ISSN: 2639-0108

Review Article

Advances in Theoretical & Computational Physics

A solution of the Third Order Homogeneous Equations

Uchida Keitaroh

Applied Mathematics Department

*Corresponding author
Uchida Keitaroh, Applied Mathematics Department, Japan.

Submitted: 10 Dec 2020; **Accepted**: 17 Dec 2020; **Published**: 30 Dec 2020

Construction and characteristics of the hyper-exponential functions

The form of hyper-exponential functions of n-order

$$exph_j^n((x; f(x)))$$

n: order.
j: the number of seed.
x: variable.
$f(x)$: *any function that is defined in an interval that contains zero.*

$$Seed\ (x; j) = \frac{x^j}{j!}\quad (j = 0, 1, 2, 3 \ldots n-1)$$

The seed of a hyper-exponential function means the first term of the series.

The most important characteristic is as follows.

$$y = exph_j^n(x; f(x))$$
$$\frac{d^n y}{dx^n} = f(x)\, y$$

Definition of hyper-exponential functions of second-order

$$x, g(x) \in R$$

$$k_0 = 1$$

$$k_{i+1}(x) = \int_0^x \int_0^x dx^2\, g(x) \times k_i(x)$$

$$exph_0^2(x; g(x)) = \sum_{i=0}^{\infty} k_i(x)$$

and

$$k_0 = x$$
$$k_{j+1}(x) = \int_0^x \int_0^x dx^2\, g(x) \times k_j(x)$$

$$exph_1^2(x; g(x)) = \sum_{j=0}^{\infty} k_j(x)$$

The characteristics are as follows.

$$v_0(x) = exph_0^2(x; g(x))$$
$$\frac{d^2 v_0(x)}{dx^2} = g(x) v_0(x)$$

and

$$v_1(x) = exph_1^2(x; g(x))$$
$$\frac{d^2 v_1(x)}{dx^2} = g(x) v_1(x)$$

and

The Wronskian is 1.

$$\begin{vmatrix} v_0(x) & v_1(x) \\ v_0'(x) & v_1'(x) \end{vmatrix} = v_0(x) v_1'(x) - v_1(x) v_0' = 1$$

A solution of the Third Order Homogeneous Equations

I show a solution of the following differential equation with variable coefficients A(x), B(x) and C(x).

$$x \in R,$$
$$A(x), B(x), C(x) \in R$$

$A(x)$, $B(x)$ and $C(x)$: *any differentiable function that is defined in an interval that contains zero.*

$$y''' + A(x) y'' + B(x) y' + C(x) y = 0 \text{ - - - ①}$$

I set

$$x \in R$$
$$u(x) = exph_0^1(x; f(x)) = exp\left\{\int_0^x f(x)\, dx\right\}$$
$$v_0(x) = exph_0^2(x; g(x))$$
$$v_1(x) = exph_1^2(x; g(x))$$

I set

h(x):any differentiable function that is defined in an interval that contains zero.

$$\left.\begin{aligned} & k_0 = 1 \\ & k_{i+1}(x) = \int_0^x \left\{ u v_1 \int_0^x u^{-1} v_0 h(x) k_i(x) dx - u v_0 \int_0^x u^{-1} v_1 h(x) k_i(x) dx \right\} dx \\ & y = \sum_{i=0}^{\infty} k_i(x) = 1 + \sum_{i=0}^{\infty} \int_0^x \left\{ u v_1 \int_0^x u^{-1} v_0 h(x) k_i(x) dx - u v_0 \int_0^x u^{-1} v_1 h(x) k_i(x) dx \right\} dx \end{aligned}\right\} ②$$

Suppose that the right hand side of ② is uniformly convergent.

$$y' = \sum_{i=0}^{\infty} \left\{ u v_1 \int_0^x u^{-1} v_0 h(x) k_i(x) dx - u v_0 \int_0^x u^{-1} v_1 h(x) k_i(x) dx \right\}$$

$$\begin{aligned} y'' = f(x) &\sum_{i=0}^{\infty} \left\{ u v_1 \int_0^x u^{-1} v_0 h(x) k_i(x) dx - u v_0 \int_0^x u^{-1} v_1 h(x) k_i(x) dx \right\} \\ &+ \sum_{i=0}^{\infty} \left\{ u\, v_1' \int_0^x u^{-1} v_0 h(x) k_i(x) dx - u\, v_0' \int_0^x u^{-1} v_1 h(x) k_i(x) dx \right\} \\ &+ \sum_{i=0}^{\infty} (v_1 v_0 - v_0 v_1)\, h(x) k_i(x) \end{aligned}$$

$$y'' = f(x) y' + \sum_{i=0}^{\infty} \left\{ u\, v_1' \int_0^x u^{-1} v_0 h(x) k_i(x) dx - u\, v_0' \int_0^x u^{-1} v_1 h(x) k_i(x) dx \right\}$$

$$\begin{aligned} y''' = f'(x)\, y' + f(x)\, y'' & \\ &+ f(x) \sum_{i=0}^{\infty} \left\{ u\, v_1' \int_0^x u^{-1} v_0 h(x) k_i(x) dx - u\, v_0' \int_0^x u^{-1} v_1 h(x) k_i(x) dx \right\} + g(x) y' \\ &+ \sum_{i=0}^{\infty} (v_1'\, v_0 - v_0'\, v_1) h(x) k_i(x) \end{aligned}$$

$$y''' = f'(x) y' + f(x) y'' + f(x)\{y'' - f(x) y'\} + g(x) y' + h(x) y$$

$$y''' - 2 f(x) y'' + \{f(x)^2 - f'(x) - g(x)\} y' - h(x) y = 0 \quad - - - ③$$

From ① and ③

A (x) = -2f (x)

B (x) = f (x)² - f ' (x) - g(x)

C (x) = -h(x)

From the above

$$f(x) = -\frac{A(x)}{2}$$

$$g(x) = \frac{A(x)^2}{4} + \frac{A'(x)}{2} - B(x)$$

$$h(x) = -C(x)$$

∴

② is one of the solutions of ①.

References

1. *Kumahara K, Saitoh S, Uchida K (2009) Normal solutions of linear ordinary differential equations of the second order, International Journal of Applied Mathematics 22: 981-996.*
2. *Uchida K (2017) Introduction to Hyper exponential function and differential equation revised first edition, eBookland. (In Japanese).*
3. *Uchida K (2018) Hyper Exponential Function, Advances in Theoretical & Computational Physics 1: 1-3.*
4. *Uchida K (2019) How to Generate the Hyper Exponential Functions, Advances in Theoretical & Computational Physics 2: 1-2.*
5. *Uchida K (2020) How to execute repeated integral to generate hyper-exponential functions, Advances in Theoretical & Computational Physics 3: 218- 219.*
6. *Videos on YouTube*
   - *Introduction of hyper-exponential function (2018).*
   - *How to solve the second order linear homogeneous equation with variable coefficients (2020).*
   - *How to derive the hyper-exponential function of second-order from the second order liner homogeneous equation with variable coefficients (2020).*