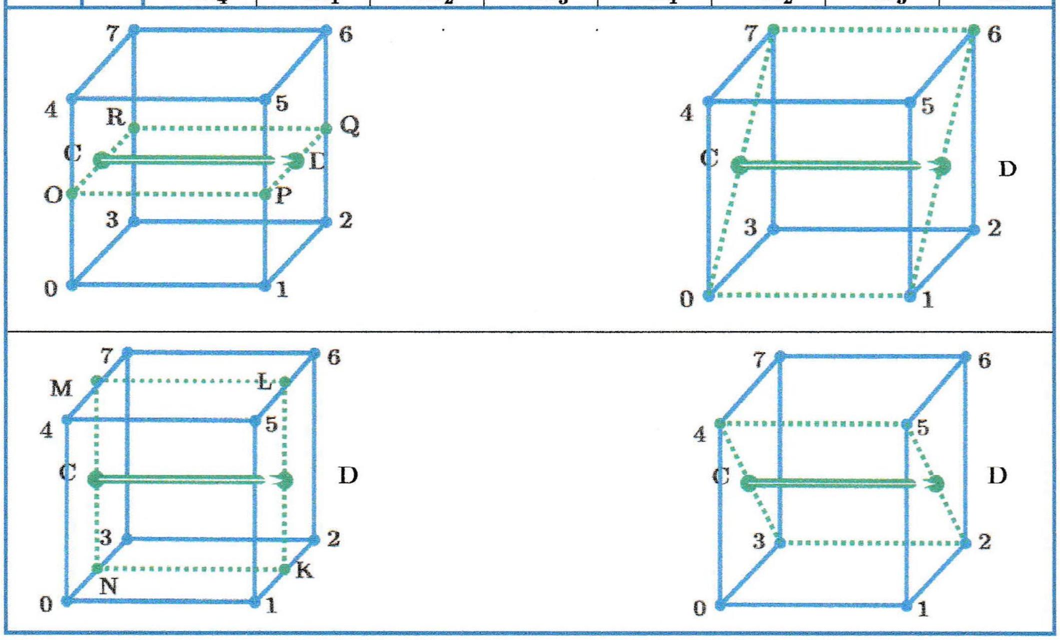
The first sub−group based on direction **(AB)** follows immediately with **(64) elements**, which belonging to multiplications of operations(column(0)) and operations(row(0)):

	★	id	R_1	R_2	R_3	m_1	m_2	m_3	m_4
AB	id	$\begin{smallmatrix}4&5&6&7\\0&1&2&3\end{smallmatrix} = id$	$\begin{smallmatrix}7&4&5&6\\3&0&1&2\end{smallmatrix} = R_1$	$\begin{smallmatrix}6&7&4&5\\2&3&0&1\end{smallmatrix} = R_2$	$\begin{smallmatrix}5&6&7&4\\1&2&3&0\end{smallmatrix} = R_3$	$\begin{smallmatrix}7&6&5&4\\3&2&1&0\end{smallmatrix} = m_1$	$\begin{smallmatrix}4&7&6&5\\0&3&2&1\end{smallmatrix} = m_2$	$\begin{smallmatrix}5&4&7&6\\1&0&3&2\end{smallmatrix} = m_3$	$\begin{smallmatrix}6&5&4&7\\2&1&0&3\end{smallmatrix} = m_4$
	R_1	$\begin{smallmatrix}7&4&5&6\\3&0&1&2\end{smallmatrix} = R_1$	$\begin{smallmatrix}6&7&4&5\\2&3&0&1\end{smallmatrix} = R_2$	$\begin{smallmatrix}5&6&7&4\\1&2&3&0\end{smallmatrix} = R_3$	$\begin{smallmatrix}4&5&6&7\\0&1&2&3\end{smallmatrix} = id$	$\begin{smallmatrix}6&5&4&7\\2&1&0&3\end{smallmatrix} = m_4$	$\begin{smallmatrix}7&6&5&4\\3&2&1&0\end{smallmatrix} = m_1$	$\begin{smallmatrix}4&7&6&5\\0&3&2&1\end{smallmatrix} = m_2$	$\begin{smallmatrix}5&4&7&6\\1&0&3&2\end{smallmatrix} = m_3$
	R_2	$\begin{smallmatrix}6&7&4&5\\2&3&0&1\end{smallmatrix} = R_2$	$\begin{smallmatrix}5&6&7&4\\1&2&3&0\end{smallmatrix} = R_3$	$\begin{smallmatrix}4&5&6&7\\0&1&2&3\end{smallmatrix} = id$	$\begin{smallmatrix}7&4&5&6\\3&0&1&2\end{smallmatrix} = R_1$	$\begin{smallmatrix}5&4&7&6\\1&0&3&2\end{smallmatrix} = m_3$	$\begin{smallmatrix}6&5&4&7\\2&1&0&3\end{smallmatrix} = m_4$	$\begin{smallmatrix}7&6&5&4\\3&2&1&0\end{smallmatrix} = m_1$	$\begin{smallmatrix}4&7&6&5\\0&3&2&1\end{smallmatrix} = m_2$
	R_3	$\begin{smallmatrix}5&6&7&4\\1&2&3&0\end{smallmatrix} = R_3$	$\begin{smallmatrix}4&5&6&7\\0&1&2&3\end{smallmatrix} = id$	$\begin{smallmatrix}7&4&5&6\\3&0&1&2\end{smallmatrix} = R_1$	$\begin{smallmatrix}6&7&4&5\\2&3&0&1\end{smallmatrix} = R_2$	$\begin{smallmatrix}4&7&6&5\\0&3&2&1\end{smallmatrix} = m_2$	$\begin{smallmatrix}5&4&7&6\\1&0&3&2\end{smallmatrix} = m_3$	$\begin{smallmatrix}6&5&4&7\\2&1&0&3\end{smallmatrix} = m_4$	$\begin{smallmatrix}7&6&5&4\\3&2&1&0\end{smallmatrix} = m_1$
NKLM	m_1	$\begin{smallmatrix}7&6&5&4\\3&2&1&0\end{smallmatrix} = m_1$	$\begin{smallmatrix}4&7&6&5\\0&3&2&1\end{smallmatrix} = m_2$	$\begin{smallmatrix}5&4&7&6\\1&0&3&2\end{smallmatrix} = m_3$	$\begin{smallmatrix}6&5&4&7\\2&1&0&3\end{smallmatrix} = m_4$	$\begin{smallmatrix}4&5&6&7\\0&1&2&3\end{smallmatrix} = id$	$\begin{smallmatrix}7&4&5&6\\3&0&1&2\end{smallmatrix} = R_1$	$\begin{smallmatrix}6&7&4&5\\2&3&0&1\end{smallmatrix} = R_2$	$\begin{smallmatrix}5&6&7&4\\1&2&3&0\end{smallmatrix} = R_3$
0264	m_2	$\begin{smallmatrix}4&7&6&5\\0&3&2&1\end{smallmatrix} = m_2$	$\begin{smallmatrix}5&4&7&6\\1&0&3&2\end{smallmatrix} = m_3$	$\begin{smallmatrix}6&5&4&7\\2&1&0&3\end{smallmatrix} = m_4$	$\begin{smallmatrix}7&6&5&4\\3&2&1&0\end{smallmatrix} = m_1$	$\begin{smallmatrix}5&6&7&4\\1&2&3&0\end{smallmatrix} = R_3$	$\begin{smallmatrix}4&5&6&7\\0&1&2&3\end{smallmatrix} = id$	$\begin{smallmatrix}7&4&5&6\\3&0&1&2\end{smallmatrix} = R_1$	$\begin{smallmatrix}6&7&4&5\\2&3&0&1\end{smallmatrix} = R_2$
GHIJ	m_3	$\begin{smallmatrix}5&4&7&6\\1&0&3&2\end{smallmatrix} = m_3$	$\begin{smallmatrix}6&5&4&7\\2&1&0&3\end{smallmatrix} = m_4$	$\begin{smallmatrix}7&6&5&4\\3&2&1&0\end{smallmatrix} = m_1$	$\begin{smallmatrix}4&7&6&5\\0&3&2&1\end{smallmatrix} = m_2$	$\begin{smallmatrix}6&7&4&5\\2&3&0&1\end{smallmatrix} = R_2$	$\begin{smallmatrix}5&6&7&4\\1&2&3&0\end{smallmatrix} = R_3$	$\begin{smallmatrix}4&5&6&7\\0&1&2&3\end{smallmatrix} = id$	$\begin{smallmatrix}7&4&5&6\\3&0&1&2\end{smallmatrix} = R_1$
1573	m_4	$\begin{smallmatrix}6&5&4&7\\2&1&0&3\end{smallmatrix} = m_4$	$\begin{smallmatrix}7&6&5&4\\3&2&1&0\end{smallmatrix} = m_1$	$\begin{smallmatrix}4&7&6&5\\0&3&2&1\end{smallmatrix} = m_2$	$\begin{smallmatrix}5&4&7&6\\1&0&3&2\end{smallmatrix} = m_3$	$\begin{smallmatrix}7&4&5&6\\3&0&1&2\end{smallmatrix} = R_1$	$\begin{smallmatrix}6&7&4&5\\2&3&0&1\end{smallmatrix} = R_2$	$\begin{smallmatrix}5&6&7&4\\1&2&3&0\end{smallmatrix} = R_3$	$\begin{smallmatrix}4&5&6&7\\0&1&2&3\end{smallmatrix} = id$



A second sub–group based on direction **(CD)** follows next with **(64) elements** belonging to multiplications of operations(column(0)) and operations(row(0)):

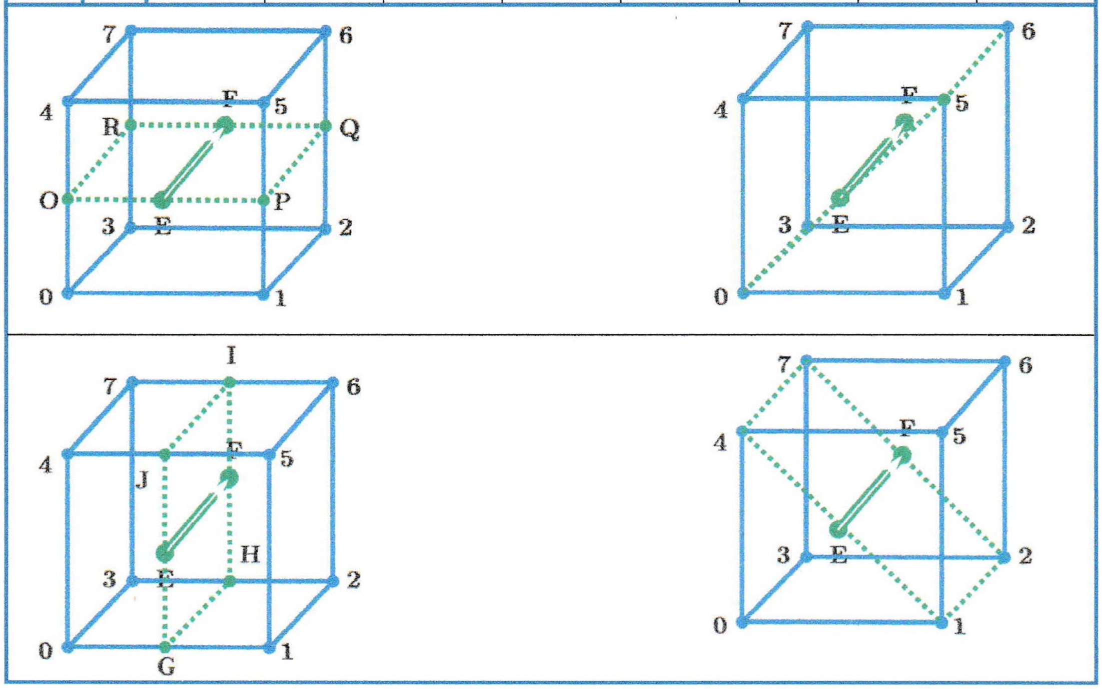
	★	id	R_1	R_2	R_3	m_1	m_2	m_3	m_4
CD	id	1 2 6 5 0 3 7 4 = id	5 1 2 6 4 0 3 7 = R_1	6 5 1 2 7 4 0 3 = R_2	2 6 5 1 3 7 4 0 = R_3	5 6 2 1 4 7 3 0 = m_1	1 5 6 2 0 4 7 3 = m_2	2 1 5 6 3 0 4 7 = m_3	6 2 1 5 7 3 0 4 = m_4
	R_1	5 1 2 6 4 0 3 7 = R_1	6 5 1 2 7 4 0 3 = R_2	2 6 5 1 3 7 4 0 = R_3	1 2 6 5 0 3 7 4 = id	6 2 1 5 7 3 0 4 = m_4	5 6 2 1 4 7 3 0 = m_1	1 5 6 2 0 4 7 3 = m_2	2 1 5 6 3 0 4 7 = m_3
	R_2	6 5 1 2 7 4 0 3 = R_2	2 6 5 1 3 7 4 0 = R_3	1 2 6 5 0 3 7 4 = id	5 1 2 6 4 0 3 7 = R_1	2 1 5 6 3 0 4 7 = m_3	6 2 1 5 7 3 0 4 = m_4	5 6 2 1 4 7 3 0 = m_1	1 5 6 2 0 4 7 3 = m_2
	R_3	2 6 5 1 3 7 4 0 = R_3	1 2 6 5 0 3 7 4 = id	5 1 2 6 4 0 3 7 = R_1	6 5 1 2 7 4 0 3 = R_2	1 5 6 2 0 4 7 3 = m_2	2 1 5 6 3 0 4 7 = m_3	6 2 1 5 7 3 0 4 = m_4	5 6 2 1 4 7 3 0 = m_1
OPQR	m_1	5 6 2 1 4 7 3 0 = m_1	1 5 6 2 0 4 7 3 = m_2	2 1 5 6 3 0 4 7 = m_3	6 2 1 5 7 3 0 4 = m_4	1 2 6 5 0 3 7 4 = id	5 1 2 6 4 0 3 7 = R_1	6 5 1 2 7 4 0 3 = R_2	2 6 5 1 3 7 4 0 = R_3
0167	m_2	1 5 6 2 0 4 7 3 = m_2	2 1 5 6 3 0 4 7 = m_3	6 2 1 5 7 3 0 4 = m_4	5 6 2 1 4 7 3 0 = m_1	2 6 5 1 3 7 4 0 = R_3	1 2 6 5 0 3 7 4 = id	5 1 2 6 4 0 3 7 = R_1	6 5 1 2 7 4 0 3 = R_2
NKLM	m_3	2 1 5 6 3 0 4 7 = m_3	6 2 1 5 7 3 0 4 = m_4	5 6 2 1 4 7 3 0 = m_1	1 5 6 2 0 4 7 3 = m_2	6 5 1 2 7 4 0 3 = R_2	2 6 5 1 3 7 4 0 = R_3	1 2 6 5 0 3 7 4 = id	5 1 2 6 4 0 3 7 = R_1
2543	m_4	6 2 1 5 7 3 0 4 = m_4	5 6 2 1 4 7 3 0 = m_1	1 5 6 2 0 4 7 3 = m_2	2 1 5 6 3 0 4 7 = m_3	5 1 2 6 4 0 3 7 = R_1	6 5 1 2 7 4 0 3 = R_2	2 6 5 1 3 7 4 0 = R_3	1 2 6 5 0 3 7 4 = id



Finally one obtains a sub–group based on direction **(EF)** which follows next with **(64) elements** belonging to all multiplications of operations(column(0)) and operations(row(0)):

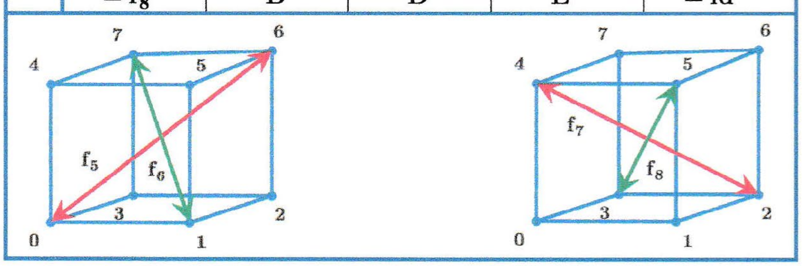
	★	id	R_1	R_2	R_3	m_1	m_2	m_3	m_4
EF	id	2 6 7 3 0 1 5 4 = id	3 2 6 7 4 0 1 5 = R_1	7 3 2 6 5 4 0 1 = R_2	6 7 3 2 1 5 4 0 = R_3	3 7 6 2 4 5 1 0 = m_1	2 3 7 6 0 4 5 1 = m_2	6 2 3 7 1 0 4 5 = m_3	7 6 2 3 5 1 0 4 = m_4
	R_1	3 2 6 7 4 0 1 5 = R_1	7 3 2 6 5 4 0 1 = R_2	6 7 3 2 1 5 4 0 = R_3	2 6 7 3 0 1 5 4 = id	7 6 2 3 5 1 0 4 = m_4	3 7 6 2 4 5 1 0 = m_1	2 3 7 6 0 4 5 1 = m_2	6 2 3 7 1 0 4 5 = m_3
	R_2	7 3 2 6 5 4 0 1 = R_2	6 7 3 2 1 5 4 0 = R_3	2 6 7 3 0 1 5 4 = id	3 2 6 7 4 0 1 5 = R_1	6 2 3 7 1 0 4 5 = m_3	7 6 2 3 5 1 0 4 = m_4	3 7 6 2 4 5 1 0 = m_1	2 3 7 6 0 4 5 1 = m_2
	R_3	6 7 3 2 1 5 4 0 = R_3	2 6 7 3 0 1 5 4 = id	3 2 6 7 4 0 1 5 = R_1	7 3 2 6 5 4 0 1 = R_2	2 3 7 6 0 4 5 1 = m_2	6 2 3 7 1 0 4 5 = m_3	7 6 2 3 5 1 0 4 = m_4	3 7 6 2 4 5 1 0 = m_1
OPQR	m_1	3 7 6 2 4 5 1 0 = m_1	2 3 7 6 0 4 5 1 = m_2	6 2 3 7 1 0 4 5 = m_3	7 6 2 3 5 1 0 4 = m_4	2 6 7 3 0 1 5 4 = id	3 2 6 7 4 0 1 5 = R_1	7 3 2 6 5 4 0 1 = R_2	6 7 3 2 1 5 4 0 = R_3
0563	m_2	2 3 7 6 0 4 5 1 = m_2	6 2 3 7 1 0 4 5 = m_3	7 6 2 3 5 1 0 4 = m_4	3 7 6 2 4 5 1 0 = m_1	6 7 3 2 1 5 4 0 = R_3	2 6 7 3 0 1 5 4 = id	3 2 6 7 4 0 1 5 = R_1	7 3 2 6 5 4 0 1 = R_2

GHIJ	m_3	6 2 3 7 1 0 4 5 $= m_3$	7 6 2 3 5 1 0 4 $= m_4$	3 7 6 2 4 5 1 0 $= m_1$	2 3 7 6 0 4 5 1 $= m_2$	7 3 2 6 5 4 0 1 $= R_2$	6 7 3 2 1 5 4 0 $= R_3$	2 6 7 3 0 1 5 4 $= id$	3 2 6 7 4 0 1 5 $= R_1$
1274	m_4	7 6 2 3 5 1 0 4 $= m_4$	3 7 6 2 4 5 1 0 $= m_1$	2 3 7 6 0 4 5 1 $= m_2$	6 2 3 7 1 0 4 5 $= m_3$	3 2 6 7 4 0 1 5 $= R_1$	7 3 2 6 5 4 0 1 $= R_2$	6 7 3 2 1 5 4 0 $= R_3$	2 6 7 3 0 1 5 4 $= id$



In addition **(4) flip–operations** $(F_5 \hat{} F_6 \hat{} F_7 \hat{} F_8)$ with respect to the space–diagonals of the cube will have be taken into consideration. The properties of these operations are summarized in the next table:

★	id	f_5	f_6	f_7	f_8
id	4 5 6 7 0 1 2 3 $= id$	2 3 6 1 0 7 4 5 $= f_5$	2 3 0 7 6 1 4 5 $= f_6$	4 3 0 1 6 7 2 5 $= f_7$	2 5 0 1 6 7 4 3 $= f_8$
f_5	2 3 6 1 0 7 4 5 $= f_5$	4 5 6 7 0 1 2 3 $= id$	4 5 0 1 6 7 2 3 A	2 5 0 7 6 1 4 3 $= id$	4 3 0 7 6 1 2 5 B
f_6	2 3 0 7 6 1 4 5 $= f_6$	4 5 0 1 6 7 2 3 A	4 5 6 7 0 1 2 3 $= id$	2 5 6 1 0 7 4 3 C	4 3 6 1 0 7 2 5 D
f_7	4 3 0 1 6 7 2 5 $= f_7$	2 5 0 7 6 1 4 3 $= id$	2 5 6 1 0 7 4 3 C	4 5 6 7 0 1 2 3 $= id$	2 3 6 7 0 1 4 5 E
f_8	2 5 0 1 6 7 4 3 $= f_8$	4 3 0 7 6 1 2 5 B	4 3 6 1 0 7 2 5 D	2 3 6 7 0 1 4 5 E	4 5 6 7 0 1 2 3 $= id$



Thus finally **(25) symmetry–operations** in total will make up the symmetry–group of a cube.

3.3. Symmetries of a Hyper-Cube.

If one replaces in a cube:

- Each pair of parallel planes involved in one of the rotations $(\mathbf{R_1} \vee \mathbf{R_2} \vee \mathbf{R_3})$ by a quadruple of cubes (from hyper–cube's structure) with surfaces parallel to a perpendicular common axis of rotation out of $(\alpha\beta \vee \gamma\delta \vee \varepsilon\zeta)$,

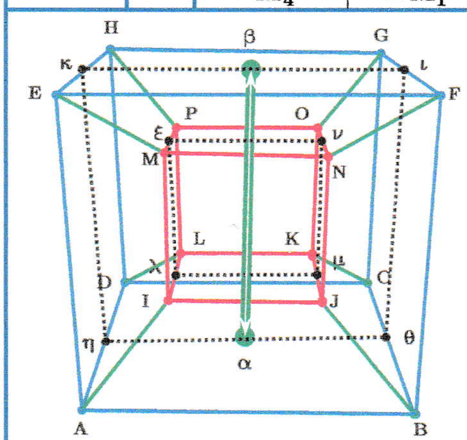
- Each mirror–plane of a cube by a **3–dimensional object** with a pair of parallel planes suitable for a further more mirror–operation,

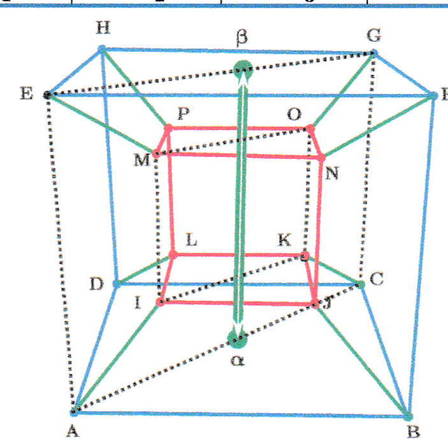
(3) symmetry–sub–groups of a hyper–cube are obtained, each isomorphic with the symmetry–group of a square and a symmetry–sub–groups of a cube. Each symmetry–sub–group of the hyper–cube consists of:

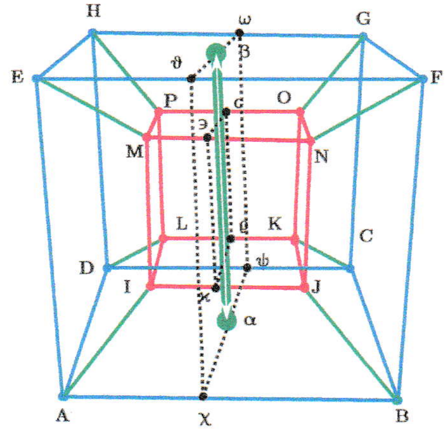
- **Right–turning rotations** $(R_1 \wedge R_2 \wedge R_3)$, around a $(\alpha\beta \vee \gamma\delta \vee \varepsilon\zeta)$–axis,
- **Mirror–operation** $(M_1 \wedge M_2 \wedge M_3 \wedge M_4)$ with respect to the appropriate mirror–objects.

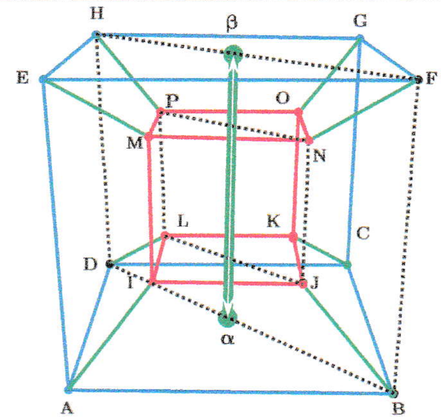
The first sub–group based on direction **(αβ)** follows immediately with **(64) permutations** according to all multiplications of operations(column(0)) and of operations(row(0)):

	★	id	R_1	R_2	R_3	M_1	M_2	M_3	M_4
αβ	id	E F G H M N O P I J K L A B C D = id	H E F G P M N O L I J K D A B C $= R_1$	G H E F O P M N K L I J C D A B $= R_2$	F G H E N O P M J K L I B C D A $= R_3$	H G F E P O N M L K J I D C B A $= M_1$	E H G F M P O N I L K J A D C B $= M_2$	F E H G N M P O J I L K B A D C $= M_3$	G F E H O N M P K J I L C B A D $= M_4$
	R_1	H E F G P M N O L I J K D A B C $= R_1$	G H E F O P M N K L I J C D A B $= R_2$	F G H E N O P M J K L I B C D A $= R_3$	E F G H M N O P I J K L A B C D = id	G F E H O N M P K J I L C B A D $= M_4$	H G F E P O N M L K J I D C B A $= M_1$	E H G F M P O N I L K J A D C B $= M_2$	F E H G N M P O J I L K B A D C $= M_3$
	R_2	G H E F O P M N K L I J C D A B $= R_2$	F G H E N O P M J K L I B C D A $= R_3$	E F G H M N O P I J K L A B C D = id	H E F G P M N O L I J K D A B C $= R_1$	F E H G N M P O J I L K B A D C $= M_3$	G F E H O N M P K J I L C B A D $= M_4$	H G F E P O N M L K J I D C B A $= M_1$	E H G F M P O N I L K J A D C B $= M_2$
	R_3	F G H E N O P M J K L I B C D A $= R_3$	E F G H M N O P I J K L A B C D = id	H E F G P M N O L I J K D A B C $= R_1$	C H E F O P M N K L I J C D A B $= R_2$	E H G F M P O N I L K J A D C B $= M_2$	F E H G N M P O J I L K B A D C $= M_3$	G F E H O N M P K J I L C B A D $= M_4$	H G F E P O N M L K J I D C B A $= M_1$
λ μ ν ξ η θ ι κ	M_1	H G F E P O N M L K J I D C B A $= M_1$	E H G F M P O N I L K J A D C B $= M_2$	F E H G N M P O J I L K B A D C $= M_3$	G F E H O N M P K J I L C B A B $= M_4$	E F G H M N O P I J K L A B C D = id	H E F G P M N O L I J K D A B C $= R_1$	G H E F O P M N K L I J C D A B $= R_2$	F G H E N O P M J K L I B C D A $= R_3$
I K O M A C G E	M_2	E H G F M P O N I L K J A D C B $= M_2$	F E H G N M P O J I L K B A D C $= M_3$	G F E H O N M P K J I L C B A D $= M_4$	H G F E P O N M L K J i D C B A $= M_1$	F G H E N O P M J K L I B C D A $= R_3$	E F G H M N O P I J K L A B C D = id	H E F G P M N O L I J K D A B C $= R_1$	G H E F O P M N K L I J C D A B $= R_2$
ϰ ϱ c ϶ χ ψ ω ϑ	M_3	F E H G N M P O J I L K B A D C $= M_3$	G F E H O N M P K J I L C B A D $= M_4$	H G F E P O N M L K J I D C B A $= M_1$	E H G F M P O N I L K J A D C B $= M_2$	G H E F O P M N K L I J C D A B $= R_2$	F G H E N O P M J K L I B C D A $= R_3$	E F G H M N O P I J K L A B C D = id	H E F G P M N O L I J K D A B C $= R_1$
J N P L B F H D	M_4	G F E H O N M P K J I L C B A D $= M_4$	H G F E P O N M L K J I D C B A $= M_1$	E H G F M P O N I L K J A D C B $= M_2$	F E H G N M P O J I L K B A D C $= M_3$	H E F G P M N O L I J K D A B C $= R_1$	G H E F O P M N K L I J C D A B $= R_2$	F G H E N O P M J K L I B C D A $= R_3$	E F G H M N O P I J K L A B C D = id



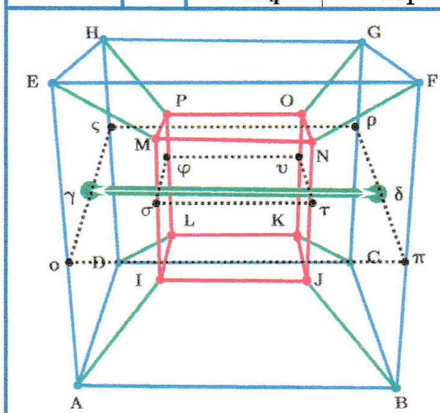


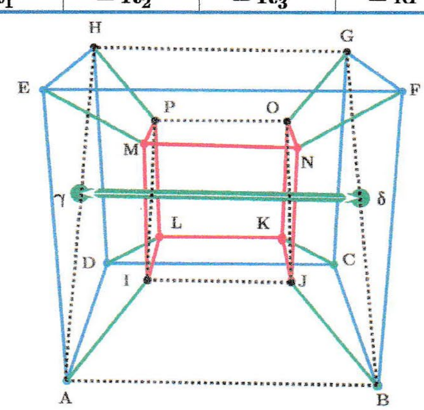


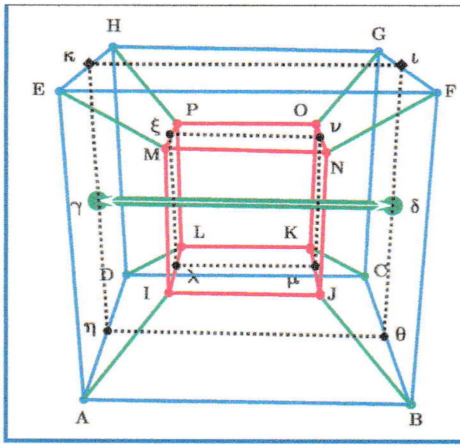


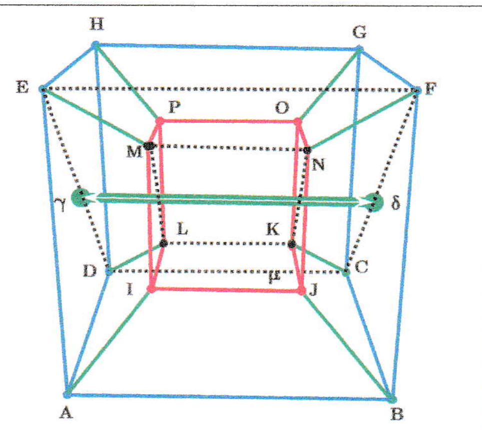
A second sub–group based on direction **(γδ)** follows next with **(64) permutations** according to all multiplications of operations(olumn(0)) and of operations(row(0)):

	★	id	R_1	R_2	R_3	M_1	M_2	M_3	M_4
γδ	id	B C G F J K O N I L P M A D H E = id	F B C G N J K O M I L P E A D H = R_1	G F B C O N J K P M I L H E A D = R_2	C G F B K O N J L P M I D H E A = R_3	F G C B N O K J M P L I E H D A = M_1	B F G C J N O K I M P L A E H D = M_2	C B F G K J N O L I M P D A E H = M_3	G C B F O K J N P L I M H D A E = M_4
	R_1	F B C G N J K O M I L P E A D Π = R_1	G F B C O N J K P M I L Π E A D = R_2	C G F B K O N J L P M I D Π E A = R_3	B C G F J K O N I L P M A D Π E = id	G C B F O K J N P L I M H D H E = M_4	F G C B N O K J M P L I E H D A = M_1	B F G C J N O K I M P L A E H D = M_2	C B F G K J N O L I M P D A E N = M_3
	R_2	G F B C O N J K P M I L H E A D = R_2	C G F B K O N J L P M I D H E A = R_3	B C G F J K O N I L P M A D H E = id	F B C G N J K O M I L P E A D H = R_1	C B F G K J N O L I M P D A E H = M_3	G C B F O K J N P L I M H D A E = M_4	F G C B N O K J M P L I E H D A = M_1	B F G C J N O K I M P L A E H D = M_2
	R_3	C G F B K O N J L P M I D H E A = R_3	B C G F J K O N I L P M A D H E = id	F B C G N J K O M I L P E A D Π = R_1	G F B C O N J K P M I L Π E A D = R_2	B F G C J N O K I M P L A E Π D = M_2	C B F G K J N O L I M P D A E Π = M_3	G C B F O K J N P L I M H D A E = M_4	F G C B N O K J M P L i E H D A = M_1
σ τ υ φ ο π ρ ς	M_1	F G C B N O K J M P L I E H D A = M_1	B F G C J N O K I M P L A E H D = M_2	C B F G K J N O L I M P D A E H = M_3	G C B F O K J N P L I M H D A E = M_4	B C G F J K O N I L P M A D Π E = id	F B C G N J K O M I L P E A D Π = R_1	G F B C O N J K P M I L Π E A D = R_2	C G F B K O N J L P M I D H E A = R_3
I J O P A B G H	M_2	B F G C J N O K I M P L Λ E H D = M_2	C B F G K J N O L I M P D A E H = M_3	G C B F O K J N P L I M H D A E = M_4	F G C B N O K J M P L I E H D A = M_1	C G F B K O N J L P M I D H E A = R_3	B C G F J K O N I L P M A D H E = id	F B C G N J K O M I L P E A D Π = R_1	G F B C O N J K P M I L Π E A D = R_2
λ μ ν ξ η θ ι κ	M_3	C B F G K J N O L I M P D A E H = M_3	G C B F O K J N P L I M H D A E = M_4	F G C B N O K J M P L I E H D A = M_1	B F G C J N O K I M P L A E H D = M_2	G F B C O N J K P M I L H E A D = R_2	C G F B K O N J L P M I D H E A = R_3	B C G F J K O N I L P M A D H E = id	F B C G N J K O M I L P E A D H = R_1
L K L M D C F E	M_4	G C B F O K J N P L I M H D A E = M_4	F G C B N O K J M P L I E H D A = M_1	B F G C J N O K I M P L A E H D = M_2	C B F G K J N O L I M P D A E H = M_3	F B C G N J K O M I L P E A D H = R_1	G F B C O N J K P M I L H E A D = R_2	C G F B K O N J L P M I D H E A = R_3	B C G F J K O N I L P M A D H E = id



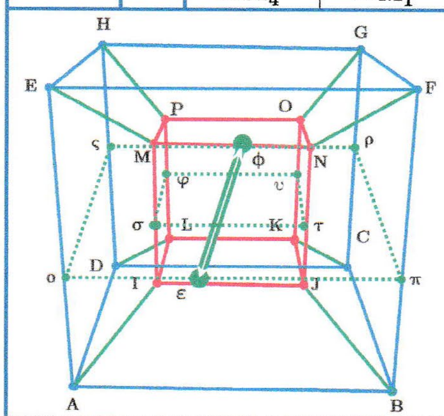


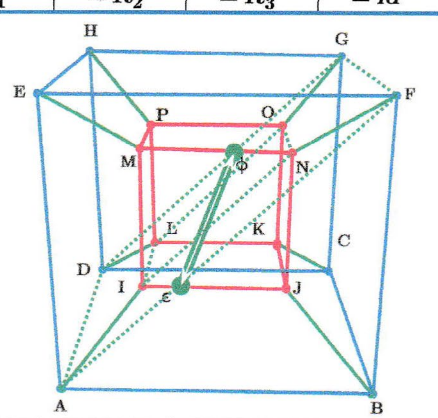




And finally a sub–group based on direction **(εζ)** with **(64) permutations** will follow according all multiplications of operations(column(0)) and operations(row(0)):

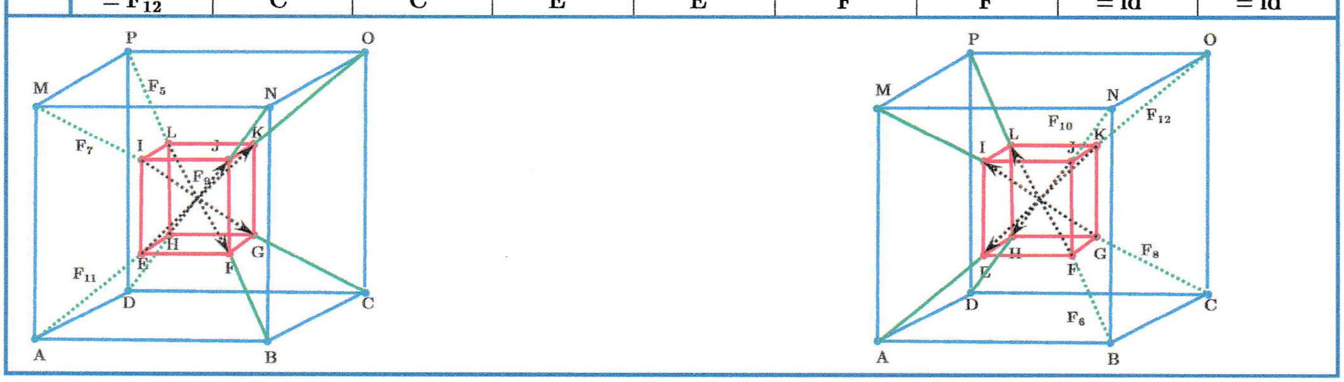
	★	id	R_1	R_2	R_3	M_1	M_2	M_3	M_4
εζ	id	A B F E I J N M L K O P D C G H = id	E A B F M I J N P L K O H D C G = R_1	F E A B N M I J O P L K G H D C = R_2	B F E A J N M I K O P L C G H D = R_3	E F B A M N J I P O K L H G C D = M_1	A E F B I M N J L P O K D H G C = M_2	B A E F J I M N K L P O C D H G = M_3	F B A E N J I M O K L P G C D H = M_4
	R_1	E A B F M I J N P L K O H D C G = R_1	F E A B N M I J O P L K G H D C = R_2	B F E A J N M I K O P L C G H D = R_3	A B F E I J N M L K O P D C G H = id	F B A E N J I M O K L P G C D H = M_4	E F B A M N J I P O K L H G C D = M_1	A E F B I M N J L P O K D H G C = M_2	B A E F J I M N K L P O C D H G = M_3
	R_2	F E A B N M I J O P L K G H D C = R_2	B F E A J N M I K O P L C G H D = R_3	A B F E I J N M L K O P D C G H = id	E A B F M I J N P L K O H D C G = R_1	B A E F J I M N K L P O C D H G = M_3	F B A E N J I M O K L P G C D H = M_4	E F B A M N J I P O K L H G C D = M_1	A E F B I M N J L P O K D H G C = M_2
	R_3	B F E A J N M I K O P L C G H D = R_3	A B F E I J N M L K O P D C G H = id	E A B F M I J N P L K O H D C G = R_1	F E A B N M I J O P L K G H D C = R_2	A E F B I M N J L P O K D H G C = M_2	B A E F J I M N K L P O C D H G = M_3	F B A E N J I M O K L P G C D H = M_4	E F B A M N J I P O K L H G C D = M_1
σ τ υ φ ο π ρ ς	M_1	E F B A M N J I P O K L H G C D = M_1	A E F B I M N J L P O K D H G C = M_2	B A E F J I M N K L P O C D H G = M_3	F B A E N J I M O K L P G C D H = M_4	A B F E I J N M L K O P D C G H = id	E A B F M I J N P L K O H D C G = R_1	F E A B N M I J O P L K G H D C = R_2	B F E A J N M I K O P L C G H D = R_3
I N Q E A F G D	M_2	A E F B I M N J L P O K D H G C = M_2	B A E F J I M N K L P O C D H G = M_3	F B A E N J I M O K L P G C D H = M_4	E F B A M N J I P O K L H G C D = M_1	B F E A J N M I K O P L C G H D = R_3	A B F E I J N M L K O P D C G H = id	E A B F M I J N P L K O H D C G = R_1	F E A B N M I J O P L K G H D C = R_2
ϰ ϱ ϲ ϶ χ ψ ω ϑ	M_3	B A E F J I M N K L P O C D H G = M_3	F B A E N J I M O K L P G C D H = M_4	E F B A M N J I P O K L H G C D = M_1	A E F B I M N J L P O K D H G C = M_2	F E A B N M I J O P L K G H D C = R_2	B F E A J N M I K O P L C G H D = R_3	A B F E I J N M L K O P D C G H = id	E A B F M I J N P L K O H D C G = R_1
J K P M B C H E	M_4	F B A E N J I M O K L P G C D H = M_4	E F B A M N J I P O K L H G C D = M_1	A E F B I M N J L P O K D H G C = M_2	B A E F J I M N K L P O C D H G = M_3	E A B F M I J N P L K O H D C G = R_1	F E A B N M I J O P L K G H D C = R_2	B F E A J N M I K O P L C G H D = R_3	A B F E I J N M L K O P D C G H = id





In addition to these (21) symmetry–operations **(8) flip–operations** will have be considered, due to the (8) quaternion– diagonals of the hypercube:

★	id	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
id	E F G H M N O P I J K L A B C D = id	C D A P G H E L K F I J O B M N = F_5	C D A P G H E L K F I J O B M N = F_6	M D A B I H E F K L G J O P C N = F_7	M D A B I H E F K L G J O P C N = F_8	C N A B G J E F K L I H O P M D = F_9	C N A B G J E F K L I H O P M D = F_{10}	C D O B G H K F E L I J A P M N = F_{11}	C D O B G H K F E L I J A P M N = F_{12}
F_5	C D A P G H E L K F I J O B M N = F_5	E F G H M N O P I J K L A B C D = id	E F G H M N O P I J K L A B C D = id	C N O B G J K F E L I H A P M D A	C N O B G J K F E L I H A P M D A	M D O B I H K F E L G J A P C N B	M D O B I H K F E L G J A D C N B	M N A B I J E F K L G H O P C D C	M N A B I J E F K L G H O P C D C
F_6	C D A P G H E L K F I J O B M N = F_6	E F G H M N O P I J K L A B C D = id	E F G H M N O P I J K L A B C D = id	C N O B G J K F E L I H A P M D A	C N O B G J K F E L I H A P M D A	M D O B I H K F E L G J A P C N B	M D O B I H K F E L G J A D C N B	M N A B I J E F K L G H O P C D C	M N A B J E F K L G H O P C D C
F_7	M D A B I H E F K L G J O P C N = F_7	C N O B G J K F E L I H A P M D A	C N O B G J K F E L I H A P M D A	E F G H M N O P I J K L A B C D = id	E F G H M N O P I J K L A B C D = id	C D O P G H K L E F I J A B M N D	C D O P G H K L E F I J A B M N D	C N A P G J E L K F I H O B M D E	C N A P G J E L K F I H O B M D E
F_8	M D A B I H E F K L G J O P C N = F_8	C N O B G J K F E L I H A P M D A	C N O B G J K F E L I H A P M D A	E F G H M N O P I J K L A B C D = id	E F G H M N O P I J K L A B C D = id	C D O P G H K L E F I J A B M N D	C D O P G H K L E F I J A B M N D	C N A P G J E L K F I H O B M D E	C N A P G J E L K F I H O B M D E
F_9	C N A B G J E F K L I H O P M D = F_9	M D O I H K F E L G J A P C N B	M D O B I H K F E L G J A P C N B	C D O P G H K L E F I J A B M N D	C D O P G H K L E F I J A B M N D	E F G H M N O P I J K L A B C D = id	E F G H M N O P I J K L A B C D = id	M D A P I H E L K F G J O B C N F	M D A P I H E L K F G J O B C N F
F_{10}	C N A B G J E F K L I H O P M D = F_{10}	M D O B I H K F E L G J A P C N B	M D O I H K F E L G J A P C N B	C D O P G H K L E F I J A B M N D	C D O P G H K L E F I J A B M N D	E F G H M N O P I J K L A B C D = id	E F G H M N O P I J K L A B C D = id	M D A P I H E L K F G J O B C N F	M D A P I H E L K F G J O B C N F
F_{11}	C D O B G H K F E L I J A P M N = F_{11}	M N A B I J E F K L G H O P C D C	M N A B I J E F K L G H O P C D C	C N A P G J E L K F I H O B M D E	C N A P G J E L K F I H O B M D E	M D A P I H E L K F G J O B C N F	M D A P I H E L K F G J O B C N F	E F G H M N O P I J K L A B C D = id	E F G H M N O P I J K L A B C D = id
F_{12}	C D O B G H K F E L I J A P M N = F_{12}	M N A B I J E F K L G H O P C D C	M N A B I J E F K L G H O P C D C	C N A P G J E L K F I H O B M D E	C N A P G J E L K F I H O B M D E	M D A P I H E L K F G J O B C N F	M D A P I H E L K F G J O B C N F	E F G H M N O P I J K L A B C D = id	E F G H M N O P I J K L A B C D = id



Together with **(200) symmetry–operations** for the **(8) inner cubes** of a hyper–cube, **(232) symmetry–operations** in total have to be counted for a hyper–cube and are responsible for its symmetry–group.

3.4. Symmetry-Group of the related JULIA-Network.

The **(16) different fixed–points** ($\mathbb{H}_{[1^\frown J\in\langle 0,15\rangle]}$) by definition from above will form a hyper–cube in quaternion–space. Thus a **probe–point** moving from **($\mathbb{H}_{[1^\frown M]}$) to ($\mathbb{H}_{[1^\frown N]}$)** by execution of a hyper–cube's symmetry–operations will change its **($\mathbb{N}$)** fluently from **($\mathbb{N}_{[M]}$)** to **($\mathbb{N}_{[N]}$)**. Due to the fact, that each of the images or pre–images must follow **equations (2.3^1. ∧ 2.3^2)** in any position of the probe, they will always be adapted in relation to the probe's location. Therefore the probe in essence mediates between the JULIA–sets with fixed–points ($\mathbb{H}_{[1^\frown M]}$) and ($\mathbb{H}_{[1^\frown N]}$).

In summery one may say, that the related JULIA–network under the action of any symmetry–operation of a hyper–cube will remain completely in itself. Thus, related JULIA–network and the symmetry–operations of a hyper–cube will built a symmetry–group.

4. Summary.

The iteration of **sequence (1^3.)** in quaternion–space – with restrictions from MANDELBROT–set on the complex components of its iteration–constant – resulted in a network of (3) sets. An unbounded escape–set (with trajectories escaping to infinity) accompanied by a set caught in a limited area (prisoner–set, whose trajectories tended to a sink–point) and the boundary–set of the prisoner–set built by points acting repulsively on points from escape– and prisoner–set as well.

The iteration stopped if the sink–point of the prisoner–set and a fixed repeller–point on JULIA–set had been obtained, that is, when equality between the iteration's predecessor– and successor–state had been reached. A Quaternion–condition for this stop–event (the fixed–point–condition) could be formulized and – by taking into account the HAMILTONian rules – could be separated into three sub–conditions (according to the quaternion–space's complex subspaces). Every one of these sub–conditions could subsequently be solved independently. On base of these results it became possible to express the quaternion fixed–points of prisoner– and JULIA–set as well.

With knowledge of the fixed–repeller–point of a JULIA–set it became possible to describe the structure of the JULIA–set by the set of images and pre–images, which are obtained from forward– or backward–iteration relative to the repeller.

Fixed–points and JULIA–set of the network, obtained by iterative execution of **sequence (1^3.)** will only depended on the choice of the actual iteration–constant. Therefore, (16) constants appropriately chosen from black part of the MANDELBROT–set will make it possible to arrange the repeller–fixed–points of the iteratively obtained JULIA–sets in the square–points of a hyper–cube. Fixed–points and their JULIA–sets positioned this way will then represent a related JULIA–network. The set of quaternion–points of the related JULIA–network together with the symmetry–operations of a hyper–cube will form the symmetry–group of the related JULIA–network.

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