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Review Article

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The Physical Metric in General Relativity, Size and Gravitational Lensing of Black Holes and Neutron Stars

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Abstract
The physical metric in general relativity has been introduced by the author as the exact solution of Einstein equation that fits the observation of time delay of the solar system. The necessary and sufficient condition for the physical metric is that the speed of light on the angular direction is unchanged from that of vacuum. A striking result is that the radius size of a black hole and a neutron star (summarized as compact object hereafter) is found to be 2.60 times greater than predicted by the Schwarzschild metric. In this article, the gravitational lensing of the compact object at long distance is absent. In other words, it is a feature result of the physical metric that the effect of the gravitational lensing for compact object by itself disappears at long distance. It will be shown that the recent observations are consistent with the prediction of the physical metric.

I. Introduction

Schwarzschild found the exact solution of Einstein equation within a year of Einstein's proposal of general relativity [1]. However, the so called Schwarzschild metric does not fit the observational data of time delay experiment of the solar system of Shapiro [2, 3]. Only the Post Newtonian Approximation gives a good fit to the observational data [4]. Since Schwarzschild showed infinite number of exact solutions of Einstein equation, it is natural to search whether there exists an exact solution that fits the observational data. The author found such a solution and calls it the physical metric [5]. It is a metric in which speed of light on the spherical direction is equal to the value in vacuum. Since the angular direction is perpendicular to the radial direction, which is the direction of gravity, it is natural to call this metric as the physical metric. In Section II, summary of the physical metric is given. In Section III, the gravitational lensing of compact object is discussed. In Section IV, recent observations of compact objects are compared with the prediction of the physical metric. Section V is the summary.

II. The Physical Metric

The author found that the physical metric for the spherically symmetric and static (SSS) metric [1, 5].

$$ds^2 = e^{\nu(r)}\, dt^2 - e^{\lambda(r)}\, dr^2$$

$$- e^{\mu(r)}\, r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2), \quad (1)$$

is given with the condition

$$\omega = e^{\nu(r)} = e^{\mu(r)} \quad (2)$$

From the coordinate transformation of the Schwarzschild metric, one gets

$$r_s/r = \sqrt{\omega}\ (1-\omega) \quad (3)$$

where $r_s = 2GM$ is the Schwarzschild radius for the mass M of black hole and neutron star.

Using

$$\frac{d\omega}{dr} r = \frac{2\,\omega(1-\omega)}{3\,\omega-1} \quad (4)$$

one gets

$$e^{\lambda(r)} = \left(\frac{d}{dr}(r\sqrt{\omega})\right)^2/\omega = (2\,\omega/(3\omega-1))^2. \quad (5)$$

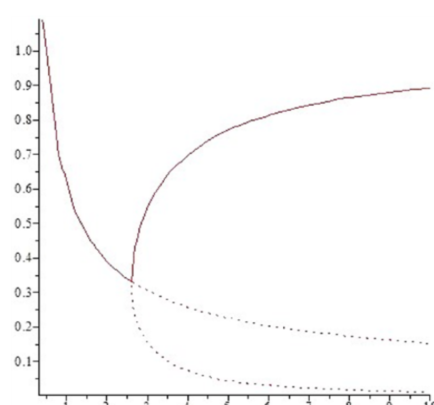

Figure 1: The metric function, $\omega = g_{00}(r)$, as a function of r/r_s in the physical metric

In Figure 1, the graph of $\omega = g_{00}(r)$ is given. The internal solution is derived in ref. 5. From Figure 1 or Equation (3), one gets the size of compact object is given by

$$R = \frac{3\sqrt{3}}{2} r_s = 2.60\, r_s \qquad (6)$$

In other words, the radius size of compact object in the physical metric is 2.60 times of the Schwarzschild radius. Since the physical metric is the exact solution of Einstein equation that fits the observation, the size of compact object should be given by Equation (6).

At r = R, the gravitational force is infinitely large attractive, and the gravitational redshift factor [5] is $1/\sqrt{3}$. One may call R in Equation (6) as extended horizon. Figure 1 is the vacuum solution and it shows that the gravity inside the extended horizon is repulsive.

It is worth noticing that light emitted from compact object is sent on the tangential direction with speed of c. It is a characteristic of the physical metric.

III. The Geodesic Equations and Gravitational Lensing

The geodesic equations for Equation (1) and Equation (2),

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0 \qquad (7)$$

where Γ^a_{bc} is the Chritoffel symbol, can be written explicitly

$$\frac{d^2 t}{d\tau^2} + \nu'(r)\frac{dt}{d\tau}\frac{dr}{d\tau} = 0 \qquad (8)$$

$$\frac{d^2 \phi}{d\tau^2} + (\mu'(r) + 2/r)\frac{d\phi}{d\tau}\frac{dr}{d\tau} = 0, \qquad (9)$$

$$\frac{d^2 r}{d\tau^2} + \frac{\nu'(r)e^{\nu(r)}}{2e^{\lambda(r)}}\frac{dt}{d\tau}\frac{dt}{d\tau} + \frac{\lambda'(r)}{2}\frac{dr}{d\tau}\frac{dr}{d\tau} - \frac{e^{\mu(r)}(r\mu'(r)+2)}{2e^{\lambda(r)}}\frac{d\phi}{d\tau}\frac{d\phi}{d\tau} = 0. \quad (10)$$

Here we assumed that $\theta = \pi/2$ and $d\theta = 0$.
Equation (8) and Equation (9) are solved giving

$$\frac{dt}{d\tau} = e^{-\nu(r)}, \qquad (11)$$

$$\frac{d\phi}{d\tau} = \frac{b}{r^2 e^{\mu(r)}}. \qquad (12)$$

Since τ is an arbitrary invariant parameter, the integration constant in Equation (11) can be chosen to be 1.
The parameter b in Equation (12) is an integration constant.

For Equation (10), multiplying $2e^{\lambda(r)}\, dr/d\tau$, one gets

$$\frac{d}{d\tau}\left(e^{\lambda(r)}\left(\frac{dr}{d\tau}\right)^2 - e^{-\nu(r)} + \frac{b^2 e^{-\mu(r)}}{r^2}\right) = 0. \quad (13)$$

The solution of Equation (13) gives

$$\frac{dr}{d\tau} = \pm e^{-\frac{\lambda(r)}{2} - \frac{\nu(r)}{2}}\sqrt{\left|1 - \frac{b^2}{r^2} - E\right|}, \qquad (14)$$

where the integration constant, E = 0 for light propagation. Then

$$\frac{d\phi}{dr} = \frac{\frac{d\phi}{d\tau}}{\frac{dr}{d\tau}} = \pm\frac{b}{r}e^{\frac{\lambda(r)}{2} - \frac{\nu(r)}{2}}/\sqrt{|b^2 - r^2|}. \qquad (15)$$

For black hole and neutron star,

$$b = R = 2.60\, r_s \qquad (16)$$

is an appropriate choice.
If one chooses the origin of the coordinate at the observational point on Earth, one gets

$$\lambda(r) = \nu(r) = 0 \quad \text{for small r.} \qquad (17)$$

Expanding Eq. (15) for small r, one gets

$$\frac{d\phi}{dr} = \pm\left(\frac{1}{r} + \frac{r}{2b^2} + \cdots\right). \qquad (18)$$

The integration by r gives

$$\varphi(r) = \ln r + \frac{r^2}{4b^2} + \cdots. \qquad (19)$$

The first term is divergent and should be cancelled by the similar term coming from the other end of the integration. Since there does not exist a linear term, the effect of the gravitational lensing by compact object should disappear at a long distance away from compact object. The second term should imply that the direction of the gravitational lensing is tangential to the direction of the compact object.

In Section II, the outcome of the physical metric is the radius size of compact object to be extended horizon, $R = 2.60\, r_s$, and the attractive force of gravity at the extended horizon to be infinitely large. In this Section, we have concluded that the gravitational lensing for the light emitted from compact object is absent. As a result, observed size of compact object should be just $R = 2.60\, r_s$. The difference from the ordinary case of gravitational lensing is that light emitted from compact object is sent on the tangential direction on the surface, while light in the ordinary gravitational lensing is sent in an arbitrary direction, which reaches the observational point.

IV. Comparison with the Observation

The largest black hole at the center of M87 is observed [6] giving mass $M = 6.5 \pm 0.2 \pm 0.7 * 10^9\, M_{\odot}$, and distance, d=16.4±0.5 Mpc. Converting the units, one gets

$$\frac{r_s}{d} = \frac{1.95\; 10^{15} cm}{5.07\; 10^{25}\, cm} = 3.85 * 10^{-11} rad = 7.94\; \mu as. \qquad (20)$$

Then, the observed diameter, 42 ± 3 µas, produces a diameter of 5.29 ± 0.57 r_s. This should be compared with our prediction for diameter, 2x 2.60 r_s = 5.20 r_s. This is a remarkable agreement.

The black hole at the center of the Milky Way [7], Sagittarius A*, is reported with mass, M = 4.154 ± 0.014 106 M_Θ, and distance, d = 8178 ± 13 pc, which give

$$\frac{r_s}{d} = \frac{1.245\ 10^{12} cm}{2.53\ 10^{22}\ cm} = 4.92\ 10^{-11} rad = 10.1\ \mu as. \qquad (21)$$

Then, the prediction of the physical metric for diameter is 5.20 r_s. The observed value is ~ 6 r_s. While this is consistent, further data is required.

For neutron star, the prediction for radius size for the Mass of 1.4 M_Θ, is given by 2.60 * 1.4 * 3 km = 10. 9 km. The derived value is in the range of 11 km [8].

V. Summary

The author has introduced the physical metric in general relativity as the exact solution of Einstein equation that fits the observational value of time delay of the solar system. The prediction of the physical metric is that the radius size of a black hole and a neutron star is 2.60 times Schwarzschild radius and that this value is not changed by the gravitational lensing effect. Comparison with the observational data of black holes at the centers of M87 and the Milky Way, as well as that of neutron star gives a remarkable agreement with this prediction.

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