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The MHD Plasma Flow Near the Heliopause Stagnation Region with A View On Kinetic Consistency

Hans J. Fahr

Argelander Institute for Astronomy, The University of Bonn, Auf dem Huegel 71, D-53121 Bonn, Germany

***Corresponding author**
Hans J Fahr, Argelander Institute for Astronomy, The University of Bonn, Auf dem Huegel 71, D-53121 Bonn (Germany).

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Abstract
Most of the representative space plasma systems in our cosmic environment, - outside of stellar interiors, - like heliospheric, interstellar, or intergalactic plasmas etc., are collision-free or, at least, only weakly collision-determined systems. Nevertheless, these plasmas consist of at least two very different particle species, namely ions and electrons, i.e. particles with very disparate masses and opposite electric charges. If in these systems concerted fluid motions are arranged by electro-magnetic or gravitational forces or by inner forces like pressure gradients, then it must be asked how this combined electron-ion system finds its common internal dynamics. In most text book literature this problem is treated by considering the plasma as a mono-fluid system in which the massive protons and the nearly massless electrons are electrically closely bound together and move as an electrically neutral couple with an identical bulk velocity. Under these conditions the well-known Bernoulli law is derived for the standard MHD. If the electron pressure, however, does compete with the energy density of the ion bulk motion, then a two-fluid situation occurs, and the resulting bulk motion of the charge-neutral plasma needs to be determined on the basis of the kinetic conditions of the two different plasma fluids. In the following we shall exactly study this specific situation.

Introduction
The stagnation flow close to the heliopause
Let us start our considerations from a standard MHD view for example on those two plasma flows which approach the heliopause a) from the solar side and b) from the opposite, the interstellar side. This MHD flow pattern should serve us here as the basis of an advanced study of the requested underlying kinetic basis which is needed to bring the whole concept concerning MHD flux conservation requirements and concerning required pressure equilibria to a satisfying consistency.

We start here assuming that the heliosheath plasma flow near the stagnation region can be represented by a 2+1 - dimensional incompressible, stationary flow with a frozen-in magnetic field $\vec{B}$ which is everywhere parallel to the flow $\vec{V}$ (e.g. see field-aligned flows studied by [1]. The incompressibility due to the very low Mach numbers of the flows on both sides of the heliopause can thereby be taken as a good approximation and implies constant, though different densities ρ_1=const on the solar side, and $\rho_2=const$ on the interstellar side, however with different densities $\rho_1 < \rho_2$ on the two adjacent sides of the heliopause. Without substantial complications the assumption of plasma incompressibility can also be given up in favour of plasma compressibility as has been shown by Kleimann [2], but in this article here we do want to care for another more important complication, namely the kinetic substructure of the MHD system which is needed to guarantee a consistency of the plasma flow system.

Theoretical description of the counterflow system near the heliopause stagnation point
To start the business in the more extended neighbourhood of the heliopause stagnation point, the plasma flow system can be well described in the x, z-plane by the following equations for the bulk velocity $\vec{V}=\vec{V}(x, z)$ and the frozen-in magnetic field $\vec{B}=\vec{B}$ (x, z) as derived by Baranov et al. (1992) with the following set of equations which could serve here as a start-off for further studies:

$$V_{x,1}=\beta_1 x$$
$$V_{z,1}=-\beta_1 z$$
$$V_{x,2}=\beta_2 x$$
$$V_{z,2}=-\beta_2 z$$
$$V_{y,1}=V_{y,2}=0$$

Assuming here at first that the ion pressure is dominant compared to the electron pressure with $P_i >> P_e$, i.e. reducing the plasma flow problem to that of a "mono-fluid"- situation, then, with the use of Bernoulli's law, assuming that everywhere $\vec{V}$ is parallel to $\vec{B}$ (i.e. field-aligned flows!), one furthermore finds for a curl-free flow [3, 4]:

Solar pressure: $P_1= -(\rho_1/2)\,(V^2_{x,1}+ V^2_{z,1}) + C1$

Interstellar pressure: $P_2= -(\rho_1/2)\,(V^2_{x,2}+ V^2_{z,2}) + C2$

where the indices 1, 2 mark quantities on the solar and on the interstellar side, respectively. $C_{1,2}$ and $\beta_{1,2}$ hereby represent constants. This description may now serve as a basic background for more detailed kinetic studies in the frame of this MHD plasma flow system which consists in elaborating on specific kinetic conditions which additionally must be fulfilled to achieve a consistency as we are going to demonstrate further below.

At first we may recognize when keeping to the monofluid assumption that the electron pressure is always negligible, then at the stagnation point {z=±0, x=0} we find from the above equations that at the stagnation point bulk velocities vanish, i.e. $V_1(x=+0, z=0) = V_2(x=-0, z=0)=0$. This means that the stagnation pressures inside and outside of the heliopause according to the above equations, derived with a mono-fluid Bernoulli law, on both sides of the heliopause would be different, unless it is strictly set: $C_1=C_2$ The initial condition $C_1 \gtrless C_2$ would rather indicate a pressure disequilibrium and consequently an instability of the heliopause at and near this point, i.e. just excluding a stationary solution which we primarily here are looking for.

At this point it is perhaps advice to extend our considerations rather now to a two-fluid approach with electrons and protons both contributing their comparable parts P_e and P_i to the total pressure $P=P_e+P_i$, as is really the actual situation for the heliosheath plasma [5-9]. The mass density, however, can hereby be kept in the monofluidal form with $\rho=\rho_i$, since electrons, though perhaps highly pressurized, do nevertheless not contribute to the mass density, i.e. neither do contribute to the mass, nor, in most cases, to the momentum flows of the plasma. One thus under these auspices would then gain from Bernoulli´s law, if ions and electrons at least can be assumed to move with identical bulk velocities $V_i=V_e$, (i.e. to avoid charge accumulations) the following relations for the total plasma pressures in the form [3].

Solar pressure: $[P_{e,1}+P_{i,1}] = -(\rho_1/2)\,(V^2_{x,1}+V^2_{z,1}) + C_1$
Interstellar pressure: $[P_{e,2}+P_{i,2}] = -(\rho_2/2)\,(V^2_{x,2}+V^2_{z,2}) + C_2$

Flow pattern near the stagnation region

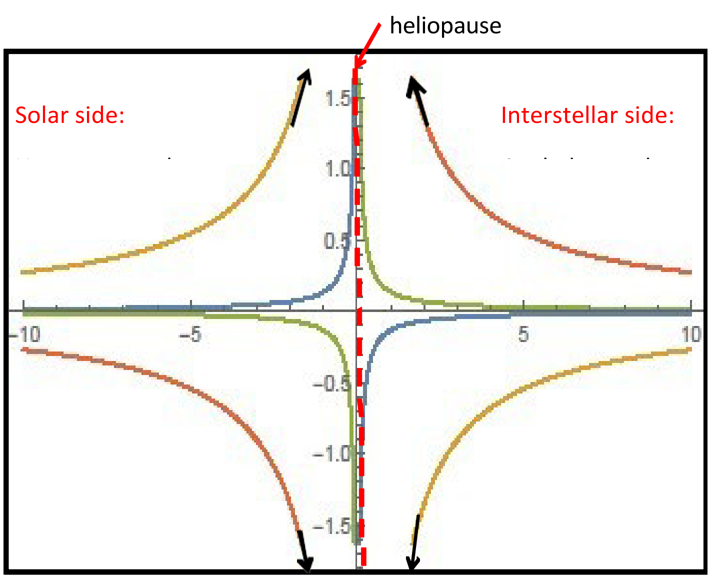

Figure 1: MHD-streamline pattern in the vicinity of the heliopause derived with the above system of equations

Especially for the stagnation point in order to guarantee heliopause stability and flow stationarity this would then lead to the following interesting requests:
Solar stagnation pressure: $[P_{e,1}+P_{i,1}]_S = C_1$
Interstellar stagnation pressure: $[P_{e,2}+P_{i,2}]_S = C_2$

Again an equilibrium between the two total stagnation pressures on the two sides of the heliopause is only possible with $C_1=C_2$, however, this time not requiring ion pressures or electron pressures to be identical on both sides, but only the sums of these pressures need to establish an equilibrium. On the other hand, this means that with the given solution for the velocity field $\vec{V}(x, z)$ one would prescribe the total pressure $P(x, z)=P_e(x, z)+P_i(x, z)$ at each point $\{x, z\}$ on any streamline inside and outside of the heliopause, however, without any statement on the individual partial pressures.

Kinetic aspects of the stagnation flow problem

Generally, for HD- or MHD- plasma physics theories the underlying, kinetic distribution functions of the particles do not play a pronounced role. What counts for these theories are the velocity moments of these distribution functions, like density, bulk velocity, pressure, heat flux etc., and it is generally assumed that these velocity-space moments can be calculated and related to each other on the basis of underlying Maxellians, and hence that only the moments of these Maxwellians do play a role further on, disregarded the realistic, actual distribution function underlying these moments. It has, however, already since quite some time been recognized that in most plasma physics scenarios Maxwellian distribution functions can hardly be justified, since most of these scenarios are collision-free systems, and hence deviations from Maxwellian distributions towards non-equilibrium distribution functions have to be recognized. For instance, due to the counterflow situation between neutral interstellar gas and the heliosheath proton plasma a coupling of the two gas flows leads to a charge-exchange - induced coupling between the two flowing media which leads to the appearance of non-Maxwellian distribution functions [10-14].

Also the kinetic state of the plasma electrons is far from being collision-determined, rather determined by wave-particle interactions or electron-impact ionizations leading to non-equilibrium electron distributions like kappa-functions and their moments [15, 16, 8, 9].

For the adequate theoretical description of the kinetic electron state, expecting at least isotropic distribution functions in the bulk frame (i.e. $\vec{V}$-frame), one can use the following phase-space transport equation which describes the evolution of the isotropic electron distribution function $f_e=f_e\,(s, v)$ in the bulk frame $\vec{V}(s)$ along heliosheath flow lines, with s being the streamline coordinate, - taking into account in our following case.

here: a) convective changes, b) magnetic cooling, and c) velocity space diffusion due to non-linear electron-whistler wave interactions - in the following form [7-9].

$$\frac{df_e}{dt} = U\frac{df_e}{ds} = \left|\frac{\partial f_e}{\partial t}\right|_{mag} + \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 D_{e,vv}\frac{\partial f_e}{\partial v}\right)$$

An equivalent, analogue kinetic equation, however written for ion-Alfvén wave-driven velocity space diffusion processes, holds for the ions and is given by

$$\frac{df_i}{dt} = V\frac{df_i}{ds} = \left|\frac{\partial f_i}{\partial t}\right|_{mag} + \frac{1}{v^2}\frac{\partial}{\partial v}(v^2 D_{i,vv}\frac{\partial f_i}{\partial v})$$

One way to proceed now with this system of differential equations is to prescribe the distribution function types $f_{i,e}$, e.g. by assuming them to be kappa-functions with separate parameters $x_{i,e}=x_{i,e}$ (s) and $\Theta_{i,e} = \Theta_{i,e}(s)$ for electrons and ions as functions of the streamline coordinate s and then try to solve this system by integrating it along different streamlines, thereby for instance assuming for ions $\Theta_i^2 = \Theta_i^2(s) = \alpha i U^2(s)$ as done in Fahr to take into account the charge exchange effect , i.e. resonant exchange of plasma ions with ions of interstellar origin [7, 17]. One has to take into account that ions, i.e protons, different from electrons, undergo charge exchange processes with interstellar H-atoms which in balance corresponds to a process where, with the locally given charge exchange rate, hot heliosheath protons are replaced by cold "interstellar protons", i.e. re-charged interstellar neutral H- atoms [14].

Different from that, electrons undergo electron-impact ionizations of interstellar H- and He-atoms and loose energy by this cooling process [15, 8, 9, 18]. This means there exists charge-exchange cooling for heliosheath protons and electron-impact cooling for heliosheath electrons. This is different both for protons and electrons on the interstellar side, since electrons are not expected to be hot and energetic enough to ionize there, i.e. they are not able to ionize H- or He- atoms, and on the other hand protons are comoving with the ambient neutral H-atoms since being common members of the same interstellar wind, unless there exists a strong outer interstellar bow shock [19, 20]. This, if included in our above considerations, will certainly complicate the treatment of the upper differential equations, but also indicate that electrons and protons have to be treated as separate fluids which evolve along the streamlines in different ways, perhaps finally raising the question, when electrons and protons would even start to locally interact kinetically with each other [21-25].

As has been shown by Fahr and Dutta-Roy or Fahr and Heyl using kappa-functions both for protons and electrons these above differential equations can be integrated over velocity space leading to corresponding pressure transport equations which describe the independent evolution both of the electron pressure $P_e(s)$ and the ion pressure $P_i(s)$ as function of the streamline coordinate $\underline{s}$ with the following equations [7,8]:

$$P_e(s) = P_{e0}(\frac{B}{B_0})^{4/3}\exp[10D_{e,0}\int_{s_0}^{s}\frac{ds}{V}] + P_e^S$$

and:

$$P_i(s) = P_{i0}(\frac{B}{B_0})^{4/3}\exp[10D_{i,0}\int_{s_0}^{s}\frac{ds}{V}] + P_i^S$$

where P_e^S and P_i^S denote streamline constants. Hereby $B=B(s)$ is the magnitude of the magnetic field at the streamline point s, and, if no wave-particle driven velocity-space diffusion, neither of electrons nor protons, takes place, then the total pressure would simply be expressed by the following expression:

$$P(s) = P_e(s) + P_i(s) = (P_{e0} + P_{i0})(\frac{B}{B_0})^{4/3} + P_e^S + P_i^S$$

meaning that the total pressure $P(s)$ varies along the streamline determined by the varying magnetic field magnitude along the streamline according to $B^{4/3}=B^{4/3}$(s), however, in such a way that the ratio of $P_e(s)/P_i(s)$ always equals the initial ratio P_{e0}/P_{i0} at the origin of this streamline $s=s_0$ at the termination shock. That expresses the fact that without the operation of velocity-space diffusion this ratio, beginning at the termination shock, would be identical over the whole heliosheath. This interestingly enough would then furthermore mean that the total pressure at the stagnation point $s_S= \{x_S=0, z_S=0\}$ because of $B(s_S)=V(s_S)=0$ would amount to $P(s_S)= P_e^S+P_i^S$.

Attempts to derive information's on the total pressure in the heliosheath from sound velocity data could then be seen in a new light [8, 9, 26]. This result would, however, become more complicate, if the above diffusion terms would be operative and sufficiently non-thermal distribution functions would be generated by efficient wave-particle interactions. This in fact could also help to establish a pressure equilibrium perpendicular to the streamlines which also is needed for a stable streamline system.

Together with Bernoulli´s law one can then, contrary to the above result, obtain the following relation:

$$[P_{e,1} + P_{i,1}] = -(\rho_1/2)(V_{x,1}^2(s) + V_{z,1}^2(s)) + C_1 = P_{e0}(\frac{B}{B_0})^{4/3}\exp[10D_{e,0}\int_{s_0}^{s}\frac{ds}{V}] + P_{i0}(\frac{B}{B_0})^{4/3}\exp[10D_{i,0}\int_{s_0}^{s}\frac{ds}{V}] + C^S$$

In case that the heliosheath electron pressure becomes dominant with $P_e >> P_i$, due to the volume work that under these conditions is solely done by the electrons, one should instead of the above equation for $P_e(s)$ use the following pressure transport equation [9]:

$$P_e(s) = P_{e0}(\frac{B}{B_0})^{4/3}(\frac{V}{V_0})^{\frac{2\pi+3}{3}}\exp[10D_{e,0}\int_{s_0}^{s}\frac{ds}{V}] + C_e^s$$

This solution shows that the electron pressure decreases with the plasma bulk velocity proportional to $V(s)^{2\pi+3/3} = V(s)^{3.09}$, however, furthermore showing that, in addition to that, frozen-in magnetic fields B enforcing the conservation of magnetic particle moments and wave-electron diffusion may independently and additionally modify the electron pressure along the streamline. An open question hereby is whether or not the diffusion constant can be kept as a constant or has to be kept open to a variation with the streamline coordinates s which would complicate the upper integrals accordingly. Perhaps by controlling the pressure along the streamline one could iteratively arrive at a consistent solution for permited values of the diffusion coefficients, - an otherwise yet unknown and presently untreatable quantity.

Use of isobaric Kappa-functions

In view of the problems which we have discussed in the sections above, it may be evident that a better approach towards the solution of these indicated problems must consist in a better perception of the fact that plasmas which were considered here are collision-free, non-thermal and non-equilibrium systems where on the kinetic level of the problem not Maxwellian, but typical non-equilibrium distribution functions like kappa-functions, Holtzmark functions or power-laws have to be expected. These non-equilibrium phenomena in recent years have quite effectively been studied with the help of isotropic kappa-distributions in the reference frame of the plasma bulk motion V which are generally given in the following form

$$f_\kappa(v) = \frac{n}{\pi^{3/2}\kappa^{3/2}\Theta^3}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)}[1+\frac{v^2}{\kappa\Theta^2}]^{-(\kappa+1)}$$

Here n denotes the particle density, k and Θ denote two independent, typical kappa-function parameters, and $\Gamma = \Gamma(\underline{x})$ means the well known mathematical Gamma-function. The above distribution function $f_K(v)$ is typical for deviations from the normally expected thermodynamically, collision-dominated equilibrium situation which latter would be characterized by a Maxwellian distribution that reappears from the upper distribution for the degenerated case $x \rightarrow \infty$. When one now, on the basis of the above distribution function f_K(v), calculates the associated pressure moment $P_K(f_K$(v) by carrying out the corresponding velocity-space integration, one obtains the following expression [27, 28]:

$$P_\kappa = \frac{4\pi m}{3}\int_0^\infty f_\kappa(v)v^4 dv = \frac{m}{2}n\Theta^2\frac{\kappa}{\kappa-3/2}$$

with m denoting the particle mass.

This shows, however, that kappa distributions with different kappa-function parameters x and Θ do nevertheless lead to the same pressure moment P_K, if the x- associated parameter Θ is a specific function of x, i.e. if $\Theta = \Theta(x)$, and if this function is given by

$$\Theta^2(\kappa) = 2P_\kappa\frac{\kappa-3/2}{mn\kappa} = \Theta^2_{\kappa,M}\frac{\kappa-3/2}{\kappa}$$

where $\Theta^2_{K,M}$ denotes the average squared velocity spread of the associated Maxwellian for $x \rightarrow \infty$. All Kappa functions with the validity of the upper relation between x and Θ represent interestingly enough the same pressure $P_K = P_{K,M} = (1/2)nm\Theta^2_{K,M}$ and thus can be called "isobaric" Kappa functions. The set of these distributions consequently is given by the following type of functions [29]:

$$f_\kappa^M(v) = \frac{n}{\pi^{3/2}\kappa^{3/2}[2P_{\kappa,M}\frac{\kappa-3/2}{mn\kappa}]^{3/2}}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)}[1+\frac{v^2}{2\kappa P_{\kappa,M}\frac{\kappa-3/2}{mn\kappa}}]^{-(\kappa+1)}$$

which can also be expressed in the form:

$$f_\kappa^M(v) = \frac{n}{\pi^{3/2}(\kappa-3/2)^{3/2}\Theta^3_{\kappa,M}}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)}[1+\frac{v^2}{\Theta^2_{\kappa,M}(\kappa-3/2)}]^{-(\kappa+1)}$$

with x being the only free parameter. The above types of isobaric functions $f^M_K(v)$ are shown below in Figure 1:

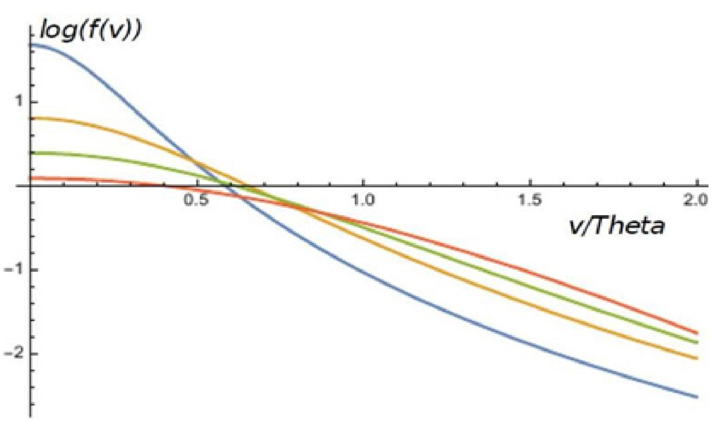

Figure 2: Isobaric kappa distributions: log $[f^M_K(v)]$ as function of $(v/\Theta_{K,M})$ for kappa values 1,6 ; 3.0; 5,0; 10.0.

The above introduced isobaric kappa functions may be of special importance, since in a kinetic multifluid view of the stability of the stagnation flow pattern not only the pressures along the streamlines count, but for the stability of the flow system also especially a pressure equilibrium perpendicular to the streamlines must be established. This we shall discuss in more detail in the next section.

Streamlines and Pressure Equilibrium Perpendicular to Streamlines

The line element s on the streamline is given through its increment $ds = \sqrt{dx^2 + dz^2}$ yielding the following relation between streamline coordinates and bulk velocity components:

$$\frac{dx}{dz} = \frac{V_x dt}{V_z dt} = \frac{V_x}{V_z} = \frac{\beta x}{-\beta z} = -\frac{x}{z}$$

from where one derives the individual streamline "k" with its streamline constant C_k given by the following expression:

$$ln(z)+\ln(x) = C_k$$

leading to the following solution for individual streamline "k":

$$z = \frac{1}{x}exp\ [C_k]$$

If one requests the streamlines to have its origin at the solar wind termination shock, then we can fix the streamline constant C_k with the initial streamline coordinates $\{x_0, z_o\}$ there. So, e.g. for solar wind maximum conditions, the TS shock front is given by a quasi-spherical ellipsoid through $R^2_{TS} = x^2_{TS} + z^2_{TS}$ with $R_{TS} \simeq 90$AU [30]. Then for each individual streamline one can fix the streamline constant Ck with the help of the upper definition by the coordinates x_{TS}; z_{TS} of the streamline origin at the termination shock via the relation:

$$X^2_{TS}(R^2_{TS} - x^2_{TS}) = \exp\ [2C_k]$$

At each streamline point $\{x_k, z_k = exp\ [C_k]/x_k\}$ the bulk velocity components $V_{x,k}$ and $V_{z,k}$ are known as function of the space coordinates, and hence also known are there the bulk velocity magnitudes $Vk = \sqrt{V^2_{x,k}+V^2_{z,k}} = \beta x\sqrt{1+\exp(2C_k)/x^4}$, and furthermore also the field magnitudes B, when keeping to the parallelity condition $B = \alpha V$ by the relation

$$B_k = \alpha V_k$$

For the calculation of the needed, remaining integral in the above given expressions for the pressures P_i and P_e one thus for instance obtains for electrons;

$$\exp[10D_{e,0}\int_{s_0}^{s}\frac{ds_k}{V}] = \exp[10D_{e,0}\int_{x_0}^{x}\frac{dx\sqrt{1+\exp(2C_k)/x^4}}{V_x\sqrt{1+V_z^2/V_x^2}}] = \exp[10D_{e,0}\int_{x_0}^{x}\frac{dx}{V_{x,k}}] = \exp[\frac{10D_{e,0}}{\beta}[\ln x - \ln x_0]]$$

This delivers the solution for the above derived pressure transport requirements. In terms of the ion pressure it would require:

$$P_i(s) = P_{i0}(\frac{B}{B_0})^{4/3}\exp[\frac{10D_{i,0}}{\beta}[\ln x - \ln x_0]]$$

This should be compared with the pressure request from the Bernoulli relation which would be:

$$P_i(x,z) = -(\rho_1/2)(V_{x,1}^2 + V_{z,1}^2) + C_1 = -(\rho_1/2)\beta^2[x^2 + \exp[2C_k]/x^2] + C_1$$

Using the upper and the upper next equation one hence can find the following result:

$$-(\rho_1/2)\beta^2[x^2 + \exp[2C_k]/x^2] + C_1 = P_{i0}(\frac{B}{B_0})^{4/3}\exp[\frac{10D_{i,0}}{\beta}[\ln x - \ln x_0]]$$

which for the termination shock at $x = x0$ leads to:

$$-(\rho_1/2)\beta^2[x_0^2 + \exp[2C_k]/x_0^2] + C_1 = P_{i0}$$

and allows to fix the constant C_1 by the solar wind termination shock conditions:

$$C_1 = P_{i0} + (\rho_1/2)\beta^2[x_0^2 + \exp[2C_k]/x_0^2]$$

To establish the complete stability of the streamline system a pressure equilibrium perpendicular to the local streamline vector $\vec{V} = \vec{V}(x, z)$ must be established (see e.g. Baumjohann and Treumann, 1996) which implies the validity of the following relation, with μ counting the direction perpendicular to $\vec{V}$, i.e. $\vec{V} \bullet \vec{\mu} = 0$:

$$\frac{d}{d\mu}[\frac{B^2}{8\pi} + (P_e + P_i)] = 0$$

Keeping to the earlier relations that have already been derived we first can then obtain:

$$B^2 = \alpha^2 V^2 = \alpha^2[\beta^2 x^2(1 + \frac{\exp(2C_k)}{x^4})]$$

and, when expressing the streamline constant C_k with the associated termination shock coordinate x_{TS}, one finds:

$$B^2 = (\alpha\beta)^2 x^2(1 + \frac{x_{TS}^2(R_{TS}^2 - x_{TS}^2)}{x^4})$$

Requiring now pressure equilibrium perpendicular to the streamlines would thus request the following relation to be valid:

$$\frac{d}{d\mu}\frac{B^2}{8\pi} = -\frac{d}{d\mu}(P_e + P_i)$$

Hereby the vectors $\vec{d\mu}$ and $\vec{V}$ are orthogonal yielding:

$$(\vec{V} \cdot \vec{d\mu}) = V_x d\mu_x + V_z d\mu_z = 0$$

or meaning:

$$d\mu_x = -\frac{V_z}{V_x}d\mu_z$$

and leading to:

$$\frac{d}{d\mu}\frac{B^2}{8\pi} = -\frac{V_z}{V_x}\frac{d}{dx}\frac{B^2}{8\pi} + \frac{d}{dz}\frac{B^2}{8\pi} = -\frac{d}{d\mu}(P_e + P_i)$$

This implies that the orthogonal pressure gradient (i.e the μ- gradient) should be:

$$\frac{d}{d\mu}(P_e + P_i) = \frac{V_z}{V_x}\frac{d}{dx}\frac{B^2}{8\pi} - \frac{d}{dz}\frac{B^2}{8\pi} = \frac{V_z}{V_x}\frac{d}{dx}\frac{B^2}{8\pi} - \frac{d}{dx}\frac{B^2}{8\pi}\frac{dx}{dz}$$

or

$$\frac{d}{d\mu}(P_e + P_i) = \frac{d}{dx}(\frac{B^2}{8\pi}) \cdot [\frac{V_z}{V_x} - \frac{dx}{dz}] = (\alpha\beta)^2[2x - \frac{2\exp(2C_k)}{x^3}] \cdot [\frac{V_z}{V_x} + \frac{x^2}{\exp[C_k]}]$$

and when replacing the streamline connection $z = exp(C_k)/x$ then leads to the following expression:

$$\frac{d}{d\mu}(P_e + P_i) = 2(\alpha\beta)^2[x - \frac{z^2}{x}] \cdot [-\frac{z}{x} + \frac{1}{z}] = 2(\alpha\beta)^2(1 - \frac{z^2}{x^2})(-z + \frac{x}{z})$$

Taking two closely adjacent streamlines with the streamline constants C_k and $C_k+\delta C_k$ and integrating along these two streamlines the pressures $P_e(s)$ and $P_i(s)$, one can check step by step in s the μ-gradient of this pressure $d/d\mu\, [Pe(s) + Pi(s)]$ and can take care that these pressures maintain pressure equilibrium with the magnetic field pressure by regulating for instance the velocity space diffusion through adaptation of the diffusion coefficient D_0 to finally knit together a mesh system of consistent solutions.

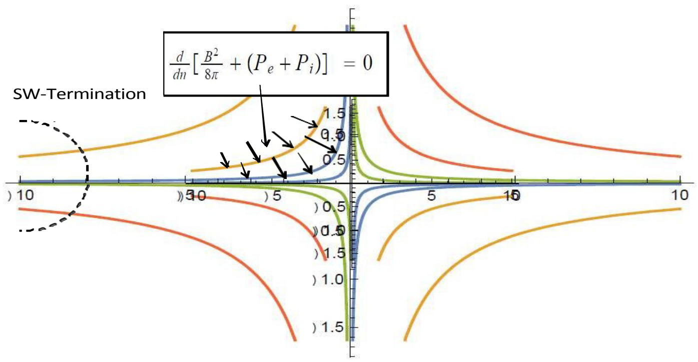

Figure 3: Streamlines with origin at the termination shock taking care of pressure equilibrium perpendicular to the lines

The Multi-Fluid Plasma Flow System

At a first glance it appears self-suggestive that in the solar rest frame the change of the ion pressure is responsible for the change of the plasma momentum flow connected with the ion mass density $\rho_i = M_L$ according to the following relation:

$$\frac{dP_i}{ds} = -\frac{d}{ds}[n\frac{M}{2}V^2]$$

If, however, there is a competing or even dominant electron pressure Pe, then a corresponding term dp/ds should be introduced on the left side of the above equation, but the question raises itself what to introduce in addition on the right hand side in this case.

In the monofluid case a streaming potential flow approximation with $\sqrt{\rho}\,\vec{V} = -\text{grad}\,\Phi$ is permitted and appropriate, if the plasma can be assumed to behave incompressible, i.e. $d\rho/ds = 0$, but for the case of dominating electron pressures this would bring up the counter-intuitive result that in the heliosheath regions downstream from the upwind termination shock, where bulk velocities according to standard models and measurements decrease by about 30 percent, the electron pressures should, in reaction to that, even continue to increase in these regions. This counterintuitive result is provocated by the exaggerated assumption that exclusively the electron pressure dictates the plasma flow. Perhaps in addition one should also face the problem here, that under the discussed situation of a plasma determined in its dynamics both by the pressures of two independent plasma fluids, i.e. the electron and the ion fluid, single fluid solutions derived from one common streaming potential Φ may not be applicable anymore. What to do under these conditions?

For better clarification of this point, we look at this situation from a slightly different view, following the standard thermodynamically procedure which states that the work done by the pressure at a change of the co-moving plasma volume ΔW is reflected by an associated change of the internal energy ϵ of that volume. This requires that in the Solar Rest Frame (SRF) the following equation has to be valid

$$-(P_e + P_i)\frac{d\Delta W}{ds} = \frac{d}{ds}[(\epsilon_i + \epsilon_e)\Delta W]$$

where ΔW, as explained in Fahr and Dutta-Roy [7], denotes the co-moving plasma volume on the streamline, i.e. a fluid volume that locally co-moves with the plasma bulk velocity $\vec{V}$. Hereby the indices "*i, e*" indicate ion- or electron- related quantities as pressures and internal energies, respectively.

Now one must take into account the fact that in the SRF the ion energy density is given by $\epsilon_i = nMV^2/2 + (3/2\pi)P_i$ while the electron energy density only is given by $\epsilon_e = (3/2\pi)P_e$ (i.e. strongly subsonic electron flow!). When furthermore assuming, in order to start the business from some concrete basis, that the electron pressure competes with the ion pressure, i.e. $P_e \simeq P_i$, will then bring us to the following net equation:

$$-(P_e + P_i)\frac{d\Delta W}{ds} = \frac{d}{ds}[(nMV^2/2 + \frac{3}{2\pi}(P_e + P_i)) \cdot \Delta W]$$

When additionally recognizing here that for an incompressible flow, as given in case of a strongly subsonic flow, the commoving plasma volume is given by the following relation $\Delta W = \Delta W_0 \cdot (V_0/V)$ [7]. Then the above equation simplifies into:

$$-(P_e + P_i)\frac{d\frac{1}{V}}{ds} = \frac{d}{ds}[\frac{1}{2}nMV] + \frac{3}{2\pi}\frac{d}{ds}((P_e + P_i)\frac{1}{V})$$

which for $n = const$ leads to

$$-(P_e + P_i)\frac{d\frac{1}{V}}{ds} = \frac{1}{2}nM\frac{d}{ds}V + \frac{3}{2\pi}(P_e + P_i)\frac{d}{ds}\frac{1}{V} + \frac{3}{2\pi}\frac{1}{V}\frac{d(P_e + P_i)}{ds}$$

and further simplifies to:

$$\frac{2\pi + 3}{2\pi}(P_e + P_i)\frac{dV}{ds} - \frac{3}{2\pi}V\frac{d(P_e + P_i)}{ds} = \frac{1}{2}nMV^2\frac{dV}{ds}$$

If one now considers the situation that the electron pressure P_e strongly dominates over the ion pressure P_i, then this equation reduces to:

$$\frac{2\pi + 3}{2\pi}P_e\frac{dV}{ds} - \frac{3}{2\pi}V\frac{dP_e}{ds} = \frac{1}{2}nMV^2\frac{dV}{ds}$$

This equation seems to require a solution of the form $P = P(U)$, and with $dU/ds \gtrless 0$ leads to

$$\frac{2\pi + 3}{2\pi}P_e - \frac{3}{2\pi}V\frac{dP_e}{dV} = \frac{1}{2}nMV^2$$

Obviously the solution of the upper differential equation requires P_e to be a function of U, tentatively by the following representation

$$P = P_0 \cdot (V/V_0)^2 + C_e$$

When inserting this into the upper differential equation, one then finds the requirement

$$\frac{2\pi - 3}{2\pi}(P_{eo}\frac{V^2}{V_0^2} + C) = \frac{1}{2}nMV^2$$

or yielding the initial electron pressure in the form:

$$P_{eo} + C = \frac{2\pi}{2\pi - 3}\frac{1}{2}nMV_0^2 = 0.96 \cdot nMV_0^2$$

On the other hand, if the ion pressure is dominant at the beginning with $P_i \gg P_e$ then one obtains:

$$\frac{2\pi + 3}{2\pi}P_i\frac{dV}{ds} - \frac{3}{2\pi}V\frac{dP_i}{ds} = \frac{1}{2}nMV^2\frac{dV}{ds}$$

and now one would obtain a solution $P_i = P_{i0}(V/V0)^2 + C_i$ but now implying just the opposite to the above solution, that now the ion pressure had to fall off with V^2. Of course none of these extreme cases will be valid on the stagnation streamline when approaching the stagnation point with $V = 0$.

With that result, coming now back to the fact that the electron pressure performs thermosdynamical work, when pumping down the streamline the electron plasma, one must conclude that without any interaction of ions and electrons, this energy, which has to be thermodynamically expended, has to be taken from the internal thermal energy ϵ_e of the electrons themselves. This leads to the following term describing the decrease of the electron thermal energy

$$-P_e\frac{d\Delta W}{ds} = \frac{d}{ds}[\epsilon_e\Delta W] = \frac{d}{ds}[\frac{3}{2\pi}P_e\Delta W]$$

Together with the relation for the commoving fluid volume in incompressible flows $\Delta W = \Delta W_0 \cdot (V_0/V)$ this leads to the following expression:

$$-P_e\frac{d}{ds}\frac{1}{V} = \frac{d}{ds}[\frac{3}{2\pi}P_e\frac{1}{V}] = \frac{3}{2\pi}[P_e\frac{d}{ds}\frac{1}{V} + \frac{1}{V}\frac{dP_e}{ds}]$$

simplifying to

$$-\frac{2\pi}{3}(1 + \frac{3}{2\pi})V\frac{d}{ds}\frac{1}{V} = \frac{1}{P_e}\frac{dP_e}{ds}$$

and consequently yielding a pressure change due to volume work as given by:

$$\frac{d\ln P_e}{ds} = \frac{2\pi + 3}{3}\frac{d\ln V}{ds}$$

When this is combined with the other terms in the pressure transport equation (Equation 14), then leads to the following completed form [7]:

$$\frac{d\ln P_e}{ds} = \frac{4}{3}\frac{d\ln B}{ds} + \frac{10D_0}{V} + \frac{2\pi + 3}{3}\frac{d\ln V}{ds}$$

This differential equation can be integrated and leads to the following completed solution for the electron pressure in case the electron pressure dominates:

$$P_e(s) = P_{e0}(\frac{B}{B_0})^{4/3}(\frac{V}{V_0})^{\frac{2\pi+3}{3}}\exp[10D_0\int_{s_0}^{s}\frac{ds}{V}] + C_e^s$$

This solution shows that the electron pressure decreases with the plasma bulk velocity proportional to $V(s)^{2\pi+3/3} = V(s)^{3.09}$, however, furthermore showing that, in addition to that, frozen-in magnetic fields enforcing the conservation of magnetic particle moments and wave-electron diffusion may independently and additionally modify the electron pressure [31-44].

Conclusions

In this paper we have shown that classical monofluid MHD theory delivers straightforward solutions for the magnetic field configuration and the plasma flow in the heliosheath, in our case the one approaching the region near the heliopause stagnation point. Then we demonstrate that monofluid solutions in fact cannot be accepted as valid solutions of the problem in the heliosheath region, because it turns out that electrons beyond the solar wind termination shock develop their own independent pressures which are comparable with or even dominant over the proton pressures. Under these conditions the electron pressures become a dynamically relevant quantity which strongly co-influences the resulting plasma dynamics, i.e. a two-fluid treatment of the plasma flow is required. In order to be able to describe electrons and protons as independent, but coupled fluids one, however, has to pay a look on the kinetic level of the underlying plasma system. We derive kinetic transport equations for electrons and protons describing the evolution of their kinetic distribution functions along streamlines, and when converting them into pressure transport equations can arrive at independent solutions for the pressures of electrons and protons as functions of the streamline coordinate s. Finally, we need to consider the state of a MHD pressure equilibrium perpendicular to the streamlines and doing this we can fix the kinetic substructure of the system.

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