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A Novel Approach of Digital Filter Unit Noise Gain Minimization Based on Variable Function Optimization

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Abstract
In this work, a new and improved optimization approach that minimize the unit noise gain in state-space of digital filters is presented. The main idea is the formulation of the unit noise gain in the form of several variables function then can get the equivalent state-space by optimization method. From this new generalized matrix, a new state space is derived and guaranty the minimization of unit noise gain. From various simulations operated on different order digital filters, we show the superiority of the proposed Several Variables Function Optimization (SVFO) algorithm over existing method.

Key Words: *Digital filters, minimization, unit noise gain, Several Variables Function Optimization (SVFO), state-space representation, unit noise gain.*

Introduction

The unit noise gain minimization of digital filters has been the subject of several researches during passed decades [1-7]. It is very important in the design of digital filters, to get a filter with minimum unit noise gain, where it is referred by the optimal filter structure [2, 8 and 9].

The problem of minimize the Round off noise subject to l2-scaling constraints in state-space of a digital filters solved by the literature [2, 10]. However, in this paper an improved optimizing method is presented. It is about finding a new formulation of the unit noise gain in the equivalent digital filters state-space, in term of several variables function, then extracting the equivalent state-space representation by optimization method (11–14). A generalized matrix that is verified for any order equal or greater than 2 is suggested and its efficiency to guaranty the minimization of unit noise gain. Different simulations examples are considered and prove the effectiveness of the proposed SVFO based method over the existing ones.

Preliminaries

Consider the state-space representation of any digital filter given by

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n) \\ y(n) &= Cx(n) + Du(n), \end{aligned} \tag{1}$$

Where $x(n) \in \Re^n$ is the state vector, $u(n)$ and $y(n)$ are the scalar input and output respectively. A, B, C and D are real matrices with appropriate dimensions. Its corresponding transfer function $H(z)$ is given as

$$H(z) = C(zI - A)^{-1}B + D, \tag{2}$$

Where I is the identity matrix of $(n \times n)$ dimension.
The unit noise gain G_0 is the sum of products of corresponding diagonal elements in K_0 and W_0 defined respectively by

$$\begin{aligned} K_0 &\triangleq \sum_{j=0}^{\infty} A^j B B^t (A^j)^t, \\ W_0 &\triangleq \sum_{j=0}^{\infty} (A^j)^t C^t C A^j, \end{aligned} \tag{3}$$

And are satisfying the Lyapunov matrix equations

$$K_0 = AK_0A^t + BB^t, \tag{4}$$

$$W_0 = A^tW_0A + C^tC.$$

Minimization unit noise gain based on SVFO

It is well known that for any state-space digital filter modelling (1), conventionally denoted by (A,B,C,D), there is an infinity of realizations of the form $(T^{-1}AT,T^{-1}B,CT,D)$, that are all equivalent under any non-singular state transformation T. The corresponding unit noise gain is given by

$$G \triangleq tr(T^tW_0T), \tag{5}$$

Where $tr(X)$ denotes the trace of the matrix X.
The new realization with minimum unit noise gain $(A,\bar{B},\bar{C},\bar{D})$ is given by

$$(A,\bar{B},\bar{C},\bar{D}) = (\Delta\ T^{-1}AT,T^{-1}B,CT,D).$$

The problem of the unit noise gain digital filter minimization (2) is to reach (5) to a minimum value, and is subject to the constraint,

$$T^{-1}K_0T^{-t} = \begin{bmatrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{bmatrix}. \tag{6}$$

Let $T = T_0RT_1$, where $K_0 = T_0T_0^t$, and R an arbitrary orthogonal matrix. The equation (6) becomes

$$T_1^{-1}T_1^{-t} = \begin{bmatrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{bmatrix}. \tag{7}$$

Let the matrix T_1 be

$$T_1^{-1} = \begin{bmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{bmatrix} \tag{8}$$

Where f_i, $i = 1,..,n$ are real functions that verify the following condition:

$$f_1^2 + f_2^2 + .. + f_n^2 = 1.$$

We suggest to put the generalized order matrix T_1^{-1} as

$$T_1^{-1} = \begin{bmatrix} \sqrt{1-(n-1)x_1^2} & x_1 & \cdots & x_1 \\ \sqrt{1-(n-1)x_2^2} & x_2 & \cdots & x_2 \\ \vdots & \vdots & \vdots & \vdots \\ \sqrt{1-(n-1)x_n^2} & x_n & \cdots & x_n \end{bmatrix} \tag{9}$$

For all $n \geq 2$. After substituting the expression of the matrix T in equation (5), the unit noise gain becomes a set of functions with several variables, for each matrix R.

To minimize the unit noise gain (G), it is necessary to take

$$R = orth(\begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_2 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \vdots & \vdots \\ t_n & t_n & \cdots & t_n \end{bmatrix}),$$

Where t_i, $i = 1,..,n$ are real variables and orth(χ) is the orthonormal basis of matrix χ. The substitution the matrix T in 5 produces a function with several variables, it can be handle with the optimization methods by using this way we can obtain the minimum of the unit noise gain (G) [11-14].

Numerical Examples

We take several examples of different orders to prove the efficacy of SVFO method, and compare it with previous works if any.

Example 1

Let a low-pass digital filter represented by the state-space matrices

$$A = \begin{bmatrix} 1.8857 & -0.8961 \\ 1.0000 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.0090 & 0.0002 \end{bmatrix}.$$

From (4), the matrices K_0 and W^0 are constructed as

$$K_0 = \begin{bmatrix} 462.2966 & 459.7509 \\ 459.7509 & 462.2966 \end{bmatrix}$$

$$W_0 = \begin{bmatrix} 0.0398 & -0.0355 \\ -0.0355 & 0.0320 \end{bmatrix}$$

It's the unit noise gain is ($G_0 = 33.18$).
From the SVFO based approach, the matrix T is computed as

$$T = \begin{bmatrix} 1.2731 & -20.8817 \\ -1.2730 & -22.0612 \end{bmatrix},$$

And the matrix R

$$R = \begin{bmatrix} -0.7275 & 0.6861 \\ 0.6861 & 0.7275 \end{bmatrix},$$

Which yield to

$$\bar{A} = \begin{bmatrix} 0.9428 & 0.0640 \\ -0.1121 & 0.9428 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0.4035 \\ -0.0233 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 0.0112 & -0.1941 \end{bmatrix},$$

And the controllability and observability Grampians are given by

$$\bar{W} = \begin{bmatrix} 0.2313 & -0.1072 \\ -0.1072 & 0.2313 \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} 1.0000 & -0.4632 \\ -0.4632 & 1.0000 \end{bmatrix}$$

The obtained minimum unit noise gain via the proposed approach is $G^{-} = 0.46$.

Example 2

Consider the state-space of the third-order digital filter in (2)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.4537681314 & -1.556161235 & 1.974861148 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T,$$

$$C = \begin{bmatrix} 0.231752363 & 0.023016947 & 0.079306721 \end{bmatrix},$$

With the matrices K_0 and W_0

$$K_0 = \begin{bmatrix} 17.0620 & 14.8866 & 9.6028 \\ 14.8866 & 17.0620 & 14.8866 \\ 9.6028 & 14.8866 & 17.0620 \end{bmatrix},$$

$$W_0 = \begin{bmatrix} 0.0481 & -0.1193 & 0.0954 \\ -0.1193 & 0.3111 & -0.2500 \\ 0.0954 & -0.2500 & 0.2310 \end{bmatrix}.$$

This permits to compute the unit noise gain $G_0 = 10.07$.
Achieving the Cholesky factorization of K_0, we can obtain the matrix T_0 as

$$T_0 = \begin{bmatrix} 4.1306 & 0 & 0 \\ 3.6040 & 2.0183 & 0 \\ 2.3248 & 3.2246 & 1.1222 \end{bmatrix}.$$

The proposed matrix T_1 is given as

$$T_1 = \begin{bmatrix} \sqrt{1-2x_1^2} & x_1 & x_1 \\ \sqrt{1-2x_2^2} & x_2 & x_2 \\ \sqrt{1-2x_3^2} & x_3 & x_3 \end{bmatrix}.$$

Using the proposed the SVFO algorithm, the matrix T is

$$T = \begin{bmatrix} -5.1161 & 0.1776 & 2.5208 \\ -3.7938 & 0.9185 & -0.4634 \\ -2.3420 & 0.0248 & -2.2613 \end{bmatrix}$$

The new state-space representation corresponding to the minimum unit noise gain is

$$\bar{A} = \begin{bmatrix} 0.6414 & 0.0853 & 0.4260 \\ -0.0032 & 0.6284 & -0.3467 \\ -0.2031 & 0.4933 & 0.7051 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} -0.1663 \\ -0.8275 \\ -0.2791 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} -0.3916 & 0.0272 & -0.1316 \end{bmatrix},$$

Which yield to

$$\bar{W} = \begin{bmatrix} 0.2174 & 0.0093 & 0.1328 \\ 0.0093 & 0.2146 & 0.0995 \\ 0.1328 & 0.0995 & 0.2207 \end{bmatrix},$$

$$\bar{K} = \begin{bmatrix} 1.0000 & 0.0230 & 0.6153 \\ 0.0230 & 1.0000 & 0.4617 \\ 0.6153 & 0.4617 & 1.0000 \end{bmatrix}.$$

The unit noise gain G^{-} is then minimum and equal to 0.65.

Example 3

Consider a fourth-order state-space of digital filter (*15*),

$$A = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ -0.3870 & -1.4674 & -2.4967 & -2.2258 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.2404 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.0183 & -0.2191 & 0.1419 & -0.2714 \end{bmatrix}.$$

The unit noise gain is $G_0 = 22.68$, the obtained optimal filter structure achieved by the proposed SVFO algorithm is given by the state matrices

$$\bar{A} = \begin{bmatrix} -0.4905 & -0.4103 & -0.3852 & 0.5444 \\ 0.4076 & -0.5203 & -0.2865 & -0.1231 \\ 0.5523 & 0.3060 & -0.6331 & 0.0926 \\ -0.2634 & 0.0559 & -0.2915 & -0.5819 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} -0.1506 \\ -0.5294 \\ 0.2394 \\ 0.7413 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 0.0202 & -0.2784 & -0.2560 & -0.2001 \end{bmatrix},$$

With the minimum unit noise gain ($\bar{G} = 0.44$), this yields to

$$\bar{W} = \begin{bmatrix} 0.0470 & 0.0341 & 0.0064 & 0.0309 \\ 0.0341 & 0.1640 & 0.0803 & 0.1148 \\ 0.0064 & 0.0803 & 0.1294 & 0.0356 \\ 0.0309 & 0.1148 & 0.0356 & 0.1002 \end{bmatrix},$$

$$\bar{K} = \begin{bmatrix} 1.0000 & 0.1947 & -0.4545 & -0.3319 \\ 0.1947 & 1.0000 & 0.4108 & -0.1693 \\ -0.4545 & 0.4108 & 1.0000 & 0.3952 \\ -0.3319 & -0.1693 & 0.3952 & 1.0000 \end{bmatrix}.$$

Example 4

Let the low-pass filter in (*16*) where the numerator *b* and denominator *a* are computed using the function of Matlab as *[b,a] = butter*(4,0.05), the unit noise gain $G_0 = 1.41 \times 10^5$. The state-space matrices of the corresponding digital filter are

$$\bar{A} = \begin{bmatrix} 0.8782 & 0.0300 & 0.0794 & -0.0694 \\ -0.0633 & 0.9386 & 0.1266 & 0.0406 \\ 0.0038 & -0.0851 & 0.8759 & 0.1256 \\ 0.1113 & -0.1119 & 0.0352 & 0.8970 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0.4637 \\ -0.2107 \\ 0.1240 \\ -0.0366 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 0.0754 & 0.1739 & 0.0327 & 0.0581 \end{bmatrix},$$

With the unit noise gain $\bar{G} = 0.26$, where

$$\bar{W} = \begin{bmatrix} 0.0361 & 0.0783 & -0.0104 & 0.0020 \\ 0.0783 & 0.1799 & -0.0067 & 0.0138 \\ -0.0104 & -0.0067 & 0.0318 & 0.0191 \\ 0.0020 & 0.0138 & 0.0191 & 0.0153 \end{bmatrix},$$

$$\bar{K} = \begin{bmatrix} 1.0000 & -0.1478 & 0.6383 & 0.5679 \\ -0.1478 & 1.0000 & 0.1400 & 0.0695 \\ 0.6383 & 0.1400 & 1.0000 & 0.9627 \\ 0.5679 & 0.0695 & 0.9627 & 1.0000 \end{bmatrix}.$$

Example 5

Generating the digital filter of order 5 by the Matlab function [*b,a*] = *butter*(5,0.2), its corresponding unit noise is $G_0 = 278.44$.

The matrix K_0 and W_0 are

$$K_0 = \begin{bmatrix} 147.7967 & 134.7808 & 100.1946 & 55.1312 & 12.4249 \\ 134.7808 & 147.7967 & 134.7808 & 100.1946 & 55.1312 \\ 100.1946 & 134.7808 & 147.7967 & 134.7808 & 100.1946 \\ 55.1312 & 100.1946 & 134.7808 & 147.7967 & 134.7808 \\ 12.4249 & 55.1312 & 100.1946 & 134.7808 & 147.7967 \end{bmatrix},$$

$$W_0 = \begin{bmatrix} 0.2025 & -0.4145 & 0.3599 & -0.1469 & 0.0236 \\ -0.4145 & 0.8764 & -0.7701 & 0.3183 & -0.0515 \\ 0.3599 & -0.7701 & 0.6831 & -0.2839 & 0.0462 \\ -0.1469 & 0.3183 & -0.2839 & 0.1188 & -0.0194 \\ 0.0236 & -0.0515 & 0.0462 & -0.0194 & 0.0032 \end{bmatrix}.$$

The proposed approach permits to get

$$\bar{A} = \begin{bmatrix} 0.7216 & -0.2565 & -0.4525 & 0.0688 & -0.0138 \\ 0.1976 & 0.6315 & 0.0544 & -0.3064 & 0.0825 \\ 0.2442 & -0.0585 & 0.5697 & -0.2127 & -0.1972 \\ -0.1956 & 0.0207 & 0.2334 & 0.5168 & 0.4226 \\ -0.0780 & -0.3134 & 0.3040 & 0.0946 & 0.5359 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0.0278 \\ 0.5385 \\ -0.7271 \\ -0.2392 \\ -0.2035 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} -0.2940 & 0.0851 & 0.1032 & -0.0869 & -0.1317 \end{bmatrix},$$

The corresponding gain is $\bar{G} = 0.37$, which yield to

$$\bar{W} = \begin{bmatrix} 0.2568 & -0.0142 & -0.0180 & 0.0118 & 0.0541 \\ -0.0142 & 0.0375 & 0.0357 & -0.0181 & -0.0191 \\ -0.0180 & 0.0357 & 0.0356 & -0.0201 & -0.0208 \\ 0.0118 & -0.0181 & -0.0201 & 0.0143 & 0.0141 \\ 0.0541 & -0.0191 & -0.0208 & 0.0141 & 0.0219 \end{bmatrix},$$

$$\bar{K} = \begin{bmatrix} 1.0000 & -0.1223 & -0.2500 & -0.0659 & 0.0273 \\ -0.1223 & 1.0000 & -0.1132 & -0.8761 & -0.8598 \\ -0.2500 & -0.1132 & 1.0000 & -0.0759 & 0.0169 \\ -0.0659 & -0.8761 & -0.0759 & 1.0000 & 0.9643 \\ 0.0273 & -0.8598 & 0.0169 & 0.9643 & 1.0000 \end{bmatrix}.$$

Example 6

Considering a digital filter of order 6, using the function [*b*,*a*] = butter(6,0.2), the corresponding unit noise gain is G_0 = 2419 and the state-space representation obtained by the SVFO method is

$$\bar{A} = \begin{bmatrix} 0.8849 & 0.2374 & 0.7390 & 0.5474 & 0.0958 & 0.4193 \\ -0.1351 & 0.6071 & -0.6598 & 0.0089 & -0.0836 & -0.2739 \\ -0.1929 & -0.4667 & 0.0303 & -0.1406 & -0.6782 & -0.1935 \\ -0.4878 & -0.2312 & -0.1982 & 0.6907 & 0.2323 & 0.3063 \\ 0.0914 & -0.0326 & 0.5936 & 0.0349 & 0.7170 & 0.1878 \\ -0.2275 & -0.0554 & -0.5260 & -0.1442 & -0.1154 & 0.6494 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} -0.0106 \\ 0.4565 \\ -0.0276 \\ -0.2099 \\ 0.6727 \\ 0.3902 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 0.2338 & 0.0826 & -0.0463 & -0.1644 & -0.1920 & 0.1575 \end{bmatrix},$$

With the gain $\bar{G}$ = 0.56 and

$$\bar{W} = \begin{bmatrix} 0.1167 & 0.0422 & -0.0194 & -0.0721 & -0.0766 & 0.0471 \\ 0.0422 & 0.0623 & -0.0235 & -0.0014 & -0.0471 & 0.0019 \\ -0.0194 & -0.0235 & 0.0202 & -0.0393 & 0.0074 & 0.0094 \\ -0.0721 & -0.0014 & -0.0393 & 0.2359 & 0.0873 & -0.0840 \\ -0.0766 & -0.0471 & 0.0074 & 0.0873 & 0.0760 & -0.0444 \\ 0.0471 & 0.0019 & 0.0094 & -0.0840 & -0.0444 & 0.0484 \end{bmatrix},$$

$$\bar{K} = \begin{bmatrix} 1.0000 & 0.2161 & -0.3226 & -0.0821 & -0.1788 & 0.2098 \\ 0.2161 & 1.0000 & -0.6518 & 0.2961 & -0.0043 & 0.9139 \\ -0.3226 & -0.6518 & 1.0000 & -0.3700 & -0.1853 & -0.6513 \\ -0.0821 & 0.2961 & -0.3700 & 1.0000 & -0.1694 & 0.2895 \\ -0.1788 & -0.0043 & -0.1853 & -0.1694 & 1.0000 & -0.0092 \\ 0.2098 & 0.9139 & -0.6513 & 0.2895 & -0.0092 & 1.0000 \end{bmatrix}.$$

Results and Interpretation

It is clear from the simulation results, in Table I, that the new state-space structure ($A,\bar{B},\bar{C},\bar{D}$) has a very low and minimum unit noise gain ($\bar{G}$) than that (G_0) of the initial state-space (A,B,C,D). For example 2, the works in obtained the same result as the authors results, but with more expensive calculations than those via proposed solution [10, 15]. In the example 3, the obtained results via the proposed SVFO approach is best than in in sense of minimum unit noise gain (0.44 < 0.80 and 0.26 < 0.55). The present work gives good results for the order 5 and 6 as shown in example 5 and example 6. Besides to that, the computational time is posed in Table I for eventual comparison with any future works, for the six examples [15, 16].

Conclusion

A new and improved method of optimization is proposed to minimize the unit noise gain of any digital filtrate proposed method converted the problem of the minimization unit noise gain from a matrix form to a function of several variables easier to deal with it and could be improved in future work.

	Example 1	Example 2	Example 3	Example 4	Example 5	Example 6
Unit noise gain G_0	33.19	10.07	22.68	1.42×10^5	278.44	2419
Unit noise gain		0.65 (10,15)	0.80 (15)	0.55 (16)		
Unit noise gain $\bar{G}$ SVFO	0.46	0.65	0.44	0.26	0.366	0.56
Computational Time (sec.)	2.00	3.89	17.27	15.61	43.66	44.97

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