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Still the New Classical Relativistic Equation of Charge Motion in an Electromagnetic Field

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Abstract
The non-relativistic Goedecke, 1975 equation, which describes the motion of a point charge considering the radiation reaction, has no "runaway" solutions. A "physical" method of covariant generalization of this equation is proposed, a special case of which is based on the Lorentz transformations in a coordinate–free covariant representation. Two equivalent forms of a new classical relativistic equation of motion of a point charge are obtained. It is shown that the Abraham–Lorentz–Dirac (ALD) and the Mo–Papas (MP) equations are approximate consequences of the presented

Keywords: Abraham–Lorentz–Dirac Equation, Goedecke (1975) Equation, Radiation Reaction

1. Introduction

1.1. Preface

This paper is based upon the materials of my old preprint from 1978, which has undergone, on my part, a thorough incite and revision. Therefore, despite the fact that a lot of time has passed since the first publication, I considered it possible to call this work "Still the new classical relativistic...".

1.2. Covariantization the Retarded Classical Equation

Let us consider the non-relativistic equation of motion of a point charge e with mass m in an external electromagnetic field $\vec{\mathbf{E}}, \vec{\mathbf{H}}$ (Goedecke, 1975):

$$\ddot{\vec{\mathbf{r}}}(t-\tau_0)=\frac{e}{m}\left\{\vec{\mathbf{E}}+\left[\dot{\vec{\mathbf{r}}}(t)\vec{\mathbf{H}}\right]\right\}, \qquad (1)$$

where $\tau_0=\frac{2}{3}\frac{e^2}{m}$, and the speed of light $c=1$. The dot above the symbol indicates the time derivative, $\frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathbf{r}}(t)=\dot{\vec{\mathbf{r}}}(t)$. The problem of the electromagnetic mass of a point charge is not considered here; it is assumed that the mass has been renormalized in any suitable way.

Equation (1) can easily be reduced to the non-relativistic Abraham-Lorentz (AL) equation (Jackson, 1975), which describes the motion of a charge under the action of an external electromagnetic field and the reaction of its proper electromagnetic field, if we expand the left-hand side of (1) in a series in the small parameter τ_0 and restrict ourselves to the linear in τ_0 term of this expansion. A special feature of equation (1), unlike the AL equation, is the absence of runaway solutions, but there is a pre-acceleration effect which is not the subject of our consideration here. The point is to find a relativistic generalization of equation (1) for which it would be convenient utilize a 4-dimensional notation (see Appendix A).

However, direct replacement of 3-dimensional spatial vectors $\dot{\vec{\mathbf{r}}}(t)$, $\ddot{\vec{\mathbf{r}}}(t-\tau_0)$, $\vec{\mathbf{E}}, \vec{\mathbf{H}}$ with their 4-dimensional counterparts $v^i(\tau)$, $\dot{v}^i(\tau-\tau_0)$,and the tensor of the external electromagnetic field $F^{ij}(\tau)$, respectively, does not immediately lead to the correct result. Indeed, the left and right sides of equation (1) in covariant notation have the form:

$$\dot{v}^i(\tau-\tau_0), \quad (2)$$

$$\frac{e}{m}F^{ij}(\tau)v_j(\tau). \quad (3)$$

The scalar product of the 4-Lorentz force (3) by the 4-velocity $v^i(\tau)$, gives an identical zero due to the skew symmetry of the electromagnetic field tensor with respect to rearranging indexes, $F^{ij}=-F^{ji}$:

$$\frac{e}{m}F^{ij}(\tau)v_i(\tau)v_j(\tau)\equiv 0. \quad (4)$$

But the result of multiplying (2) by the 4-velocity vector $v_i(\tau)$ is not identically zero, in the general case,

$$\dot{v}^i(\tau-\tau_0)v_i(\tau)\not\equiv 0, \quad (5)$$

since the 4-speed of the charge is $v^i(\tau)\equiv \mathrm{d}x^i/\mathrm{d}\tau$ and 4-acceleration of the charge $\dot{v}^i(\tau-\tau_0)\equiv \mathrm{d}v^i(\tau-\tau_0)/\mathrm{d}\tau$, taken at different moments of their proper time $\tau - \tau_0$ and τ, refer to different inertial instantaneous-accompanying reference frames, and, therefore, do not always have to be orthogonal to each other. Here, $\mathrm{d}\tau=\mathrm{d}t\sqrt{1-(\mathrm{d}\vec{\mathbf{r}}(t)/\mathrm{d}t)^2}$, and $\vec{\mathbf{r}}(t)$ be the spatial radius vector of the charge position. One of the ways to obtain a relativistic analogue of an equation of type (1) is to perform the orthogonalization procedure on the 4-vectors $v^i(\tau)$,$\dot{v}^i(\tau-\tau_0)$, like obtaining the covariant ALD equation, (Dirac, 1938), from the AL equation, (Landau & Lifshitz, 1975).

To do this, to the 4-vector $\dot{v}^i(\tau-\tau_0)$ we add the 4-vector

$$-\dot{v}^k(\tau-\tau_0)v_k(\tau)v^i(\tau), \quad (6)$$

after that, the left side of the equation under constraction takes the form

$$\dot{v}^i(\tau-\tau_0)-\dot{v}^k(\tau-\tau_0)v_k(\tau)v^i(\tau). \quad (7)$$

In other words, the 4-vector $\dot{v}^j(\tau-\tau_0)$ be multiplied by the orthogonal projector

$$\delta^i_j - v^i(\tau)v_j(\tau). \quad (8)$$

As a result, equation (Sorg, 1976) follows from (7):

$$\dot{v}^i(\tau-\tau_0)-\dot{v}^k(\tau-\tau_0)v_k(\tau)v^i(\tau)=\frac{e}{m}F^{ij}(\tau)v_j(\tau). \quad (9)$$

It is easy to check the orthogonality of both parts of equation (9) to the 4-velocity $v^i(\tau)$ directly. However, this method of orthogonalization has the disadvantage that it is difficult to give it a physical meaning, since (9) is not a strictly covariant generalization of equation (1) - orthogonal projection does not preserve the norm of the 4-vector $\dot{v}^i(\tau-\tau_0)$. In (Sermyagin, 1978), another method was proposed that allows us to talk about a consistent relativistic approach to obtaining covariant generalizations of non-relativistic delayed-motion dynamics equations. The physical reason that $v^i(\tau)$ and $\dot{v}^i(\tau-\tau_0)$ are not orthogonal is that they refer, generally speaking, to two different instantaneously accompanying frames of reference moving in an inertial laboratory the frame of reference an observer with different 4-velocities, $v^i(\tau)\not\equiv v^i(\tau-\tau_0)$. Therefore, the "physical" procedure for orthogonalizing the quantities $v^i(\tau)$ and $\dot{v}^i(\tau-\tau_0)$ consists in reducing these quantities to the same inertial instantaneous frame of reference, for that it is necessary to perform the Lorentz transformation $\Lambda(u\Rightarrow v)$ 4-acceleration vectors $\dot{v}^i(\tau-\tau_0)$ from the reference frame, 4-velocity of which in the laboratory reference frame is $v^i(\tau-\tau_0)\equiv u$, to the reference frame whose 4-velocity is $v^i(\tau)\equiv v$. As result of this physical orthogonalization, the correct covariant relativistic generalization of equation (1) takes the form:

$$\boldsymbol{\Lambda}^i_k \dot{v}^k(\tau-\tau_0)=\frac{e}{m}F^{ik}(\tau)v_k(\tau), \tag{10}$$

where the Lorentz transformation $\boldsymbol{\Lambda}^i_k$ is defined as $\boldsymbol{\Lambda}^i_k v^k(\tau-\tau_0)= v^i(\tau)$.
We should immediately note that due to the fact that

$$\boldsymbol{\Lambda}^i_k \boldsymbol{\Lambda}^{-1}{}^k_j = \mathbf{I}^i_j, \tag{11}$$

where $\boldsymbol{\Lambda}^{-1}{}^k_j \equiv \boldsymbol{\Lambda}^{-1}\ v \Rightarrow u$ is the Lorentz transformation, the inverse of the transformation $\boldsymbol{\Lambda}$, and $\mathbf{I}^i_j$ is the identical transformation,

$$\mathbf{I}^i_j \equiv \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}, \tag{12}$$

the equation

$$\dot{v}^i(\tau-\tau_0) = \frac{e}{m}\boldsymbol{\Lambda}^{-1}{}^i_j F^{jk}(\tau)v_k(\tau) \tag{13}$$

be equivalent to (10).

Using index-free notation and explicit expressions for the Lorentz transformations Λ and Λ^{-1}, the derivation of which is given in Appendix B, from from (33) and (34) we obtain a relativistic generalization of equation (1) in two equivalent forms:

$$m\dot{u} - m\dot{u}\cdot v\frac{u+v}{1+u\cdot v} = eF\cdot v, \tag{14}$$

$$m\dot{u} = eF\cdot v - eF\cdot v\cdot u\frac{u+v}{1+u\cdot v}, \tag{15}$$

where $u \equiv v(\tau-\tau_0);\ v \equiv v(\tau);\ \tau_0 = \frac{2}{3}\frac{e^2}{m};\ c = 1.$

It is important to note that the Lorentz transformations used to derive (14) and (15) do not contain spatial rotations. Therefore, strictly speaking, (14) and (15) describe one-dimensional motions only, while equations (10) and (13) are applicable to the general situation. An adequate covariant representation of such general Lorentz transformations will be published separately.

The fact that equations (14) and (15) do not have “runaway” solutions for $F = 0$ is evident from the general form of these equations (10) and (13), when the representations of the operators $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}^{-1}$ be not specified. For (13) and (15), this is obvious; under the condition $F = 0$, we act on the left and right sides of equations (10) and (14) with the operator $\boldsymbol{\Lambda}^{-1}$, we will have then $\dot{u} = 0$, i.e. the equation of motion of a free particle.

Note the connection of (14) and (15) with the ALD and MP equations, which be approximate consequences of the presented theory. To do this, we decompose the quantities depending on the lagging argument $\tau - \tau_0$ by degrees of delay τ_0 and discard the terms proportional to τ_0 to a power higher than the first. We will then obtain from (14) the equation ALD,

$$\dot{v} - \tau_0(\ddot{v} + \dot{v}\cdot\dot{v}v) = \frac{e}{m}F\cdot v. \tag{16}$$

In this case, it follows from (15):

$$\dot{v} - \tau_0\ddot{v} = \frac{e}{m}F\cdot v + \tau_0\frac{e}{m}F\cdot v\cdot\dot{v}v. \tag{17}$$

Using the ratio $\ddot{v} = \frac{e}{m}\frac{\mathrm{d}}{\mathrm{d}\tau}F\cdot v$, assuming that $\dot{F}\cdot v = 0$, we obtain from (17) the equation MP (Mo & Papas, 1971):

$$\dot{v} = \frac{e}{m} F \cdot v + \tau_0 \frac{e}{m} (F \cdot \dot{v} + F \cdot v \cdot \dot{v} v). \tag{18}$$

For the first time, equations (10), (13), (14) and (15) were presented in (Sermyagin, 1978). The investigation of solutions to these equations will be published separately.

2. Discussion and Conclusion

We have achieved the stated goal of obtaining a covariant generalization of equation (1) through proper physical covariantization and got the new relativistic equation in two equivalent forms, (10), (13), (14) and (15). Since the Lorentz transformations used do not contain spatial rotations, the resulting equations (14) and (15) be of a special case describing one-dimensional motion in the external electromagnetic field. An analysis of general (10) and (13) will be published later.

2.1. Appendix A: Coordinate & Index-Free Notation and Normalizing

As usual, the summation rule is used (without explicitly specifying the sign of the sum Σ) for repeated indexes that take values 0,1,2,3:

$$x^k y_k \equiv \sum_{i,j=0}^{3} x^i \, x^j \eta_{ij}, \tag{19}$$

where η_{ij} denotes the metric tensor of the Minkowski space,

$$\eta_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{20}$$

Where not stated otherwise, the 4-velocity vectors are denoted by small Latin letters with no indexes and no indicated dependence on the proper time τ, $u \equiv v^i(\tau - \tau_0)$, $v \equiv v^i(\tau)$. Inner (scalar) product of pair 4-vectors is indicated with a dot between these vectors; in the case of 4-vectors: $x^k y_k \Rightarrow x \cdot y$, in the case of the second rank tensor and 4-vector: $F^{ik} v_k \equiv F^i_k \, v^k \equiv F^{ik} \eta_{kj} \, v^j \Rightarrow F \cdot v$. 4-speed normalization:

$$v \cdot v = 1. \tag{21}$$

The orthogonality of 4-velocity and 4-acceleration is a consequence of normalization (21). Indeed, differentiating (21) by proper time τ, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} v \cdot v = 2 v \cdot \dot{v} \equiv 0. \tag{22}$$

2.2. Appendix B: Covariantization the Lorentz Transformation

The possibility of factoring Lorentz transformations between two inertial reference frames, in the formalism of tetrad, that is, 4-dimensional basis vectors, was established by Bazansky (Bazanski, 1965). An explicit form of Lorentz transformations, without spatial rotations, as functions of 4-velocities in index-free notation, was obtained by Krause (Krause, 1977, 1978). Such Lorentz transformations, derived independently in a different way, are used in the author's work (Sermyagin, 1978) for a covariant generalization equation (1).

If $\mathbf{\Lambda}$ is a homogeneous or proper Lorentz transformation without spatial rotations, $\mathbf{\Lambda}: u \Rightarrow v$, so that

$$\cosh \varphi = u \cdot v, \tag{23}$$

then the matrix $\mathbf{M}$ of rotations in 1+3 Minkowski space corresponds to this transformation,

$$\mathbf{M} = \begin{bmatrix} \cosh \varphi & \sinh \varphi & 0 & 0 \\ \sinh \varphi & \cosh \varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{24}$$

with respect to the orthonormal local basis ϵ,

$$\epsilon = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}, \epsilon^{\mathrm{T}} = \begin{bmatrix} e^0 & e^1 & e^2 & e^3 \end{bmatrix}, \tag{25}$$

$e_0 = e^0 \equiv u$, $e_1 = e^1 \equiv \frac{v}{\sinh\varphi} - u(\coth\varphi)$, "...$^{\mathrm{T}}$" means transposition. We don't need exact expressions for e_2 and e_3 here. An arbitrary 4-vector a can be written as a basis vector expansion (25),

$$a \equiv \epsilon^T \cdot \mathcal{G}a \cdot \epsilon. \tag{26}$$

The matrix $\mathcal{G}$ is defined by the relation

$$\mathcal{G} \equiv \eta_{ij} = \epsilon\epsilon^T. \tag{27}$$

Now, using decomposition (26), we write the Lorentz transformation of the 4-vector a in general form as:

$$\begin{aligned} b &= \mathbf{\Lambda} \cdot a \\ &= \epsilon^T \mathbf{M}\mathcal{G}a \cdot \epsilon \\ &= a - a + \epsilon^T \mathbf{M}\mathcal{G}a \cdot \epsilon \\ &= a + \epsilon^T(\mathbf{M} - \mathbf{I})\mathcal{G}a \cdot \epsilon, \end{aligned} \tag{28}$$

where **I** is the identity matrix (12),

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{29}$$

We also use light font to denote identity matrix (29), $\mathbf{I} \Rightarrow \mathrm{I}$.

From (28), we obtain the Lorentz transformation in a coordinate-free covariant form:

$$\mathbf{\Lambda} = \epsilon^T \mathbf{M}\mathcal{G}\epsilon = \mathbf{I} + \epsilon^T(\mathbf{M} - \mathbf{I})\mathcal{G}\epsilon =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} u & u\cosh\varphi - \frac{v}{\sinh\varphi} & e^2 & e^3 \end{bmatrix} \left\{ \begin{bmatrix} \cosh\varphi & \sinh\varphi & 0 & 0 \\ \sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \right.$$

$$\left. \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ u\cosh\varphi - \frac{v}{\sinh\varphi} \\ e^2 \\ e^3 \end{bmatrix} =$$

$$\mathrm{I} + \begin{bmatrix} u & u\cosh\varphi - \frac{v}{\sinh\varphi} & e^2 & e^2 \end{bmatrix} \begin{bmatrix} \cosh\varphi - 1 & \sinh\varphi & 0 & 0 \\ \sinh\varphi & \cosh\varphi - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times$$

$$\times \begin{bmatrix} u \\ -u\cosh\varphi + \frac{v}{\sinh\varphi} \\ -e^2 \\ -e^3 \end{bmatrix} =$$

$$\mathrm{I} + \begin{bmatrix} u & u\cosh\varphi - \frac{v}{\sinh\varphi} \end{bmatrix} \begin{bmatrix} \cosh\varphi - 1 & \sinh\varphi \\ \sinh\varphi & \cosh\varphi - 1 \end{bmatrix} \begin{bmatrix} u \\ -u\cosh\varphi + \frac{v}{\sinh\varphi} \end{bmatrix} \qquad (30)$$

Performing matrix product in (30), remembering that $\cosh\varphi = u \cdot v$ (23), we get:

$$\mathbf{\Lambda} = \mathrm{I} - u\frac{u+v}{1+u\cdot v} + v\frac{2u\cdot vu + u - v}{1+u\cdot v}. \qquad (31)$$

The inverse transformation has the form:

$$\mathbf{\Lambda}^{-1} = \epsilon^T \mathbf{M}^{-1} \boldsymbol{\mathcal{G}} \epsilon = \mathrm{I} + \epsilon^T (\mathbf{M}^{-1} - \mathbf{I}) \boldsymbol{\mathcal{G}} \epsilon, \qquad (32)$$

$$\mathbf{M}^{-1} = \begin{bmatrix} \cosh\varphi & -\sinh\varphi & 0 & 0 \\ -\sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (33)$$

$$\mathbf{\Lambda}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} u & u\cosh\varphi - \frac{v}{\sinh\varphi} & e^2 & e^3 \end{bmatrix} \left\{ \begin{bmatrix} \cosh\varphi & -\sinh\varphi & 0 & 0 \\ -\sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right.$$

$$\left. - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ u\cosh\varphi - \frac{v}{\sinh\varphi} \\ e^2 \\ e^3 \end{bmatrix}$$

$$= \mathrm{I}$$

$$+ \begin{bmatrix} u & u\cosh\varphi - \frac{v}{\sinh\varphi} \end{bmatrix} \begin{bmatrix} \cosh\varphi - 1 & -\sinh\varphi \\ -\sinh\varphi & \cosh\varphi - 1 \end{bmatrix} \begin{bmatrix} u \\ -u\cosh\varphi + \frac{v}{\sinh\varphi} \end{bmatrix}$$

$$= \mathrm{I} - v\frac{u+v}{1+u\cdot v} + u\frac{2u\cdot vv + v - u}{1+u\cdot v}. \qquad (34)$$

The forward $\mathbf{\Lambda}$ and inverse $\mathbf{\Lambda}^{-1}$ Lorentz transformations differ by permutation of 4-velocities, which is obvious from a comparison of their form:

$$\mathbf{\Lambda} = \mathrm{I} - u\frac{u+v}{1+u\cdot v} + v\frac{2u\cdot vu + u - v}{1+u\cdot v}, \tag{35}$$

$$\mathbf{\Lambda}^{-1} = \mathrm{I} - v\frac{u+v}{1+u\cdot v} + u\frac{2u\cdot vv + v - u}{1+u\cdot v}. \tag{36}$$

We also note that

$$\mathbf{\Lambda}\cdot u = v, \qquad v\cdot\mathbf{\Lambda} = u, \qquad \mathbf{\Lambda}^{-1}\cdot v = u, \qquad u\cdot\mathbf{\Lambda}^{-1} = v. \tag{37}$$

When using a coordinate-free/index-free representation, it is necessary to keep in mind the matrix nature of the multiplied quantities and their noncommutativity. In this regard, a modified 4-vector representation formalism may be convenient, in which the Lorentz transformations look like this:

$$\mathbf{\Lambda} = \mathrm{I} - |u)\frac{(u| + (v|}{1+(u|v)} + |v)\frac{2(u|v)(u| + (u| - (v|}{1+(u|v)}, \tag{38}$$

$$\mathbf{\Lambda}^{-1} = \mathrm{I} - |v)\frac{(u| + (v|}{1+(u|v)} + |u)\frac{2(u|v)(v| + (v| - (u|}{1+(u|v)}. \tag{39}$$

An arbitrary 4-vector *p* written in bra-cket notation:

$$p^i \equiv |p), \;\; p_k \equiv (p|, \tag{40}$$

i.e., the transition between bracket vectors corresponds to the procedure of raising and lowering indices in index notation using the Minkovsky metric tensor, (20). The scalar product of 4-vectors a and *b*:

$$a^i b_i \Rightarrow a\cdot b \equiv (a|b), \tag{41}$$

tensor product of 4-vectors *a* and *b*:

$$a^i b_k \Rightarrow ab \equiv |a)(b|. \tag{42}$$

In this entry, the Lorentz transformation $\mathbf{\Lambda}: u \Rightarrow v$ looks like this:

$$\mathbf{\Lambda}|u) = \left\{\mathrm{I} - |u)\frac{(u| + (v|}{1+(u|v)} + |v)\frac{2(u|v)(u| + (u| - (v|}{1+(u|v)}\right\}|u) =$$

$$|u) - |u)\frac{(u|u) + (u|v)}{1+(u|v)} + |v)\frac{2(u|v)(u|u) + (u|u) - (u|v)}{1+(u|v)} =$$

$$|u) - |u) + |v) = |v). \tag{43}$$

Similarly, it is easy to prove that $\mathbf{\Lambda}^{-1}\cdot v = u$. The properties of covariant Lorentz transformations can be found in (Krause, 1977, 1978). Substituting (38) into (10), considering (37), we get:

$$m\mathbf{\Lambda}\cdot\dot{u} \Rightarrow m\left\{\mathrm{I} - |u)\frac{(u|+(v|}{1+(u|v)} + |v)\frac{2(u|v)(u|+(u|-(v|}{1+(u|v)}\right\}|\dot{u}) = m\left\{|\dot{u}) - |u)\frac{(u|\dot{u})+(v|\dot{u})}{1+(u|v)} + |v)\frac{2(u|v)(u|\dot{u})+(u|\dot{u})-(v|\dot{u})}{1+(u|v)}\right\} = m|\dot{u}) - m(v|\dot{u})\frac{|u)+|v)}{1+(u|v)} \Rightarrow$$

$$m\dot{u} - mv\cdot\dot{u}\frac{u+v}{1+u\cdot v} = eF\cdot v. \tag{44}$$

For the Lorentz transformation of the Lorentz force, we note that

$$F^{ik} = -F^{ki};$$
$$F^{ik}v_i v_k = F^{ik}u_i u_k \Rightarrow F \cdot u \cdot u = F \cdot v \cdot v = 0; \tag{45}$$
$$F^{ik}v_k \Rightarrow F \cdot v = |F \cdot v).$$

Substituting (39) into (13), taking into account (45), we obtain:

$$\boldsymbol{\Lambda}^{-1}F^{ik}v_k \Rightarrow \boldsymbol{\Lambda}^{-1}|F \cdot v) = \left\{\mathrm{I} - |v)\frac{(u|+(v|}{1+(u|v)} + |u)\frac{2(u|v)(v|+(v|-(u|}{1+(u|v)}\right\}|F \cdot v) = |F \cdot v) - |v)\frac{(u|F\cdot v)+(v|F\cdot v)}{1+(u|v)} + |u)\frac{2(u|v)(v|F\cdot v)+(v|F\cdot v)-(u|F\cdot v)}{1+(u|v)}$$
$$= |F \cdot v) - |v)\frac{(u|F \cdot v)}{1+(u|v)} - |u)\frac{(u|F \cdot v)}{1+(u|v)}$$
$$= |F \cdot v) - \frac{|u) + |v)}{1+(u|v)}(u|F \cdot v). \tag{46}$$

As a result, the second equivalent form of the covariant generalization of equation (1) in index-free notation takes the form:

$$m\dot{u} = e\left\{F \cdot v - \frac{u+v}{1+u \cdot v}F \cdot u \cdot v\right\}. \tag{47}$$

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