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The Application of the Single Sided Laplace Transform and Fourier Transform Embodied Within Double Sided Laplace Transform by Using Generalized Functions, Eliminating Regions of Convergence Restraints

Roland Khera*

Roland Khera, CA-19990 pear Tree Ln, Cupertino, USA

***Corresponding Author**
Roland Khera, CA-19990 pear Tree Ln, Cupertino, USA.

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Abstract
The Double-Sided Laplace Transform (DSLT) and the Fourier Transform (FT) are the same at s=jω, but the Unit Step function (Heaviside Function), U(t) does not have the same DSLT and FT at s=jω. This is now solved. It will be shown that the DSLT of f $f(t)\,\frac{\text{sgn}\,t}{2}$ *is the Single-Sided Laplace transform (SSLT) of f(t). With the use of generalized function (in particular the complex delta function and its derivatives), the DSLT can be used where ever SSLT and FT are used in engineering applications. The SSLT of an even rational function is shown to be odd, and visa-versa. The problem of the region of convergence in the "s" complex plane is eliminated by including the complex delta function for solving divergent DSLT integrals. The solution for solving divergent integrals are already well established for solving divergent integrals for FT by using the real delta function. An example is provided for solving the Phase Retrieval problem exactly by measuring the signal's even and odd components autocorrelation functions. This has not been possible with the use of the SSLT because one is then only considers cases for time greater than zero, whereas an even and odd function in time needs to be both positive and negative.*

Introduction
The DSLT F(s) of a function f(t) and it's FT are one and the same at s=jω and yet the DSLT and FT of the Unit Step function (Heaviside Function) are different at s=jω. As stated by Professor Bracewell on page 219 of reference 1, "there is a profound difference in application between the two transforms." That is why the Fourier Transform (FT) and the Laplace Transform (LT) are taught as independent separate subjects in universities. This paper unifies the two subjects under one subject, the Double-Sided Laplace Function (DSLT). This was not possible up to now because the complex delta function is not all that well appreciated [1]. The complex delta function makes it possible to solve divergent integrals. The solution for solving divergent integrals is already well established in Fourier transforms using the real delta function. This is now carried over to the DSLT by including the complex delta function, which has not been fully appreciated for its applications. At present FT had more functions that could be transformed than LTs, but now with this inclusion of the complex delta function, all functions that have FT will also have DSLT, simply by replacing "s" with jω and vice-versa [2-8].

Under present knowledge, there was no LT for F(t)—the inverse LT—where as in FT, the inverse for FT is 2πF (-ω). The theory of symmetry [Ref 2. p 14] is now carried over to DSLT where it will be shown that the DSLT of F(t) is 2πjf(-s).

II. Mathematical inconsistency between the FT and LT using present knowledge

The FT of U(t), the Unit Step Function, is now $\pi\delta(\omega)+\frac{1}{j\omega}$

Yet the DSLT of U(t) is given as $\frac{1}{s}$, but at the s=jω it is simply $\frac{1}{j\omega}$ for FT. That is the delta term that is missing. In fact, the complex delta function πjδ(s) is the one absent.

The DSLT of U(t) is derived as follows:
The Unit Step Function U(t) is composed of two components:

$$U(t)=\frac{1}{2}+\frac{\text{sgn}\,t}{2} \qquad (1)$$

Therefore, to find the DSLT of U(t) one needs to find the DSLT of Unity and sgn t.

DSLT of Unity
The DSLT of unity is given by:

$$F(s)=\int_{-\infty}^{\infty}1.\,e^{-st}dt=\lim_{R\to\infty}\int_{-R}^{R}e^{-st}\,dt= \qquad (2)$$

$$\lim_{R\to\infty}\frac{e^{sR}-e^{-sR}}{s}=2\lim_{R\to\infty}\frac{\sinh(sR)}{s}=$$

$$2\lim_{R\to\infty}\frac{\sin(js)R}{(js)}$$

Let $js = s_1$ so that (2) is equal to:

$$2\lim_{R\to\infty}\frac{\sin(s_1 R)}{s_1} \qquad (3)$$

But (3) is equal to 2πδ(s1). Here δ(s1) is the Dirac delta function as evidenced by

Therefore, the DSLT of unity is 2πδ(js), an even function. In fact similar to the above method, it can be shown that F(s) = F(-s). Also, there is no restraint on "s" in the complex plane for the integral to exist.

It will now be shown that jδ(js) = δ(s)

The definition of the[1] generalized Dirac Delta Function is given in (4) below

$$\int_{-\infty}^{\infty}\delta(t)\emptyset(t)dt = \emptyset(0) \qquad (4)$$

Where Ø(t) is a smooth test function having all derivatives. This definition is unaltered if one also defines it as (5) below:

$$\int_{-\infty}^{\infty}\delta(t)\emptyset(jt)dt = \emptyset(o) \qquad (5)$$

The purpose of (5) is to show that the integral result is unaltered if one integrates instead along the imaginary axis where the authors of reference 6 state that generalized function may be invariant with respect to all rotation [9].

Also, all generalized functions can be expressed as derivatives of the delta function.

Now transforming (5) so that one integrates along the imaginary axis use the property of Integration by putting variable $jt = t_1$.

Therefore (5) is equal to: (δ(t), is an even function, therefore δ(-t) = δ(t)).

$$\phi(o) = \int_{+j\omega}^{-j\omega}\delta(jt_1)\phi(t_1)(-j)dt_1 = \qquad (6)$$

$$\int_{-j\omega}^{j\omega}\delta(jt_1)\phi(t_1)dt_1(+j)$$

Therefore, one has:

$$j\delta(jt_1) = \delta(t_1) \qquad (7)$$

V. DSLT of sgn t

Now to find the DSLT of $\frac{\operatorname{sgn} t}{2}$ one will use the derivative property of generalized functions.
Namely [10].

$$\int_{-\infty}^{\infty}f(t)\,\phi'(t)dt = -\int_{-\infty}^{\infty}f'(t)\phi(t)dt \qquad (8)$$

And in general, one has:

$$\int_{-\infty}^{\infty}f(t)\,\phi^{(n)}(t)dt = (-1)^n\int_{-\infty}^{\infty}f^{(n)}(t)\phi(t)dt \qquad (9)$$

Where ϕ(t) is a smooth test function. Also, one will use the two properties of the Dirac delta function, namely: $\delta(t) = \frac{d}{dt}\left(\frac{\operatorname{sgn} t}{2}\right)$ and (8) above.

Therefore, one has:

$$\int_{-\infty}^{\infty}\frac{\operatorname{sgn} t}{2}e^{-st}dt = \int_{-\infty}^{\infty}\frac{\operatorname{sgn} t}{2}\frac{d}{dt}\left(\frac{e^{-st}}{-s}\right)dt = \qquad (10)$$

$$\frac{1}{s}\int_{-\infty}^{\infty}\frac{d}{dt}\left(\frac{\operatorname{sgn} t}{2}\right)e^{-st}dt =$$

$$\frac{1}{s}\int_{-\infty}^{\infty}\delta(t)e^{-st}dt = \frac{1}{s}$$

See (4) for the last step.
Again, there is no restraint on "s" in the complex plane.

Here (10) is the same result obtained for finding the Single Sided Laplace Transform (SSLT) of U(T) In fact, in general, the results obtained for finding the SSLT of a function f(t), is in fact the same as obtaining the DSLT of f(t).

Here the DSLT of U(t) is:

$$\pi j\delta(s) + \frac{1}{s} \qquad (11)$$

For the FT put $s = j\omega$ and one has the FT for U(t) as

$$\pi j\delta(j\omega) + \frac{1}{j\omega} = \pi\delta(\omega) + \frac{1}{j\omega}$$

This will require a correction to the book by van de Pol, B. and Bremmer. R.: "Operational Calculus based on the Two-Sided Laplace Integral" [11].

One might now wonder if there is an inconsistency when on the one hand the DSLT of $\frac{\operatorname{sgn} t}{2}$ is $\frac{1}{s}$, whereas the SSLT of unity is $\frac{1}{s}$.

The inconsistency shows up as follows: Given that is the inverse of DSLT of f(t), then one has:

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega}\frac{e^{st}ds}{s} \qquad (12)$$

Where σ is real and such that the poles of integrand lie to the left of the vertical line of integration parallel to the imaginary axis.

Let $s = js_1$, then $ds = jds_1$ and for $s = \sigma \pm j\infty$, then $s_1 = -j(\sigma \pm j\infty) = \mp\infty - j\sigma$.

Therefore, (12) is equal to:

$$f(t) = \frac{1}{2nj}\int_{-\infty-j\sigma}^{\infty-j\sigma}\frac{e^{js_1 t}}{s_1}ds_1 \qquad (13)$$

Therefore, f(t) is equal to $\frac{\operatorname{sgn} t}{2}$. See ref 2, p. 38 as a guide.

The DSLT of $\frac{\operatorname{sgn} t}{2}$ is $\frac{1}{s}$. That is, the SSLT of U(t) is the DSLT of $\frac{\operatorname{sgn} t}{2}$.

One can generalize this for finding the DSLT of f(t) $\frac{\operatorname{sgn} t}{2}$. For example, to find the DSLT of f(t) = t, use property (4) for generalized function as well as the delta property f(t)δ(t) = f(0) δ(t) for (t) continuous at t = 0, and also (8); and one then has:

$$\int_{-\infty}^{\infty} t\frac{\operatorname{sgn} t}{2}e^{-st}dt = \int_{-\infty}^{\infty} t\frac{\operatorname{sgn} t}{2}\frac{d}{dt}\left(\frac{e^{-st}}{-s}\right)dt=$$

$$\frac{1}{s}\int_{-\infty}^{\infty}\frac{d}{dt}\left(\frac{t.\operatorname{sgn} t}{2}\right)e^{-st}dt =$$

$$\frac{1}{s}\int_{-\infty}^{\infty}\left(t\delta(t) + \frac{\operatorname{sgn} t}{2}\right)e^{-st}dt = \frac{1}{s^2} \qquad (14)$$

Here (14) is the same as the SSLT of "t" with no restraints in "s" complex plane.

For a pulse that exist for 0 < t ≤ T and zero elsewhere, then one finds the DSLT for:

$$f(t)\left(\frac{\operatorname{sgn} t}{2} - \frac{\operatorname{sgn}(t-T)}{2}\right)$$

Since sgn t is an odd function, then if f(t) is an even rational function, f(t)sgn t is an odd function. Similarly, if f(t) is an odd rational function, then the DSLT of f(t) sgn t is an even function. That is why in SSLT for f(t) being even/odd it's SSLT is odd/even for f(t) being rational.

The integrals have been solved with no restraint on "s" in the complex plane, similar to the case for ω in the Fourier transforms. In fact, the use of generalized functions makes it possible to solve integrals that otherwise were divergent. See example 5 in pages 130 to 131 in reference 1.

Derivatives of f (t)

The first derivative with respect to "s" is given by:

$$\int_{-\infty}^{\infty} -\frac{t.\operatorname{sgn} t}{2}e^{-st}dt = \frac{-1}{s^2} \qquad (15)$$

That is, the DLST of $\frac{t.\operatorname{sgn} t}{2}$ is $\frac{1}{s^2}$ which is the DSLT of "t".

The Left-Hand Side (LHS) of (15) can be verified by using the derivative property (9) of generalized functions. That is:

$$\int_{-\infty}^{\infty}\frac{t.\operatorname{sgn} t}{2}e^{-st}dt =$$

$$\int_{-\infty}^{\infty}\frac{t.\operatorname{sgn} t}{2}\frac{d^2}{dt^2}\left(\frac{e^{-st}}{s^2}\right)dt =$$

$$\frac{1}{s^2}\int_{-\infty}^{\infty}\frac{d^2}{dt^2}\left(\frac{t.\operatorname{sgn} t}{2}\right)e^{-st}dt \qquad (16)$$

But $\frac{d^2}{dt^2}\left(\frac{t.\operatorname{sgn} t}{2}\right) = t\delta'(t) + 2\delta(t) = \frac{-t\delta(t)}{t} +$ 2δ (t) = δ(t) [See Ref 2, p274]. Therefore from (4) one has (16) equal to $\frac{1}{s^2}$.

In general, the nth derivative of (9) with respect to "s" can be proven as:

$$\int_{-\infty}^{\infty}(-t)\left(\frac{n\operatorname{sgn} t}{2}\right)e^{-st}dt = \frac{(-1)^n}{s^{n+1}}$$

Therefore, the DSLT of $\frac{t^n\operatorname{sgn} t}{2}$ is $\frac{1}{s^{n+1}}$, which is the SSLT of t_n.

Also, the DSLT of $\frac{\operatorname{sgn} t}{2}e^{at}$ (where "a" can be real or complex) is given by:

$$\int_{-\infty}^{\infty}\frac{\operatorname{sgn} t}{2}e^{at}e^{-st}dt =$$

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{\operatorname{sgn} t}{2}e^{-(s-a)t}dt =$$

$$\int_{-\infty}^{\infty}\frac{\operatorname{sgn} t}{2}\frac{d}{dt}\left(\frac{e^{-s(-a)t}}{-(s-a)}\right)dt =$$

$$\frac{1}{s-a}\int_{-\infty}^{\infty}\frac{d}{dt}\left(\frac{\operatorname{sgn} t}{2}\right)e^{-(s-a)t}dt =$$

$$\frac{1}{s-a}\int_{-\infty}^{\infty}\delta(t)e^{-(s-a)}dt \qquad (17)$$

and from (4) one has (17) equal to $\frac{1}{s-a}$ which is the SSLT of eat with no restraints of "s" in the complex plane.

In general, one can show that the nth derivative of (17) with respect to "s" will provide the DSLT of $t^n e^{at}. \frac{\operatorname{sgn} t}{2}$ as $\frac{1}{(s-a)^{n+1}}$ which is the SSLT of tneat.

The DLST of F(t) where F(s) is the LT of f(t)

In the introduction it was mentioned that under present knowledge there was no LT for F(t) where F(s) is the LT of f(t). This is because with present knowledge the F(s) was always real when "s" is real, whereas for F(t) to have a LT, its LT has to be imaginary when "s" is real.

It will be shown that the DSLT of F(t) is 2πjf (-s), which is imaginary for "s" being real. The approach is similar to the one used for FT where if f(ω) is the FT of f(t), then the FT of FT of F(t) is 2πf(-ω) [See ref. 2, p. 14]. The author calls it the Theorem of Symmetry.

As mentioned previously, the region of the convergence is eliminated with the use of generalized functions. That means the exponential sign for the LT integral can be either positive or negative, and its inverse integral term has an opposite exponential sign.

Present convention has it given below.

$$F(s) = \int_{-\infty}^{\infty} f(\tau)e^{-s\tau}d\tau \qquad (18)$$

The inverse of (18) is:

$$f(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} F(\tau)e^{t\tau}d\tau \qquad (19)$$

For the inverse of LT integral (19) is usually written as having integral limits $\sigma \pm j\infty$ in order for the integral to be within the region of convergence, but this is now not necessary as already mentioned. Also "t" and "s" are usually used as integral variables, but the "τ" variable is preferred by the author because it provides a cleared understanding. Equation (19) will be rewritten as (20) below so that it appears the same as equation (18) by replacing "t" with "-s", but with its integral limits unaltered. Generalized functions are invariant with respects to rotation [12]. That is for generalized functions such as δ(t), sgn t, etc. The DSLT result remain unaltered whether one integrates along either axis, real or imaginary.

$$2\pi j f(-s) = \int_{-j\infty}^{j\infty} F(\tau)e^{-s\tau}d\tau \qquad (20)$$

From (20) one has the DSLT of F(t) as 2πjf(-s) with no restraints on the region of convergence for integral to exist.

Put s=jω (20) to represent the FT case, and one has $2\pi j f(-j\omega) = 2\pi f(-\omega)$ (See equation (7)).

In general, all ordinary functions can be represented by generalized functions. As an example, e^{-t^2} is the same as $e - t^2(U(t) + U(-t))$. Note that U(t) $U(t) = \frac{1}{2} + \frac{\text{sgn}\, t}{2}$. Generalized function can be expressed as derivatives of the delta function [13,14].

DSLT of $\frac{1}{t^n}, n > 0$

Since the FT is the same as the DSLT at s = jω, then all functions that have a FT should also have a DSLT at s = jω, and yet functions such as[1] have a FT but no t Laplace Transform.

For finding the DSLT of $\frac{1}{t}$, the inverse DSLT (where one is to find the DSLT of $F\left(\frac{1}{t}\right)$) needs to be included for finding the DSLT of a functions, which at present is considered unrelated to the findings of the DSLT of a function. The author will call this the Procedure of Symmetry.

Procedure of Symmetry

To find the DSLT of $\frac{1}{t}$, one already has from (10) the DSLT of $\frac{\text{sgn}\, t}{2}$ as $\frac{1}{s}$. Therefore from the above Procedure of Symmetry the DSLT of $\frac{1}{t}$, will be:

$$2\pi j\left(\frac{\text{sgn}(-s)}{2}\right) = -\pi j\, \text{sgn}\, s \qquad (21)$$

There is no restrain in "s" in the complex plane.

Now it will be shown that this is the same as the FT of $\frac{1}{t}$ by replacing "s" with "ω" where $-\pi j\ \text{sgn}(j\omega) = -\pi j\, \text{sgn}\, \omega$, both being FT of $\frac{1}{t}$. This is because one has $\frac{d}{d\omega}\left(\frac{\text{sgn}(j\omega)}{2}\right) = j\delta(j\omega)$ which is equal to δ(ω) from (7).

And $\frac{d}{d\omega}\text{sgn}\,\omega$ is also equal to δ(ω). Therefore sgn (jω) = sgn ω. Therefore the FT of $\frac{1}{t}$ is $-\pi j\, \text{sgn}\, \omega$ or π(jsgn(ω)). [See Ref. 2, p 37] for the FT of sgnt; and for the FT of $\frac{1}{t}$, use the Theory of Symmetry. [Ref. 2, p 14].

Having established the DSLT then use the time differentiation of inverse DSLT equation (19) for DSLT of , n>0. Namely:

$$= \frac{d^n\left(\frac{1}{t}\right)}{dt^n} = \frac{(-1)^n}{t^n} = \frac{-1}{2\pi j}\int_{-j\infty}^{j\infty} \pi j\, \text{sgn}\, \tau \frac{\delta^n}{\delta t^n}(e^{t\tau})d\tau =$$

$$\frac{-1}{2}\int_{-j\infty}^{j\infty} \tau^n \text{sgn}\, \tau e^{t\tau} d\tau$$

And therefore, from equation (18), the DSLT of $\frac{(-1)^n}{t^n}$ is $-\pi j s^n$ sgn s, with no restraints on "s" in the complex plan for solving a divergent integral.

To find the DSLT of $\frac{1}{(t+k)^n}$, $n > 0$, and $-\infty < k < \infty$ (called Time Shifting), then the DSLT equation (18) becomes:

$$F(s) = \int_{-\infty}^{\infty} \frac{e^{-s\tau}d\tau}{(\tau+k)^n}$$

Let $\tau + k = \tau_1$ and one has

$$\int_{-\infty}^{\infty} \frac{e^{-(\tau_1 - k)s}d\tau_1}{\tau_1} = e^{ks}\int_{-\infty}^{\infty} \frac{e^{-s\tau}d\tau_1}{\tau_1} =$$

$\pi j s^n e^{ks}$sgn s

Phase Retrieval Problem

There are many situations in experimental physics and other areas of engineering where the observable quantity is F(s)F(-s) and from which one needs to extract the original signal. If a mechanism can be found to first convert the unknown signal into its even and odd components prior to measuring the quantity F(s)F(-s), then one can extract the even and odd components of the desired original signal exactly, and them sum the two. Up to now 2N ambiguities needed to be tackled in order to recover the original image, and the N large, the problem becomes enormous. N is the number of zeros in F(s), the DSLT of image.

Alternatively, 2^n is the number of different images all having the same autocorrelation function, since only F(s)F(-s) (the DSLT of the autocorrelation function Ψ(t)) is known, See reference 8.

It will be shown that if the unknown signal can first be broken up into its even and odd components prior to measuring F(s)F(-s), then it will be possible to find the original signal with just two measurements and have no ambiguity of results. This is similar to the way a function can be multiplied by e^{az} in order to recover the original signal, as has been proposed by various authors. See references (9) and (10). Here instead the function and its mirror image is added and subtracted from one another to obtain its even and odd components respectively.

Then autocorrelation function of f(t) is defined as:

$$\Psi(t) = \int_{-\infty}^{\infty} f(\tau)f(t+\tau)d\tau \qquad (22)$$

and it will be shown that if f(t) had a DSLT F(s), then the DSLT of Ψ(t) is F(s)F(-s). The DSLT Ψ(t) is given below as:

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\tau)f(t+\ \tau)d\tau\right)e^{-st}dt \quad (23)$$

Interchanging the order of integration, one has:

$$\int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{\infty} f(t+\tau)e^{-st}\,dt\right)d\tau \quad (24)$$

Let $t+\tau=t_1$. Therefore, $dt=dt_1$ and for $t=\pm\infty$, $t_1=\pm\infty$. Therefore, (20) is equal to:

$$F(s)=\int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{\infty} f(t_1)e^{-s(t_1-\tau)}dt_1\right)d\tau$$

But $\int_{-\infty}^{\infty} f(t_1)e^{-st_1}dt_1$ is the DSLT of f(t1) equal to F(s).

Therefore (21) is equal to:

$$F(s)\int_{-\infty}^{\infty} f(\tau)e^{s\tau}d\tau = F(s)\int_{-\infty}^{\infty} f(\tau)e^{-(s)\tau}d\tau =$$

$$F(s)F(-s)$$

That is, the DSLT of Ψ(t) is F(s)F(-s). Note that, as explained previously, there is no restraint for "s" in the complex plane.

For the DSLT of f (-t) one has:

$$\int_{-\infty}^{\infty} f(-t)e^{-st}dt$$

and with the transformation $t = -t_1$, (25) becomes F(-s). Therefore, for f(t) as an even or odd function, its DSLT is F(s) and $-F(s)$ respectively, and the DSLT of its autocorrelation function is F2(s) and $-F^2(s)$ respectively. Therefore no other function other than f(t) odd will have the same autocorrelation function. In terms of the zeros of F(s) this is the equivalent of them lying symmetrically with the respect to both the real and the imaginary axes of the "s" complex plane. It would not be possible to solve the retrieval problem with SSLT because one only considerers t > 0. Also, there should also not be any restraint in "s" in the complex plane, so as to be able to handle both F(s) and F (-s).

Conclusion

The DSLT of f(t) $f(t)\frac{\operatorname{sgn} t}{2}$ $\frac{\operatorname{sgn} t}{2}$ is the SSLT of f(t). Since $\frac{\operatorname{sgn} t}{2}$ is an odd function, and if f(t) is a rational even then function, then f(t) is an odd function. Therefore the SSLT of F(s) is an odd function. Similarly if f(t) is an odd rational function, its SSLT is an even function [15-19].

The region of convergence for "s" in the complex plane is eliminated. In fact, one could just as well define the DSLT with a positive exponential term instead of the current negative exponential term. Even though "s" can be complex, the condition for F(s) to be even or odd i.e., F(s) = F(-s) or F(s) = -F(-s) remains the same. That is, if $s = Re^{j\theta}$ and $-1 = e^{\pi j}$, then for F(s) even one has $F(Re^{j\theta}) = F(Re^{j(\vartheta+\pi)})$, and for F(s) odd one has $F(Re^{j\vartheta}) = -F(Re^{j(\vartheta+\pi)})$.

Furthering our discussion, a unit pulse of duration T is given by $\frac{\operatorname{sgn} t}{2} - \frac{\operatorname{sgn}(t-T)}{2}$. Therefore, (26) represents:

$$F(t)\left(\frac{\operatorname{sgn} t}{2} - \frac{\operatorname{sgn}(t-T)}{2}\right) \quad (26)$$

For f(t) = 0 for t < 0 and t > T, and f(t) for 0 < t < T. The SSLT of f(t) is then given by the DSLT of (26).

For the FT replace "s" with jω. The Phase Retrieval problem is an example where it can be solved using DSLT but not the conventional SSLT that only handles cases or positive time whereas the handling of even and odd function of time requires including both positive and negative time.

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