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A Short Note on the Infinity Product Tan (z) Function

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Abstract
We present a novel derivation of the infinite product of the tangent function, Tan(z), expressed in terms of trigonometric expressions including Euler's Sinc function and Viète's formula, along with their generalizations.

Key Words and Phrases: Viète's Formula, Tan Function.

1. Introduction

Infinite product (1) was previously published by Viète in 1593, based on geometric considerations. A detailed compendium, including analysis and evaluations of infinite products of elementary functions in terms of trigonometric functions, can be found in the classical literature

$$\prod_{k=0}^{\infty} \cos\left(\pi 2^{-k-2}\right) = \frac{2}{\pi} \tag{1}$$

Viète's formula is the following infinite product of nested radicals

$$\frac{1}{8}\sqrt{\frac{1}{2}(\sqrt{2}+2)(\sqrt{\sqrt{2}+2}+2)(\sqrt{\sqrt{\sqrt{2}+2}+2}+2)...} = \frac{2}{\pi} \tag{2}$$

Viète's formula may be obtained as a special case of a formula for the sinc function that has often been attributed to Leonhard Euler[1], more than a century later:

$$\prod_{k=0}^{\infty} \cos\left(2^{-k-1} z\right) = \frac{\sin(z)}{z} \tag{3}$$

expressing each term of the product on the left as a function of earlier terms using the half-angle formula:

$$\cos\left(\frac{z}{2}\right) = \sqrt{\frac{1}{2}(\cos(z)+1)} \tag{4}$$

gives Viète's formula.
This formula has been generalization obtained through use of both Chebyshev polynomial and Fourier transformation (see Nishimura [3] and Kent E. Morrison [2]) for integer $q \geq 2$. As

$$\prod_{j=0}^{\infty} \frac{\sum_{n=1}^{q} \cos\left(\frac{(2n-1)z}{(2q)^{j+1}}\right)}{q} = \frac{\sin(z)}{z} \tag{5}$$

This can be written in more compact way

$$\prod_{k=0}^{\infty} \frac{\sin\left(zq^{-k}\right)\csc\left(zq^{-k-1}\right)}{q} = \operatorname{sinc}(z) \tag{6}$$

2. The Tan Function

We introduce two new generalizations of Euler's infinite product of the tangent function in the style of sinc(z) type, $q \geq 2$.

$$\prod_{k=0}^{\infty} 2^{-\left((q-1)q^k\right)} q^{-q^k(k(q-1)+q)} z^{(q-1)q^k} \times \tan^{-q^{k+1}}\left(\frac{1}{2} z q^{-k-1}\right) \tan^{q^k}\left(\frac{zq^{-k}}{2}\right) = \frac{2\tan\left(\frac{z}{2}\right)}{z}$$

And the equivalent convergent product

$$\prod_{k=0}^{\infty} \frac{\tan\left(\frac{zq^{-k}}{2}\right)\cot\left(\frac{1}{2}zq^{-k-1}\right)}{q} = \frac{2\tan\left(\frac{z}{2}\right)}{z} \tag{8}$$

3. Generalization

It is possible to generalize equations (7) and (8) so that n and m are integers $\{n,m\} \geq 0$ as follows:

$$\prod_{k=m}^{n-1} 2^{-\left((q-1)q^k\right)} q^{-q^k(k(q-1)+q)} z^{(q-1)q^k} \times \tan^{-q^{k+1}}\left(\frac{1}{2} z q^{-k-1}\right) \tan^{q^k}\left(\frac{zq^{-k}}{2}\right) = 2^{q^m-q^n} q^{mq^m-nq^n} z^{q^n-q^m} \times \tan^{q^m}\left(\frac{zq^{-m}}{2}\right) \tan^{-q^n}\left(\frac{zq^{-n}}{2}\right) \tag{9}$$

And

$$\prod_{k=m}^{n-1} \frac{\tan\left(\frac{zq^{-k}}{2}\right)\cot\left(\frac{1}{2}zq^{-k-1}\right)}{q} = q^{m-n}\tan\left(\frac{zq^{-m}}{2}\right)\cot\left(\frac{zq^{-n}}{2}\right) \tag{10}$$

For brevity, we employ mathematical induction to prove the case of Formula (10) when $q = 2$ and $m = 0$

Proof: starting with

$$\prod_{k=0}^{n-1} \frac{1}{2}\tan\left(2^{-k-1}z\right)\cot\left(2^{-k-2}z\right) = 2^{-n}\tan\left(\frac{z}{2}\right)\cot\left(2^{-n-1}z\right) \tag{11}$$

Step 1: Base Case $n = 1$ We need to show that the formula holds for $n = 1$: the left-hand side LHS becomes:

$$\frac{1}{2}\tan\left(2^{-1-1}z\right)\cot\left(2^{-1-2}z\right) = \frac{1}{2}\tan\left(\frac{z}{2}\right)\cot\left(\frac{z}{4}\right) \tag{12}$$

The right-hand side RHS becomes:

$$2^{-1}\tan\left(\frac{z}{2}\right)\cot\left(2^{-1-1}z\right) = \frac{1}{2}\tan\left(\frac{z}{2}\right)\cot\left(\frac{z}{4}\right) \tag{13}$$

The LHS and RHS are equal for the base case.

Step 2: Iinductive hypothesis assume that the formula is true for some arbitrary positive integer n:

$$\prod_{k=0}^{n-1} \frac{1}{2}\tan\left(2^{-k-1}z\right)\cot\left(2^{-k-2}z\right) = 2^{-k}\tan\left(\frac{z}{2}\right) \tag{14}$$

Step 3: Inductive step we need to prove that the formula holds for $n + 1$: Consider the LHS for $n + 1$:

$$\prod_{k=0}^{n} \frac{1}{2} \tan\left(2^{-k-1} z\right) \cot\left(2^{-k-2} z\right) \tag{15}$$

Multiply both sides by $\left(\frac{1}{2}\tan\left(2^{-n-1}z\right)\cot\left(2^{-n-2}z\right)\right)$:

$$\left(\prod_{k=0}^{n-1} \frac{1}{2} \tan\left(2^{-k-1} z\right) \cot\left(2^{-k-2} z\right)\right) \cdot \frac{1}{2} \tan\left(2^{-\mathrm{n}-1} z\right) \cot\left(2^{-\mathrm{n}-2} z\right) \tag{16}$$

Using the inductive hypothesis, the RHS simplifies to:

$$2^{-k} \tan\left(\frac{z}{2}\right) \cot\left(2^{-\mathrm{n}-1} z\right) \cdot \frac{1}{2} \tan\left(2^{-\mathrm{n}-1} z\right) \cot\left(2^{-\mathrm{n}-2} z\right) \tag{17}$$

Simplify the RHS:

$$2^{-\mathrm{n}-1} \tan\left(\frac{z}{2}\right) \cot\left(2^{-\mathrm{n}-1} z\right) \tag{18}$$

The RHS matches the LHS for $n + 1$.
Therefore, by mathematical induction, the given formula holds for all positive integers (n)
Example 1. Infinitely nested radicals.
Using several values for n Integers $\frac{2n\tan\left(\frac{\pi}{2n}\right)}{\pi}$ in formula (10) get the following table:

Product ∞	Result
$\frac{1}{4}\sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}}\cdots$	$\frac{4}{\pi}$
$\frac{(2-\sqrt{3})(2+\sqrt{3})\left(\frac{1}{4}\sqrt{3(2-\sqrt{2})}(1+\sqrt{2})+\frac{1}{4}(-1+\sqrt{2})\sqrt{2+\sqrt{2}}\right)}{4\sqrt{3}\left(\frac{1}{4}\sqrt{2-\sqrt{2}}(1+\sqrt{2})-\frac{1}{4}(-1+\sqrt{2})\sqrt{3(2+\sqrt{2})}\right)}\cdots$	$\frac{2\sqrt{3}}{\pi}$
$\frac{1}{4}\sqrt{\frac{(2-\sqrt{2})(2+\sqrt{2+\sqrt{2+\sqrt{2}}})}{(2+\sqrt{2})(2-\sqrt{2+\sqrt{2+\sqrt{2}}})}}\cdots$	$\frac{8\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}}{\pi}$
$\frac{\sqrt{1-\frac{2}{\sqrt{5}}}\left(\frac{1}{8}(-1+\sqrt{2})\sqrt{2+\sqrt{2}}(-1+\sqrt{5})+\frac{1}{4}(1+\sqrt{2})\sqrt{\frac{1}{2}(2-\sqrt{2})(5+\sqrt{5})}\right)}{4\left(-\frac{1}{8}\sqrt{2-\sqrt{2}}(1+\sqrt{2})(-1+\sqrt{5})+\frac{1}{4}(-1+\sqrt{2})\sqrt{\frac{1}{2}(2+\sqrt{2})(5+\sqrt{5})}\right)}\cdots$	$\frac{10\sqrt{1-\frac{2}{\sqrt{5}}}}{\pi}$
	$\frac{12(2-\sqrt{3})}{\pi}$
$\frac{1}{4}\sqrt{\frac{(2-\sqrt{2+\sqrt{2}})(2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}})}{(2+\sqrt{2+\sqrt{2}})(2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}})}}\cdots$	$\frac{16\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}}{\pi}$

Table 1

References

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