ISSN: 2834-7706

Mini Review Article Journal of Mathematical Techniques and Computational Mathematics

A property of $C^{k,\alpha}$ functions

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Submitted: 16 Feb 2023; **Accepted:** 27 Feb 2023; **Published:** 15 Mar 2023

Citation: Dalmasso, R. (2023). A property of $C^{k,\alpha}$ functions. *J Math Techniques Comput Math*, 2(3), 136-138.

Abstract

Let f be a nonnegative function of class C^k ($k \geq 2$) such that $f^{(k)}$ is Hölder continuous with exponent α in (0,1]. If $f'(x) = \cdots = f^{(k)}(x) = 0$ when $f(x) = 0$, we show that f^μ is differentiable for $\mu \in (1/(k+\alpha),1)$ and under an additional condition we show that $(f^\mu)'$ is Hölder continuous with exponent $\beta = \mu(1+\alpha) - 1$ (if $\beta \leq 1$) at $x \in [0,T]$ when $f(x) = 0$. $(f^\mu)'$ is Lipschitz continuous at x if $f(x) > 0$.

Keywords and Phrases: $C^{k,\alpha}$ functions; differentiability, 2020 Mathematics Subject Classification: 26A06, 26A24.

$C^k[a, b]$ denotes the space of functions differentiable up to order k such that the derivatives of order k are continuous on $[a, b]$ and $C^{k,\alpha}[a, b]$ denotes the space of functions in $C^k[a, b]$ such that the derivatives of order k are Hölder continuous with exponent α in (0,1]. Recall that $g : [a, b] \to \mathbb{R}$ is Hölder continuous with exponent $\alpha \in (0,1]$ at $x \in [a, b]$ if

$$\sup\{|g(y) - g(x)||y - x|^{-\alpha} \,;\, y \neq x,\, y \in [a, b]\} < \infty,$$

and that g is Hölder continuous with exponent $\alpha \in (0,1]$ in $[a, b]$ if

$$\sup\{|g(x) - g(y)||x - y|^{-\alpha} \,;\, x \neq y,\, x, y \in [a, b]\} < \infty.$$

It is well-known ([4]) that if a nonnegative function f is in $C^2[a,b]$ and if the second derivative of f vanishes at the zeros of f, then $f^{1/2}$ is in $C^1[a,b]$. Now if $f \in C^m[a,b]$ is nonnegative and if all its derivatives vanish at the zeros of f, then $f^{1/m}$ is not necessarily in $C^1[a,b]$ (See [3]). Finally let $f \in C^{k,\alpha}[a,b]$, $k \geq 1$ and $f \geq 0$. Then $f^{1/k+\alpha}$ is absolutely continuous (See [1] Lemma 1 and also Remark 2 in [2] when $k = 1$).

Now let $f \in C^{k,\alpha}[0, T]$, $T > 0$, $k \geq 2$, be such that $f^{(j)}(x) = 0$ for some $x \in [0,T]$, $j = 0,\cdots,k$. Then we define

$$N(x,y) = (y-x)^{k-1}\int_0^1 (1-s)^{k-2} f^{(k)}(sy + (1-s)x)\,ds\ ,$$

and, if $f \geq 0$,

$$D(x,y) = ((y-x)^k \int_0^1 (1-s)^{k-1} f^{(k)}(sy + (1-s)x)\,ds)^{(k+\alpha-1)/(k+\alpha)}$$

for $x, y \in [0, T]$. We have the following theorem.

Theorem. *Let $f \in C^{k,\alpha}[0,T]$, $T > 0$, $k \geq 2$, be such that $f \geq 0$. Assume that f has at least one zero in $[0, T]$. If $f'(x) = \cdots = f^{(k)}(x) = 0$ when $f(x) = 0$, then f^μ is differentiable for $\mu \in (1/(k+\alpha),1)$. If moreover $N(x,y)/D(x,y)$ is bounded for $(x,y) \in \{t \in [0,T];\, f(t) = 0\} \times \{t \in [0,T];\, f(t) > 0\}$, then $(f^\mu)'$ is Hölder continuous with exponent $\beta = \mu(k+\alpha) - 1$ at x such that $f(x) = 0$ (if $\beta \leq 1$). $(f^\mu)'$ is Lipschitz continuous at x if $f(x) > 0$.*

Proof. f^μ is clearly differentiable at $x \in [0, T]$ when $f(x) > 0$. Suppose that $f(x) = 0$. For $y \in [0, T]$ we can write

$$\begin{aligned} f(y) &= \frac{(y-x)^k}{(k-1)!}\int_0^1 (1-s)^{k-1} f^{(k)}(sy+(1-s)x)\,ds \\ &\leq \frac{|y-x|^k}{(k-1)!}\int_0^1 (1-s)^{k-1} |f^{(k)}(sy+(1-s)x)|\,ds \\ &\leq C\frac{|y-x|^{k+\alpha}}{(k-1)!}\int_0^1 (1-s)^{k-1} s^\alpha\,ds \\ &= \frac{C}{(1+\alpha)\cdots(\alpha+k)}|y-x|^{k+\alpha}\ , \end{aligned} \tag{1}$$

for some constant C, which implies that f^μ is differentiable at x.

Let $x \in [0, T]$. Suppose first that $f(x) = 0$. Then $f^{(j)}(x) = 0$ for $j = 1,\cdots,k$. Let $y \in [0, T]$ be such that $f(y) > 0$. We can write

$$f'(y) = \frac{(y-x)^{k-1}}{(k-2)!} \int_0^1 (1-s)^{k-2} f^{(k)}(sy + (1-s)x)\, ds\ ,$$

and

$$f(y) = \frac{(y-x)^{k}}{(k-1)!} \int_0^1 (1-s)^{k-1} f^{(k)}(sy + (1-s)x)\, ds\ .$$

Using (1) we get

$$\begin{aligned} |(f^\mu)'(y) - (f^\mu)'(x)| &= \mu |f(y)^{\mu-1} f'(y)| \\ &= \mu |f(y)^{\mu - \frac{1}{k+\alpha}} f(y)^{-\frac{k+\alpha-1}{k+\alpha}} f'(y)| \\ &= C_1 |f(y)^{\mu-\frac{1}{k+\alpha}} |N(x,y)|/D(x,y) \\ &\le C_2 f(y)^{\mu - \frac{1}{k+\alpha}} \le C_3 |y-x|^\beta\ , \end{aligned}$$

for some constants C_j ($j = 1,\cdots,3$) where C_2 and C_3 may depend on x. Since f^μ is C^1 near t when $f(t) > 0$, this implies that $f^\mu \in C^1[0, T]$. Suppose now that $f(x) > 0$. There exist $c, d \in [0,T]$ such that $c < d$, $x \in [c,d]$ when $x = 0$ or $x = T$ and $x \in (c,d)$ when $x \in (0,T)$ and $f(y) \ge f(x)/2$ for $y \in [c,d]$. Let $y \in [c,d]$. We have

$$\begin{aligned} &|(f^\mu)'(y) - (f^\mu)'(x)| = \mu |f(y)^{\mu-1} f'(y) - f(x)^{\mu-1} f'(x)| \\ &\le \mu (f(y)^{\mu-1} |f'(y) - f'(x)| \\ &+ |f'(x)| |f(y)^{\mu-1} - f(x)^{\mu-1}|) \\ &\le C_1 |y-x|\ , \end{aligned}$$

for some constant C_1 depending on x. Since $(f^\mu)'$ is continuous on $[0,T]$ there exists a constant C_2 depending on x such that $|(f^\mu)'(y) - (f^\mu)'(x)| \le C_2 |y-x|$ for $y \in [0,T] \setminus [c, d]$.
The proof of the theorem is complete.

Remark. The case $k = 1$ is treated in [2]. Notice that, when $k \ge 2$ and $\mu \in [1/2,1)$, f^μ is in $C^1[0, T]$: See [3, 4]. Moreover, assume that $k \ge 2$ and that $f'(0) = 0$ (resp. $f'(T) = 0$) when $f(0) = 0$ (resp. $f(T) = 0$). Then, if $\mu \in (1/2,1)$, $(f^\mu)'$ is Hölder continuous with exponent $2\mu - 1$ at x if $f(x) = 0$ and Lipschitz continuous at x if $f(x) > 0$: See [2].

Corollary. *Let $f \in C^{k,\alpha}[0,T]$, $T > 0$, $k \ge 2$. Assume that $f^{(j)}(0) = 0$ for $j = 0,\cdots,k$ and that $f^{(k)} > 0$ on $(0,\eta]$ for some $\eta \in (0,T)$ and $f^{(k)} \ge 0$ on $[\eta, T]$. Then $(f^\mu)'$ is Hölder continuous with exponent $\beta = \mu(k + \alpha) - 1$ at 0 (if $\beta \le 1$). $(f^\mu)'$ is Lipschitz continuous at $x \in (0, T]$.*

Proof. In view of the Theorem it is enough to show that $N(0, y)/D(0, y)$ is bounded on $(0,T]$. Let

$$0 < \varepsilon < \min(1, (\frac{k-1}{2||f^{(k)}||_\infty} \int_0^1 (1-s)^{k-2} f^{(k)}(sy) ds)^{\frac{1}{k-1}})\ .$$

We can write

$$\int_0^1 (1-s)^{k-1} f^{(k)}(sy) ds = \int_0^{1-\varepsilon} (1-s)^{k-1} f^{(k)}(sy) ds + \int_{1-\varepsilon}^1 (1-s)^{k-1} f^{(k)}(sy) ds\ .$$

Now we have

$$\int_0^{1-\varepsilon} (1-s)^{k-1} f^{(k)}(sy) ds \ge \varepsilon \int_0^{1-\varepsilon} (1-s)^{k-2} f^{(k)}(sy) ds\,,$$

and

$$\int_{1-\varepsilon}^1 (1-s)^{k-2} f^{(k)}(sy) ds \le \frac{\varepsilon^{k-1} ||f^{(k)}||_\infty}{k-1}\ .$$

Then

$$\int_0^1 (1-s)^{k-1} f^{(k)}(sy)ds \geq \varepsilon \int_0^1 (1-s)^{k-2} f^{(k)}(sy)ds$$

$$-\varepsilon \int_{1-\varepsilon}^1 (1-s)^{k-2} f^{(k)}(sy)ds$$

$$\geq \varepsilon \int_0^1 (1-s)^{k-2} f^{(k)}(sy)ds - \frac{\varepsilon^k}{k-1}||f^{(k)}||_\infty$$

$$\geq \frac{\varepsilon}{2} \int_0^1 (1-s)^{k-2} f^{(k)}(sy)ds \ .$$

Now, when $y > 0$, we get

$$\frac{N(0,y)}{D(0,y)} \leq y^{-\frac{\alpha}{k+\alpha}} (\frac{2}{\varepsilon})^{\frac{k+\alpha-1}{k+\alpha}} (\int_0^1 (1-s)^{k-2} f^{(k)}(sy)ds)^{\frac{1}{k+\alpha}}$$

$$\leq C_1(\varepsilon) y^{-\frac{\alpha}{k+\alpha}} (y^\alpha \int_0^1 (1-s)^{k-2} s^\alpha)^{\frac{1}{k+\alpha}} \leq C_2(\varepsilon) \ .$$

Then the result follows from the Theorem.

Example 1. Let

$$\beta_0 = 0\,, \quad \beta_j = \frac{1}{j+1}(\beta_{j-1} + \frac{1}{(j+1)!}) \ , j = 1, \cdots, k \quad \text{and} \quad T \in (0,1] \ ,$$

and let

$$f(x) = \begin{cases} -\dfrac{x^{k+1}}{(k+1)!} \ln x + \beta_k x^{k+1} & \text{if} \quad x \in (0,T] \\ 0 & \text{if} \quad x = 0 \ . \end{cases}$$

Then $f \in C^{k,\alpha}[0,T]$ for all $\alpha \in (0,1)$, $f^{(j)}(0) = 0$ for $j = 0,\cdots,k$ and $f^{(k)}(x) = -x\ln x$. Then we can apply the Corollary. Notice that here $N(0,y)/D(0,y)$ is continuous on $(0,T]$ and tends to 0 as $y \to 0$.

Example 2. For $\alpha \in (0,1]$ let $f(x) = x^{k+\alpha} g(x)$, $x \in [0,T]$ where $g \in C^{k,\alpha}[0,T]$ is such that $g > 0$ on $(0,T]$. Then $f \in C^{k,\alpha}[0,T]$, $f^{(j)}(0) = 0$ for $j = 0,\cdots,k$ and $N(0,y)/D(0,y)$ is continuous on $(0,T]$. Suppose that $g^{(j)}(0) \neq 0$ for some $j \in \{0,\cdots,k\}$ and $g^{(i)}(0) = 0$ for $i = 0,\cdots,j-1$ if $j \geq 1$. Then $N(0,y)/D(0,y) \to l$ as $y \to 0$ where $l > 0$ if $j = 0$ and $l = 0$ if $j \in \{1,\cdots,k\}$.

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