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Research Article

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To separation of variables in the Fokker–Planck equations

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It is well-known, that for separation of variables in the eigenvalue problem, the corresponding operator should be represented as a sum of operators depending on single variables. In the case of the Fokker–Planck equations, separation of variables is possible under essentially weaker conditions.

For separation of variables in the eigenvalue problem

$$\hat{L}\, P(x,y) = \lambda\, P(x,y) \tag{1}$$

the operator $\hat{L}$ should be represented as a sum of two operators $\hat{L}_x + \hat{M}_y$, depending only on x and y correspondingly.

Conditions for separation of variables in the Fokker–Planck equations appear to be essentially weaker. For example, for the equation describing the time evolution of the probability distribution $P \equiv P(x, y)$,

$$\frac{\partial P}{\partial t} = \left\{\hat{L}_{x,y} P\right\}'_x + \left\{\hat{M}_y P\right\}'_y\,, \tag{2}$$

it is sufficient that the operator $\hat{M}_y$ in the last term depends only on y, while the operator $\hat{L}_{x,y}$ remains arbitrary. Indeed, setting $P = P(x)P(y)$ and dividing by $P(x)$, one has

$$-\frac{\partial P(y)}{\partial t} + \left\{\hat{M}_y P(y)\right\}'_y = \frac{P(y)}{P(x)}\frac{\partial P(x)}{\partial t} - \frac{1}{P(x)}\left\{\hat{L}_{x,y} P\right\}'_x\,. \tag{3}$$

The left-hand side is independent of x, and can be considered as a certain function $F(y)$. Then

$$P(y)\frac{\partial P(x)}{\partial t} - \left\{\hat{L}_{x,y} P\right\}'_x = F(y)P(x) \tag{4}$$

and integration over x gives $F(y) \equiv 0$, since the left-hand side turns to zero, while the integral over $P(x)$ is equal to unity due to normalization. As a result, the left-hand side and the right-hand side of Eq. 3 turn to zero independently, and the equation for $P(y)$ is separated

$$\frac{\partial P(y)}{\partial t} - \left\{\hat{M}_y P(y)\right\}'_y = 0\,. \tag{5}$$

On the other hand, integrating (3) over y, one has

$$\frac{\partial P(x)}{\partial t} - \left\{\hat{\mathcal{L}}_x P(x)\right\}'_x = 0\,, \tag{6}$$

where

$$\hat{\mathcal{L}}_x = \int \hat{L}_{x,y} P(y) dy\,. \tag{7}$$

The given considerations are very general and applicable to any diffusion-type equation. In physical applications, such equations are written not for an abstract function P, but for the distribution of probability: correspondingly, their right-hand side is always a sum of full derivatives, in order to provide the conservation of probability. As a result, conditions for separation of variables appear to be always weaker than for equation (1). In our opinion, this fact should be mentioned in any courses of the mathematical physics; unfortunately, it is not the case.

The separation of variables in the Fokker–Planck equations was discussed in the comparatively new papers (e.g. [1, 2, 3]), but under rather restricted assumptions. The equation of type (2) arises in the theory of 1D localization, where it describes the evolution of the mutual distribution $P(\rho, \psi)$ of the Landauer resistance ρ and the phase variable $\psi = \theta - \varphi$, where θ and φ are phases entering the transfer matrix (see Eq.28 in [4] and the comments after it). Analogous situation is expected for description of quasi-1D systems in the framework of the generalized version [5] of the Dorokhov–Mello–Pereyra-Kumar equation [6, 7]. It looks probable that analogous equations arose in other fields of physics, but the fact of separation of variables was not noticed by corresponding authors. A separation of variables in the physical problem is not simply a technical trick, but a fact with serious consequences, witnessing on independence of the corresponding degrees of freedom: e.g. the separated equation for P(ψ) in the above example provides the existence of the stationary distribution for the phase variable ψ, which is only essential for the given problem.

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