ISSN: 2834-7706

Review Article

Journal of Mathematical Techniques and Computational Mathematics

A Review of Analytical Algorithms for Approximate Solutions of Ordinary Differential Equation

Mohammed Hussein[1,2*]

[1]*Education Directorate of Thi-Qar, Ministry of Education, Thi-Qar, Iraq*

[2]*Scientific Research Center, Alayen University, Nasiriyah, Iraq*

[*]Corresponding Author
Mohammed Hussein, Inc, Education Directorate of Thi-Qar, Ministry of Education, Thi-Qar, Iraq

Submitted: 2023, Sep 20; **Accepted:** 2023, Oct 12; **Published:** 2023, Oct 20

Citation: Hussein, M. (2023). A Review of Analytical Algorithms for Approximate Solutions of Ordinary Differential Equation. *J Math Techniques Comput Math, 2*(10), 438-445.

Abstract
This review article aims to provide an in-depth analysis of a wide range of analytical methods used for solving ordinary differential equations (ODEs). ODEs are fundamental in numerous scientific and engineering disciplines, making the development, and understanding of effective solution techniques crucial. We explore various approaches, including the Adomian decomposition method, homotopy perturbation method, homotopy analysis method, variational iteration method, Daftardar-Jafari method, successive approximation method, power series method, and modified Adomian decomposition method. Each method is discussed in terms of its principles, applications, advantages, limitations, and computational considerations. This comprehensive overview will serve as a valuable resource for researchers, practitioners, and students interested in solving ODEs.

Keywords: Adomian Decomposition Method; Homotopy Permutation Method; Variational Perturbation Method; Power Series Method; Successive Approximation Method; Daftardar-Jafari Method; Differential Transform Method.

1. Introduction

In the vast landscape of mathematical problem-solving, ordinary differential equations (ODEs) stand as stalwart sentinels, guarding the gateway to understanding dynamic systems in the natural and engineering sciences. These mathematical entities possess an unparalleled ubiquity, offering an elegant language to describe phenomena that evolve over time. From the gentle sway of a pendulum to the convoluted behavior of stock market dynamics, ODEs are the thread that weaves together the narrative of change and evolution. However, while ODEs serve as unifying elements in the scientific tableau, their analytical solutions often prove to be enigmatic, complex, and elusive. It is here that the alchemy of mathematical methods and algorithms comes to the forefront, providing the enchanted keys to unlock the secrets held within these equations. In this profound exploration, we embark on a journey through the intricacies of ODE solving, shedding light on eight remarkable techniques that have emerged as formidable tools in the hands of mathematicians, scientists, and engineers: the Adomian Decomposition Method, Homotopy Perturbation Method, Homotopy Analysis Method, Variational Iteration Method, Daftardar-Jafari Method, Successive Approximation Method, Power Series Method, and Modified Adomian Decomposition Method [1-3].

The challenges posed by ODEs are as diverse as the systems they represent. Whether we are studying the thermal dynamics of a reactor, predicting the spread of infectious diseases, or analyzing the intricate patterns of population growth, ODEs present us with a kaleidoscope of mathematical conundrums. Each method we explore in this comprehensive review is a testament to human ingenuity and an indispensable instrument to decipher these puzzles. As we delve into this intricate world, it is essential to recognize that these methods are not merely abstract constructs confined to the realms of academia. They are formidable allies, standing at the vanguard of scientific progress. The Adomian Decomposition Method, with its power to break complex equations into solvable components, has found applications in fields as diverse as quantum mechanics and fluid dynamics. The Homotopy Perturbation Method, born from the concept of "small parameter," has enabled the analytical exploration of nonlinear ODEs, delivering insights into chaos theory and chaotic systems. The Homotopy Analysis Method, rooted in homotopy theory, provides a versatile approach to unravel the intricate dynamics of ODEs and has been employed to analyze heat conduction in composite materials and biomechanical systems [4-6].

The Variational Iteration Method, with its emphasis on variational principles, has brought forth an elegant avenue to approximate solutions for ODEs, making significant contributions in problems involving nonlinear phenomena. The Daftardar-Jafari Method, a rising star in the realm of ODE solving, leverages the concept of Laplace transform to tackle various nonlinear problems, finding applications in fluid dynamics and heat transfer. The Successive Approximation Method, as its name suggests, employs iterative techniques to approximate solutions, offering a robust approach for a wide range of ODEs. The Power Series Method, a classic technique, unfolds ODEs into power series expansions, allowing us to tackle a multitude of problems with precision. Lastly, the Modified Adomian Decomposition Method builds upon the foundations of its predecessor to enhance convergence and efficiency, rendering it a valuable tool in various scientific domains [7,8].

The sheer breadth of these methods is a testament to the intricate nature of ODEs and the diversity of problems they encapsulate. As we traverse the landscape of ODE-solving methodologies in this article, we aim not only to demystify their inner workings but also to provide practical insights into their applications across a spectrum of disciplines Whether you are a seasoned mathematician seeking to expand your arsenal of problem-solving techniques, a curious scientist yearning to explore the boundaries of mathematical modeling, or a dedicated student embarking on a journey of mathematical discovery, this article serves as your guide. Together, we will navigate the labyrinthine world of ODEs, revealing the riches hidden within and empowering you to wield these analytical methods and algorithms with mastery and precision. Join us on this intellectual odyssey as we embark on a comprehensive review of these remarkable tools, unearthing their secrets and harnessing their power to unveil the mysteries of the natural world [9-11].

In the subsequent sections, we delve into the details of each method, presenting a comprehensive analysis of their principles, applications, computational aspects, and relative merits. By the end of this review, readers will have gained a deep appreciation for the rich landscape of analytical tools available for unraveling the intricate dynamics encapsulated within ordinary differential equations. Before we begin reviewing these methods, we must know the general formula of the differential equation that can be solved by these methods. General Form of a nonhomogeneous First-Order Nonlinear ordinary differential equation

$$\frac{d\mathcal{Y}}{d\mathcal{X}} = F(\mathcal{X}, \mathcal{Y}).$$

Here, $\mathcal{Y}$ represents the dependent variable, $\mathcal{X}$ is the independent variable, F is a nonlinear function.

2. Review of Analytical Methods

2.1 Adomian Decomposition Method (ADM) [12]

Consider the following First-Order nonlinear ordinary differential equation,

$$\mathcal{L}\big(\mathcal{Y}(\mathcal{X})\big) + \mathcal{N}\big(\mathcal{Y}(\mathcal{X})\big) = g(\mathcal{X}) \ \#(1)$$

With initial condition $\mathcal{Y}(0) = \psi$, where $\mathcal{Y}$ is an unknown function, $\mathcal{L} = \frac{d}{d\mathcal{X}}$, $\mathcal{N}(\mathcal{Y})$ expresses the nonlinear terms, and $g(\mathcal{X})$ is an inhomogeneous term. We assume that $\mathcal{L}$ is invertible and the inverse operator $\mathcal{L}^{-1}$ is given by,

$$\mathcal{L}^{-1}(.) = \int_0^t (.)\, d\mathcal{X}.$$

Applying $\mathcal{L}^{-1}$ to both sides of Eq. (1) gives,

$$\mathcal{Y}(\mathcal{X}) = \psi + \mathcal{L}^{-1} g(\mathcal{X}) - \mathcal{L}^{-1}\mathcal{N}(\mathcal{Y}) \ \#(2)$$

The Adomian decomposition method admits the decomposition of $\mathcal{Y}$ into an infinite series of components,

$$\mathcal{Y}(\mathcal{X}) = \sum_{n=0}^{\infty} \mathcal{Y}_n \ \#(3)$$

and the nonlinear term $\mathcal{N}(\mathcal{Y})$ be equated to an infinite series of polynomials

$$\mathcal{N}(\mathcal{Y}) = \sum_{n=0}^{\infty} \mathcal{A}_n, \#(4)$$

where

$$\mathcal{A}_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\textstyle\sum_{i=0}^{n} \lambda^i \mathcal{Y}_i\right)\right]_{\lambda=0},\ n = 0,1,2,\cdots$$

where $\mathcal{A}_n$ are the Adomian polynomials. Substituting Eq. (3) and Eq. (4) into Eq. (2) gives

$$\sum_{n=0}^{\infty} \mathcal{Y}_n = \psi + \mathcal{L}^{-1} g(\mathcal{X}) - \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} \mathcal{A}_n\right). \#(5)$$

The various components $\mathcal{Y}_n$ of the solution $\mathcal{Y}$ can be easily determined by using the recursive relation

$$\begin{aligned} \mathcal{Y}_0 &= \psi + \mathcal{L}^{-1}\big(g(\mathcal{X})\big), \\ \mathcal{Y}_{k+1} &= -\mathcal{L}^{-1}(\mathcal{A}_k), k \geqslant 0. \end{aligned} \#(6)$$

Thus, the approximate solution of Eq. (1) is given by

$$\mathcal{Y}(\mathcal{X}) = \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3 + \cdots = \sum_{n=0}^{\infty} \mathcal{Y}_n. \#(7)$$

2.2 Modified Adomian Decomposition Method (MADM) [13].

Consider the following First-Order nonlinear ordinary differential equation,

$$\mathcal{L}\big(\mathcal{Y}(\mathcal{X})\big) + \mathcal{N}\big(\mathcal{Y}(\mathcal{X})\big) = g(\mathcal{X}), \#(8)$$

with initial condition $\mathcal{Y}(0) = \psi$. By using the algorithm of ADM, we get

$$\sum_{n=0}^{\infty} \mathcal{Y}_n = \psi + \mathcal{L}^{-1} g(\mathcal{X}) - \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} \mathcal{A}_n\right). \#(9)$$

The modified Adomian decomposition method (MADM) applies a slight modification to ADM, such that it splits $g(\mathcal{X})$ into two parts: $g(\mathcal{X}) = g_1(\mathcal{X}) + g_2(\mathcal{X})$ as follows:

$$\begin{aligned} \mathcal{Y}_0(\mathcal{X}) &= \psi + \mathcal{L}^{-1}\big(g_1(\mathcal{X})\big), \\ \mathcal{Y}_1(\mathcal{X}) &= \mathcal{L}^{-1}(g_2(\mathcal{X})) - \mathcal{L}^{-1}(\mathcal{A}_k), \#(10) \\ \mathcal{Y}_{k+1}(\mathcal{X}) &= -\mathcal{L}^{-1}(\mathcal{A}_k),\ k \geq 1. \end{aligned}$$

Thus, the approximate solution of Eq. (8) is given by

$$\mathcal{Y}(\mathcal{X}) = \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3 + \cdots = \sum_{n=0}^{\infty} \mathcal{Y}_n. \#(11)$$

2.3 Variational Perturbation Method (VIM) [14]

Consider the following First-Order nonlinear ordinary differential equation,

$$\mathcal{L}\big(\mathcal{Y}(\mathcal{X})\big) + \mathcal{N}\big(\mathcal{Y}(\mathcal{X})\big) = g(\mathcal{X}), \#(12)$$

with initial condition $\mathcal{Y}(0) = \psi$.The variational iteration method presents a correction functional for Eq. (12) in the form

$$\mathcal{Y}_{n+1}(\mathcal{X}) = \mathcal{Y}_n(\mathcal{X}) + \int_0^{\mathcal{X}} \lambda(\xi)\left(\mathcal{L}\tilde{\mathcal{Y}}_n(\xi) + \mathcal{N}\tilde{\mathcal{Y}}_n(\xi) - g(\xi)\right) d\xi, \#(13)$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{\mathcal{Y}}_n$ is a restricted variation which means $\delta\tilde{\mathcal{Y}}_n = 0$.

Make the variation of Eq. (13), we have

$$\delta\mathcal{Y}_{n+1}(\mathcal{X}) = \delta\mathcal{Y}_n(\mathcal{X}) + \delta\int_0^{\mathcal{X}} \lambda(\xi)\left(\mathcal{L}\tilde{\mathcal{Y}}_n(\xi) + \mathcal{N}\tilde{\mathcal{Y}}_n(\xi) - g(\xi)\right) d\xi, \#(14)$$

Since Eq. (12) is ordinary differential equation of first order, then $\lambda = -1$.

Thus,

$$\mathcal{Y}_{n+1}(\mathcal{X}) = \mathcal{Y}_n(\mathcal{X}) - \int_0^{\mathcal{X}} \lambda(\xi)\left(\mathcal{L}\tilde{\mathcal{Y}}_n(\xi) + \mathcal{N}\tilde{\mathcal{Y}}_n(\xi) - g(\xi)\right) d\xi, \#(15)$$

Consequently, the solution

$$\mathcal{Y}(\mathcal{X}) = \lim_{n\to\infty} \mathcal{Y}_n(\mathcal{X}). \#(16)$$

2.4 Successive Approximation Method (SAM) [15]

Consider the following First-Order nonlinear ordinary differential equation,

$$\mathcal{L}(\mathcal{Y}(\mathcal{X})) + \mathcal{N}(\mathcal{Y}(\mathcal{X})) = g(\mathcal{X}) \#(17)$$

With initial condition $\mathcal{Y}(0) = \psi$. Applying $\mathcal{L}^{-1}$ to both sides of Eq. (17) gives,

$$\mathcal{Y}(\mathcal{X}) = \psi + \mathcal{L}^{-1}g(\mathcal{X}) - \mathcal{L}^{-1}\mathcal{N}(\mathcal{Y}) \#(18)$$

The successive approximation method consists of representing the solution of Eq. (17) as a sequence $\{\mathcal{Y}_n\}_{n=0}^{\infty}$. The method introduces the recurrence relation

$$\mathcal{Y}(\mathcal{X}) = \psi + \mathcal{L}^{-1}g(\mathcal{X}) - \mathcal{L}^{-1}\mathcal{N}(\mathcal{Y}) \#(19)$$

Where the zero approximation $\mathcal{Y}_0(\mathcal{X})$ is an arbitrary real function. Several successive approximations $u_n, n \geq 1$ will be determined as.

$$\mathcal{Y}_{n+1}(\mathcal{X}) = \psi + \mathcal{L}^{-1}g(\mathcal{X}) - \mathcal{L}^{-1}\mathcal{N}(\mathcal{Y}_n) \#(20)$$

And the solution computed as:

$$\mathcal{Y}(\mathcal{X}) = \lim_{n\to\infty} \mathcal{Y}_n(\mathcal{X}). \#(21)$$

2.5 Homotopy Permutation Method (HPM) [16]

To illustrate the basic idea of the homotopy perturbation method, we consider the following differential equation:

$$\mathcal{A}(\mathcal{Y}) - f(r) = 0, r \in \Omega, \#(22)$$

where $\mathcal{A}$ is a general differential operator, and $f(r)$ is a known analytical function. Suppose that

$$\mathcal{A}(\mathcal{Y}) = \mathcal{L}(\mathcal{Y}) + \mathcal{N}(\mathcal{Y}). \#(23)$$

Therefore Eq. (22) can be rewritten as

$$\mathcal{L}(\mathcal{Y}) + \mathcal{N}(\mathcal{Y}) - f(r) = 0. \#(24)$$

By the homotopy perturbation technique, we construct a homotopy $\mathcal{V}(r, \wp): \Omega \times [0,1] \to \mathbb{R}$ which satisfies:

$$\mathcal{H}(\mathcal{V}, \wp) = (1 - \wp)[\mathcal{L}(\mathcal{V}) - \mathcal{L}(\mathcal{Y}_0)] + \wp[\mathcal{A}(\mathcal{V}) - f(r)] = 0, \#(25)$$

or

$$\mathcal{H}(\mathcal{V}, \wp) = \mathcal{L}(\mathcal{V}) - \mathcal{L}(\mathcal{Y}_0) + \wp\mathcal{L}(\mathcal{Y}_0) + \wp[\mathcal{N}(\mathcal{V}) - f(r)] = 0, \#(26)$$

where $\wp \in [0,1]$ is an embedding parameter and $\mathcal{Y}_0$ is an initial approximation of Eq. (22) which satisfies the boundary conditions. Considering Eq. (26), we will have:

$$\mathcal{H}(\mathcal{V}, 0) = \mathcal{L}(\mathcal{V}) - \mathcal{L}(\mathcal{Y}_0) = 0\#(27)$$

and

$$\mathcal{H}(\mathcal{V}, 1) = \mathcal{A}(\mathcal{V}) - f(r) = 0\#(28)$$

According to the homotopy perturbation theory, we can first use the embedding parameter $\wp$ as a small parameter and assume that the solution of Eq. (26) can be written as a power series in $\wp$:

$$\mathcal{V}=\sum_{n=0}^{\infty}\wp^{n}\mathcal{V}_{n}=\mathcal{V}_{0}+\wp\mathcal{V}_{1}+\wp^{2}\mathcal{V}_{2}+\cdots\#(29)$$

and nonlinear terms can be decomposed as:

$$\mathcal{N}(\mathcal{V})\sum_{n=0}^{\infty}\wp^{n}\mathcal{H}_{n}(\mathcal{V})=\mathcal{H}_{0}+\wp\mathcal{H}_{1}+\wp^{2}\mathcal{H}_{2}+\cdots,\#(30)$$

Where $\mathcal{H}_n(\mathcal{V})$ are called He's polynomials that are given by

$$\mathcal{H}_{n}=\frac{1}{n!}\frac{d^{n}}{d\wp^{n}}[\mathcal{N}(\textstyle\sum_{i=0}^{n}\wp^{i}\mathcal{V}_{i})]_{\wp=0},\ n=0,1,2,\cdots$$

Setting $\wp=1$ one have the approximation solution of Eq. (26)

$$\mathcal{Y}(\mathcal{X})=\lim_{\wp\to1}\mathcal{V}=\mathcal{V}_{0}+\mathcal{V}_{1}+\mathcal{V}_{2}+\cdots.\#(31)$$

2.6 Homotopy Analysis Method (HAM) [17]

Consider the following First-Order nonlinear ordinary differential equation,

$$\mathcal{N}\big(\mathcal{Y}(\mathcal{X})\big)=0,\#(32)$$

where $\mathcal{N}$ is anon linear operator, $\mathcal{X}$ is independent variable and $\mathcal{Y}$ is an unknown function. By the HAM, we construct a homotopy which satisfies

$$(1-q)\Big(\mathcal{L}\big(\psi(\mathcal{X},q)-\mathrm{y}(\mathcal{X})\big)\Big)=qh\mathcal{H}(\mathcal{X})\mathcal{N}\big(\psi(\mathcal{X};q)\big),\#(33)$$

where $q\in[0,1]$ is the embedding parameter, $h\neq0$ is an auxiliary parameter, $\mathcal{L}$ is an auxiliary linear operator, $\psi(\mathcal{X};q)$ is an unknown function, $\mathcal{Y}_0(\mathcal{X})$ is initial guess of $\mathcal{Y}(\mathcal{X})$, and $\mathcal{H}(\mathcal{X})$ is anon-zero auxiliary function.

At $q=0$ then,

$$\psi(\mathcal{X};0)=\mathcal{Y}_{0}(\mathcal{X}).\#(34)$$

At $q=1$ then,

$$\psi(\mathcal{X};1)=\mathcal{Y}(\mathcal{X}).\#(35)$$

Expanding $\psi(\mathcal{X};q)$ in taylor series with respect to q one has

$$\psi(\mathcal{X};q)=\mathcal{Y}(\mathcal{X})+\sum_{m=1}^{\infty}\mathcal{Y}_{m}(\mathcal{X})q^{m},\#(36)$$

where

$$\mathcal{Y}_{m}(\mathcal{X})=\frac{1}{m!}\frac{\partial^{m}\psi(\mathcal{X};q)}{\partial q^{m}}\bigg|_{q=0}.\#(37)$$

The convergence of series Eq. (36) depends up on h if it is convergent at $q=1$, one has

$$\mathcal{Y}(\mathcal{X})=\mathcal{Y}_{0}(\mathcal{X})+\textstyle\sum_{m=1}^{\infty}\mathcal{Y}_{m}(\mathcal{X}).\#(38)$$

Differentiating Eq. (33) m-times with respect to q and then dividing them by $m!$ and finally set $q=0$, we get

$$\mathcal{L}[\mathcal{Y}_{m}(\mathcal{X})-\mathcal{X}_{m}\mathcal{Y}_{m-1}(\mathcal{X})]=hR_{m}(\mathcal{Y}_{m-1})\#(39)$$

where

$$R_{m}(\mathcal{Y}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\mathcal{N}\big(\psi(\mathcal{X};q)|_{q=0}\#(40)$$

and

$$X_{m}=\begin{cases}0 & m\leqslant1\\1 & m>1\end{cases}\#(41)$$

For any given nonlinear operator $\mathcal{N}$, the term $R_m(\mathcal{Y}_{m-1})$ can be easily expressed by Eq. (40). Thus, we can gain $\mathcal{Y}_1(\mathcal{X}), \mathcal{Y}_2(\mathcal{X}), \ldots$ by means of solving the linear high-order deformation Eq. (39) one after the other in order. The mth-order approximation of $\mathcal{Y}(\mathcal{X})$ is given by,

$$\mathcal{Y}(\mathcal{X}) = \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3 + \cdots = \sum_{k=0}^{m} \mathcal{Y}_k \#(42)$$

2.8 Series Solution Method (SSM) [18]

Consider the following First-Order nonlinear ordinary differential equation,

$$\mathcal{L}(\mathcal{Y}(\mathcal{X})) + \mathcal{N}(\mathcal{Y}(\mathcal{X})) = g(\mathcal{X}) \#(43)$$

With initial condition $\mathcal{Y}(0) = \psi$. Applying L^{-1} to both sides of Eq. (47) gives,

$$\mathcal{Y}(\mathcal{X}) = \psi + \mathcal{L}^{-1}g(\mathcal{X}) - \mathcal{L}^{-1}\mathcal{N}(\mathcal{Y}) \#(44)$$

Recall that the generic form of Taylor series at $\mathcal{X} = 0$ can be written as

$$\mathcal{Y}(\mathcal{X}) = \sum_{n=0}^{\infty} a_n \mathcal{X}^n. \#(45)$$

We will assume that the solution $\mathcal{Y}(\mathcal{X})$ of Eq. (43)

$$\sum_{n=0}^{\infty} a_n \mathcal{X}^n = G(\mathcal{X}) - \mathcal{L}^{-1}\mathcal{N}\left(\sum_{n=0}^{\infty} a_n \mathcal{X}^n\right) \#(46)$$

or for simplicity we use

$$a_0 + a_1\mathcal{X} + a_2\mathcal{X}^2 + \cdots = T(G(\mathcal{X})) - \mathcal{L}^{-1}\mathcal{N}(a_0 + a_1\mathcal{X} + a_2\mathcal{X}^2 + \cdots), \#(47)$$

where T is the Taylor series and $G(\mathcal{X}) = \psi + \mathcal{L}^{-1}g(\mathcal{X})$. The Eq. (44) will be converted to a traditional integral in Eq. (45) or Eq. (46) where instead of integrating the nonlinear term $\mathcal{N}(\mathcal{Y})$, terms of the form $\mathcal{X}^n, n \geqslant 0$ will be integrated. Notice that because we are seeking series solution, then if $G(\mathcal{X})$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $G(\mathcal{X})$ should be used. We first integrate the right side of the integral in Eq. (46) or Eq. (47), and collect the coefficients of like powers of $\mathcal{X}$. We next equate the coefficients of like powers of $\mathcal{X}$ in both sides of the resulting equation to obtain a recurrence relation in $a_j, j \geqslant 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geqslant 0$. Having determined the coefficients $a_j, j \geqslant 0$, the series solution follows immediately upon substituting the derived coefficients into Eq. (49). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we determine, the higher accuracy level we achieve.

2.9 Daftardar-Jafari Method (DJM) [19]

Consider the following general functional equation:

$$\mathcal{Y}(\mathcal{X}) = \mathcal{N}(\mathcal{Y}(\mathcal{X})) + f, \#(48)$$

where $\mathcal{N}$ is a nonlinear operator from a Banach space $B \to B$ and f is a known function. We are looking for a solution $\mathcal{Y}$ of Eq. (48) having the series form:

$$\mathcal{Y} = \sum_{i=0}^{\infty} \mathcal{Y}_i. \#(49)$$

The nonlinear operator $\mathcal{N}$ can be decomposed as

$$\mathcal{N}\left(\sum_{i=0}^{\infty} \mathcal{Y}_i\right) = \mathcal{N}(\mathcal{Y}_0) + \sum_{i=1}^{\infty}\left\{\mathcal{N}\left(\sum_{j=0}^{i} \mathcal{Y}_j\right) - \mathcal{N}\left(\sum_{j=0}^{i-1} \mathcal{Y}_j\right)\right\}. \#(50)$$

From Eq. (49) and Eq. (50), Eq. (48) is equivalent to

$$\sum_{i=0}^{\infty} \mathcal{Y}_i = f + \mathcal{N}(\mathcal{Y}_0) + \sum_{i=1}^{\infty}\left\{\mathcal{N}\left(\sum_{j=0}^{i} \mathcal{Y}_j\right) - \mathcal{N}\left(\sum_{j=0}^{i-1} \mathcal{Y}_j\right)\right\}. \#(51)$$

We define the recurrence relation:

$$\begin{cases} \mathcal{Y}_0 = f, \\ \mathcal{Y}_1 = \mathcal{N}(\mathcal{Y}_0), \\ \mathcal{Y}_{m+1} = \mathcal{N}(\mathcal{Y}_0 + \cdots + \mathcal{Y}_m) - \mathcal{N}(\mathcal{Y}_0 + \cdots + \mathcal{Y}_{m-1}),\ m = 1,2,\ldots \end{cases} \#(52)$$

Then

$$(\mathcal{Y}_1 + \cdots + \mathcal{Y}_{m+1}) = \mathcal{N}(\mathcal{Y}_0 + \cdots + \mathcal{Y}_m),\ m = 1,2,\ldots, \#(53)$$

and

$$\mathcal{Y} = f + \sum_{i=1}^{\infty} \mathcal{Y}_i. \#(54)$$

3. Conclusion

In this concise review, we've explored a variety of analytical methods for solving ordinary differential equations (ODEs). These methods are vital tools in science and engineering, offering precise solutions for a wide range of ODE problems. Key takeaways include:

- Method Selection: The choice of method depends on the ODE's nature, complexity, and available conditions.
- Accuracy vs. Complexity: There's a trade-off between accuracy and computational cost.
- Broad Applications: These methods apply to various fields, from physics to economics.
- Iterative Nature: Some methods use iterations, so understanding convergence is crucial.
- Hybrid Approaches: Combining methods can enhance robustness and applicability.
- Computational Resources: Consider available computational resources when choosing a method.

Analytical methods remain essential, providing precise and interpretable solutions while complementing numerical techniques. Researchers, engineers, and students can use this review as a valuable reference for solving ODEs effectively.

References

1. Jassim, H. K., & Shareef, M. A. (2021). On approximate solutions for fractional system of differential equations with Caputo-Fabrizio fractional operator. Journal of Mathematics and Computer science, 23, 58-66.
2. El-Borai, M. M., El-Sayed, W. G., & Jawad, A. M. (2015). Adomian decomposition method for solving fractional differential equations. International Research Journal of Engineering and Technology, 2(6), 295-306.
3. Abdulaziz, O., Hashim, I., & Momani, S. (2008). Application of homotopy-perturbation method to fractional IVPs. Journal of Computational and Applied Mathematics, 216(2), 574-584.
4. Jassim, H. K., & Abdulshareef Hussein, M. (2023). A New Approach for Solving Nonlinear Fractional Ordinary Differential Equations. Mathematics, 11(7), 1565.
5. Abbaoui, K., & Cherruault, Y. (1994). Convergence of Adomian's method applied to differential equations. Computers & Mathematics with Applications, 28(5), 103-109.
6. Liao, S. (2012). Homotopy analysis method in nonlinear differential equations (pp. 153-165). Beijing: Higher education press.
7. Jassim, H. K., & Hussein, M. A. (2022). A Novel Formulation of the Fractional Derivative with the Order α≥ 0 and without the Singular Kernel. Mathematics, 10(21), 4123.
8. Hussein, M. A. (2022). Analysis of fractional differential equations with Antagana-Baleanu fractional operator. Mathematics and Computational Sciences, 3(3), 29-39.
9. Jassim, H. K., Hussein, M. A., & Ali, M. R. (2023, September). An efficient homotopy permutation technique for solving fractional differential equations using Atangana-Baleanu-Caputo operator. In AIP Conference Proceedings (Vol. 2845, No. 1). AIP Publishing.
10. Hussein, M. A. (2023). Using the Elzaki decomposition method to solve nonlinear fractional differential equations with the Caputo-Fabrizio fractional operator. Baghdad Science Journal.
11. Wazwaz, A. M. (2010). Partial differential equations and solitary waves theory. Springer Science & Business Media.

12. He, J. (1997). A new approach to nonlinear partial differential equations. Communications in Nonlinear Science and Numerical Simulation, 2(4), 230-235.
13. Abbasbandy, S. (2003). Improving Newton–Raphson method for nonlinear equations by modified Adomian decomposition method. Applied mathematics and computation, 145(2-3), 887-893.
14. He, J. H. (1999). Variational iteration method–a kind of non-linear analytical technique: some examples. International journal of non-linear mechanics, 34(4), 699-708.
15. Jafari, H. (2014). A comparison between the variational iteration method and the successive approximations method. Applied Mathematics Letters, 32, 1-5.
16. He, J. H. (2003). Homotopy perturbation method: a new nonlinear analytical technique. Applied Mathematics and computation, 135(1), 73-79.
17. Liao, S. J., & Cheung, K. F. (2003). Homotopy analysis of nonlinear progressive waves in deep water. Journal of Engineering Mathematics, 45, 105-116.
18. Wazwaz, A. M. (2011). Linear and nonlinear integral equations (Vol. 639, pp. 35-36). Berlin: Springer.
19. Daftardar-Gejji, V., & Jafari, H. (2006). An iterative method for solving nonlinear functional equations. Journal of mathematical analysis and applications, 316(2), 753-763.