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Static Mantle Density Distribution 2 Improved Equation and Solution

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Abstract
Using Archimedes Principle of Sink or Buoyancy (APSB), Newton's universal gravity, buoyancy, lateral buoyancy, centrifugal force and Principle of Minimum Potential Energy (PMPE), this paper improves the derivation of equation of static mantle density distribution. It is a set of mutual related 2-D integral equations of Volterra/Fredholm type. Using method of match-trying, we find the solution. Some new results are: (1) The mantle is divided into sink zone, neural zone and buoyed zone. The sink zone is located in a region with boundaries of an inclined line, angle α_0=35°15', with apex at O(0,0) revolving around the z-axis, inside the crust involving the equator. Where no negative mass exists, while positive mass is uniformly distributed. The buoyed zone is located in the remainder part, inside the crust involving poles. Where no positive mass exists, while negative mass is uniformly distributed. The neural zone is the boundary between the buoyed and sink zones. The shape of core (in sink zone) is not a sphere. (2) The total positive mass is equal to the total negative mass. (3) The volume of BUO zone is near twice the volume of SIN zone. (4) No positive mass exists in an imagine tunnel passing through poles (the z-axis).

Keywords: Newton's law of universal gravitation, Archimedes principle of buoyancy, lateral buoyancy, Principle of minimum potential energy, method of Lagrange multipliers. Method of match-trying.

1. Introduction
Although there are many researches and books on Earth structure, e.g., [1-5]. However, most studies focus on physical and chemistry properties, dynamic analysis. Seldom paper on study of mantle distribution has been found. The study of mantle distribution does relate to the reflecting of seismic waves, and has important meaning. For example, a recent paper shows that the energy release of earthquake proportions to the square of Earth rotation velocity, and the calculation of energy release relates to seismic waves [6].

The Newton's law of universal gravitation is a part of classical mechanics and has basic importance for wide fields, especially in astronomy and gravity. Note that no point masses are rejected, and all objects with mass above on Earth are attracted to the ground no matted on large or small of mass. However, the Newton's law of universal gravitation does not consider the effect of environmental factors (such as media, temperature, pressure, motion, etc.) between the masses. For the case of masses immersed in a fluid media, buoyancy against gravity, it puts lighter object up. Which reveals that the up or down of the object depends on the resultant force of attraction and buoyancy. Which is summarized as "Archimedes Principle of sink or buoy" (APSB). The buoyancy has the same important as gravity in the study of Earth, which is emphasized in [7].

If only attraction force exists, then, all objects are attracted to the ground, the Earth becomes death. Since the buoyancy exists, as an opposite force, it keeps the system to equilibrium. The Earth being a planet with life is relying on the gravity force and buoyancy force, the later makes cycles of water to evaporation to cloud, cloud to water droplet, and water droplet to rain. The cycles brings water to everywhere on Earth to keep life existence.

The aim of this paper is to improve the derivation of static mantle density distribution obtained in and to solve the obtained equations [8]. By the method of match-trying, we find the solution, where the mantle density is uniformly distributed in different regions.

2. Recall and comment on "Static Mantle Density 1 Equation" Comment on "Static Mantle Distribution 1 Equation"
(1) The method, theory, calculation and conclusion used in are correctness, but the text needs to be simplified and revised [8].
(2) Hypotheses 3 is added.
For convenience to readers, let us repeat and improve the arrangement of briefly [8].

Theory and Calculation
2.1 Basic Hypotheses, Coordinates and Study Range
(1)The Earth is assumed to be an ellipsoid with equator radius R_e and pole radius R_p:

$$\left(\frac{r}{R_e}\right)^2 + \left(\frac{z}{R_p}\right)^2 = 1, \qquad (2.1\text{-}1)$$

(2) Mantle masses are co-here with continuously, fully filled, z-axial-symmetry and equatorial-plane-symmetry distributed incompressible non-isotropic liquid medium masses.
(3) The medium mass is considered as negative mass, while the remainder are considered as positive masses. The same sign masses are

attracted each other; while different sign masses are rejected each other.

Notation: The bold face denotes vector. A:={B | C} means A is defined by B with property C.
Let (x,y,z) be the Cartesian coordinates of the geometrical center of the Earth with origin O(0,0,0). The coordinates (x,y,z) is chosen that the z-axes is perpendicular to the equatorial plane xOy with z = 0 at xOy.

Cylindrical coordinates: Let (r,θ,z) be the cylindrical coordinates of the geometric center of the Earth. The relation between (x, y) and (r,θ) is:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} (0 \le \theta \le 2\pi, 0 \le r < \infty, -\infty < z < \infty) \quad (2.1\text{-}2)$$

(**i**, **j**, **k**) and (**e**$_r$, **e**$_\theta$, **k**) denote the unit vectors of Cartesian and cylindrical coordinates respectively. In the following, by hypotheses 2, a point f(r,θ,z) independents to θ and can be simplified by f(r,z), and discussion only focus on semi-sphere z≥0.
We study the static stable equilibrium system.

2.2 Newton's Law of Universal Gravitation

2.2.1 For two point masses

The Newton's law of universal gravitation of vector form is:

$$\mathrm{F_{fg}} = -G\frac{m_f m_g}{|h_{gf}|^2}\mathbf{h}_{gf} = -G\frac{m_f m_g}{|h_{gf}|^2}(\mathbf{h_g} - \mathbf{h_f}), \quad (2.2\text{-}1)$$

Where $\mathbf{F}_{fg}$ is the force applied on point mass f exerted by point mass g, its direction is that from f towards to g; gravitational constant

G= $6.674 \times 10^{-11}, \mathrm{N}.(\frac{m}{kg})^2$; m_f and m_g are masses of point f(x_f, y_f, z_f) and g(x_g, y_g, z_g), respectively;

$$|h_{fg}| = |h_g - h_f| = \left|\sqrt{(x_g - x_f)^2 + (y_g - y_f)^2 + (z_g - z_f)^2}\right|, \quad (2.2\text{-}2)$$

$|h_{fg}|$ is the distance between points f and g; $\mathbf{h}_f$ and $\mathbf{h}_g$ are vectors from O(0, 0) to point f and g, respectively;

$\mathbf{h_{gf}} := \frac{h_f - h_g}{|h_f - h_g|}$ is the unit vector from point g to f.

Or, F_{fg} is expressed in cylindrical components form:

$$F_{fg} = F_{rfg}e_r + F_{zfg}k, \quad (2.2\text{-}3)$$

$$F_{rfg} = G\frac{m_f m_g}{H}(r_f - r_g), \quad (2.2\text{-}4)$$

$$F_{zfg} = G\frac{m_f m_g}{H}(z_f - z_g), \quad (2.2\text{-}5)$$

$$H = \left|(r_f - r_g)^2 + (z_f - z_g)^2\right|^{3/2}, \quad (2.2\text{-}6)$$

2.2.2 For point mass and point masses group

The Newton's law of universal gravitation usually uses for point mass.

Now, we want to calculate force $\mathbf{F}_{fM}$ applied on mass m_{fp} (located at f(r_f, z_f)) exerted by all positive masses M_{fp}, except m_{fp} itself.

$$\mathbf{F}_{fM} := G\int_0^{V_{mant}^-}\frac{m_{fp}m_g}{H}(\mathbf{r_f} - \mathbf{r_g})dV_g, \quad (2.2\text{-}7)$$

Using the theorem of mean value of vector integral (or the theorem of resultant of forces)), we have

$$\mathbf{F}_{fM} := Gm_{fp}\frac{M_{fp}^-}{H_f}(\mathbf{r_f} - \int_0^{V_{mant}^-}\mathbf{r_g}\,dV_g) = Gm_{fp}\frac{M_{fp}^-}{H_f}(\mathbf{r_f} - \mathbf{0}), \quad (2.2\text{-}8)$$

$$M_{fp}^- := M_{fp} - m_{fp} = \rho_{mean}\int_0^{V_{mant}}dV_g = \rho_{mean}V_{mant}^-, \quad (2.2\text{-}9)$$

$$\int_0^{V_{mant}^-}dV_g := \int_0^{f(r_f^-, z_f^-)}dV_g + \int_{f(r_f^+, z_f^+)}^{V_{zone}}dV_g = \int_{f(r_f, z_f)}^{V_{mant}}dV_g \quad (2.2\text{-}10)$$

Where M_{fp} is a masses group of all positive mass, (located nearly at 0 (0, 0) due to symmetry), m_{fp}, except itself

(because that m_{fp} can not be attracted by itself); V_{mant} is the volume of mantle; $V_{mant}^- = V_{mant} - f(r_f, z_f) \approx V_{mant}$.

The component form of (2.2-8) is:

$$F_{rfM} = G\frac{m_{fp}M_{fp}^-}{H_f}(r_f - 0), \quad (2.2\text{-}11)$$

$$F_{zfM} = G\frac{m_{fp}M_{fp}^-}{H_f}(z_f - 0), \quad (2.2\text{-}12)$$

$$H_f = \left[r_f^2 + z_f^2\right]^{3/2}, \quad (2.2\text{-}13)$$

Similar arrangement can be used for negative mass.

2.3 Buoyancy

Archimedes's principle of buoyancy states that any object, wholly or partly, immersed in a fluid, is buoyed by a force equal to the weight of the fluid displaced by the object.

The Archimedes's principle shows that The Newton's gravity does not attracted medium mass but rejects it. This fact reveals us that the mass can be divided into positive mass and negative mass. The medium mass m_n belongs to negative mass, while the remainder m_p belongs to positive mass. The same sign mass attracts each other.
(1) The components of buoyancy $\mathbf{F}_{bz}$ in z-axis can be defined by Newton's second law, i.e., by (2.2-5),

$$F_{bz} := -m_n a_z = -\rho_n a_z dv = -G\frac{m_n m_g}{H}(z_f - z_g), \quad (2.3\text{-}1)$$

Where a_z is the component of acceleration in z-axis; ρ_n is the density of negative mass (mass per unit volume) of the media at f (r, z); dv=rd0drdz; $m_{fn} = \rho_n dv$; $m_p = \rho_p dv$. The substance of m_n must be liquid, while the substance of m_p could be gas, liquid or solid. The minus sign means the direction of buoyancy is opposed to the attraction force.

According to Archimedes's Principle of Sink or Buoy (APSB), there are three zones inside the Earth:

The sink zone, $SIN := \{\aleph_s | \rho_p > \rho_n\}$ The neural zone,

$NEU := \{\aleph_n | \rho_p = \rho_n\}$,

The buoyancy zone, $BUO := \{\aleph_b | \rho_p < \rho_n\}$

2.4 Extension the Archimedes' Principle of Buoyancy to Lateral Buoyancy

The buoyancy is firstly extended to lateral buoyancy, by logical deduction, which assumes that a rule suits for the z-axis, it is also suited for x-axis and y-axis [7]. Similar to (2.3-1) , we have:

$$F_{br} = -m_n a_r, \quad (2.4\text{-}1)$$

Where $a_r = \omega_c^2 r$.

2.5 Angular Velocity of a Point of Mantle Due to Earth Rotation

Proposition: the angular velocity of a point of mantle equals that of crust.

Proof: Suppose that the angular velocity ω_N of a point N(r, 0, z) of mantle is different to that ω_C of a point C(r+dr, z) of crust, say, $\omega_C > \omega_N$, then, a friction force $F_{friction}$ exists between C(r+dr, 0, z) and N(r, z), such that $F_{friction}$ blocks ω_C meanwhile drags ω_N, until $\omega_C = \omega_N$. Similarly, the rotating angular velocity of a point of mantle is equal to that of its neighbor.

2.6 Potential Energy inside the Earth

Potential energy is known as the capacity of doing work due to object's position static changing (with zero acceleration, because the work done by acceleration is calculated in kinetic energy). If a work w, done by a force **F**, moved from point f (r, z) to point g (r_g, z_g), then, it is calculated by

$$w = \int_f^g F \cdot d\mathbf{s} = \Delta E_p = E_p(g) - E_p(f), \quad (2.6\text{-}1)$$

Where ΔE_p denotes the change of potential energy; ds is the change of position vector **s**.

2.6.1 Work Done by Gravity, Buoyancy, Lateral Buoyancy and Centrifugal Force, for m_{pf} and m_{nf} Moving from f(r_f, z_f) to O (0,0)

The general component form of work done by multi-forces moving from f(r_f, z_f) to O (0,0), by (2.6-1) is:

$$w = \Delta E_p = \int_f^O \sum F_r \, dr + \int_f^O \sum F_z \, dz, \quad (2.6\text{-}2)$$

the x under the Σx sign denote each term of the force components. By hypotheses 2, the mantle is incompressible and continuously distributed. No vacuum or overlap exists in mantle. When the masses m_{pf} and m_{nf} move from f (r_f, z_f) to O (0,0), at the same time, it must have masses m_{po} and m_{nO} move from O to f. The potential energy is:

$$\Delta E_p = \int_f^O \sum F_{rf} \, dr + \int_f^O \sum F_{zf} \, dz + \int_O^f \sum F_{rq} \, dr + \int_O^f \sum F_{zq} \, dz, \quad (2.6\text{-}3)$$

where F_{rf}, F_{zf}, F_{rO}, and FzO, are components of forces $\mathbf{F_f}$ and $\mathbf{F_O}$ respectively.

For the case of multiple forces due to gravity, buoyancy, centrifugal force and lateral buoyancy

$$F_{rf} = (m_{pf} - m_{nf})[GM_{\bar{p}} H_f^{-1} r_f + \omega_c^2 r_f] - (m_{po} - m_{no})[GM_{\bar{p}} H_O^{-1} r_O + \omega_c^2 r_O] \quad (2.6\text{-}4)$$

$$F_{zf} = (m_{pf} - m_{nf})[GM_{\bar{p}} H_f^{-1} z_f] - (m_{po} - m_{nO})[GM_{\bar{p}} H_O^{-1} z_O], \quad (2.6\text{-}5)$$

Now, we prove the mass $m_{pO} = 0$.

Proof: By (2.2-8),

$$\mathbf{F}_{OM} = \frac{m_{pO} M_{\bar{p}}}{H_O} (\mathbf{r}_O - \mathbf{0}) = \mathbf{0}, \quad (2.6\text{-}6)$$

due to symmetry assumption. Where

$$H_O = \epsilon^3, \ (r_O - 0) = (\epsilon - \epsilon), \ \epsilon \to 0. \quad (2.6\text{-}7)$$

Using L'Hospital rule twice, if $m_{pO} \neq 0$ Then, we have

$$F_{OM} = \frac{m_{pO} M_{\bar{p}}}{\epsilon} \to \infty, \quad (2.6\text{-}8)$$

Which conflicts to (2.26), therefore $m_{pO} = \epsilon^n \to 0$. (n>1).

Similarly, $m_{nO} = 0$.

Substituting $m_{pO} = 0$ and $m_{nO} = 0$ into (2.6-4) and (2.6-5), we have:

$$F_{rf} = (m_{pf} - m_{nf})[GM_{\bar{p}} H_f^{-1}(r_f - 0) + \omega_c^2 (r_f - 0)], \quad (2.6\text{-}9)$$

$$F_{zf} = (m_{pf} - m_{nf})[GM_{\bar{p}} H_f^{-1}(z_f - 0)], \quad (2.6\text{-}10)$$

Substituting (2.6-9) and (2.6-10) into (2.6-3), we have:

$$\Delta E_p = E_p(O) - E_p(f) = m_{pf}\{\int_f^O [GM_{\bar{p}} H_f^{-1} r_f + \omega_c^2 r_f] \, dr + \int_f^O [GM_{\bar{p}} H_f^{-1} z_f] dz\}, -m_{nf}\{\int_f^O [GM_{\bar{n}} H_f^{-1} r_f + \omega_c^2 r_f] dr + \int_f^O [GM_{\bar{n}} H_f^{-1} z_f] dz\}, \quad (2.6\text{-}11)$$

In the following, m_{pf} and m_{nf} are simplified by m_p and m_n respectively.

3. Principle of Minimum Potential Energy (PMPE)

3.1 The potential energy inside the Earth

For point mass m_p and m_n moving from f (r_f, z_f) to O(0,0), the potential energy of the system is shown in (2.6-11). i.e., (3.1-1):

$$\Delta E_p(m_p, m_n) = E_p(O) - E_p(f) = -E_p(f) = -E_p = m_p\{\int_f^O [GM_{\bar{p}} H_f^{-1} r_f + \omega_f^2 r_f] \, dr + \int_f^O [GM_{\bar{p}} H_f^{-1} z_f] dz\}$$

$$-m_n\left\{\int_f^O\left[GM_n^- H_f^{-1} r_f + \omega_c^2 r_f\right]dr + \int_f^O\left[GM_n^- H_f^{-1} z_f\right]dz\right\}$$

(3.1-1)

$$\Delta m_n = \Delta m_n(r_f, z_f) = m_n(r_f, z_f) - m_p(r_f, z_f), \quad (3.1\text{-}2)$$

In SIN zone,

$$m_n(r_f, z_f) < m_p(r_f, z_f), \quad \Delta m_n < 0, \qquad f(r_f, z_f) \in \aleph_s,$$

In BUO zone,

$$m_n(r_f, z_f) > m_p(r_f, z_f), \quad 1 > \Delta m_n > 0, \quad f(r_f, z_f) \in \aleph_b,$$

3.2 PMPE

The PMPE states that the necessary and sufficient conditions of a system in stable equilibrium is its potential energy at minimum.

The actually distributed mantle density must be that which makes the potential energy to be minimum. We focus on necessary condition. Now, PMPE is expressed:

$$\min_{m_p, m_n} -E_p\left(m_p, m_n\right) = \min_{m_p, m_n} -E_p\left[m_p(r_f, z_f), m_n(r_f, z_f)\right] =$$

$$m_p\{\int_f^O\left[GM_p^- H_f^{-1} r_f + \omega_c^2 r_f\right]dr + \int_f^O\left[GM_p^- H_f^{-1} z_f\right]dz\}$$

$$-m_n\{\int_f^O\left[GM_p^- H_f^{-1} r_f + \omega_c^2 r_f\right]dr + \int_f^O\left[GM_p^- H_f^{-1} z_f\right]dz\},$$

(3.2-1)

Where E_p is considered as functions of two independent variables

m_p and m_n

Subject to

$$\int_0^{V_s^-} \rho_p(r_f, z_f)dv_s + \int_0^{V_b^-} \rho_p(r_f, z_f)dV_b = M_p^- \approx M_p,$$

(3.2-2)

$$\int_0^{V_s^-} \rho_n(r_f, z_f)dV_s + \int_0^{V_b^-} \rho_n(r_f, z_f)dV_b = M_n^- \approx M_n,$$

(3.2-3)

$$\int_0^{V_{neu}}\left[\rho_p(r_f, z_f) + \rho_n(r_f, z_f)\right]dV_{neu} = M_{neu} = 0,$$

(3.2-4)

$$M_{mant} = M_p + M_n + M_{neu}, \quad (3.2\text{-}5)$$

Where V_s^-, V_b^-, M_s^-. M_b^-, and M_{mat}^- are the volume and mass group, of SIN zone, BUO zone, NEU zone and the mantle, respectively. where point $f(r_f, z_f)$ has not been included.

Eqs. (3.2-1) --- (3.2-5) form a constraint optimization problem. Using method of Lagrange multipliers to transmit it to un-constraint optimization problem [9]. Constructing a new variable Y:

$$Y = -E_p[m_p, m_n] + K_1\left[\int_0^{V_s^-} \rho_p\, dV_s + \int_0^{V_b^-} \rho_p\, dV_b - M_p^-\right]$$

$$+K_2\left[\int_0^{V_s^-} \rho_n\, dV_s + \int_0^{V_b^-} \rho_n\, dV_b - M_n^-\right]$$

$$+K_3\left[\int_0^{V_{eur}}\left(\rho_p + \rho_n\right)dV_{eur} - M_{eur}\right], \quad (3.2\text{-}6)$$

Where multipliers K_1, K_2, K_3 are independent constants.
The necessary conditions of Y to be minimum are:

$$\frac{\partial Y}{\partial m_p} = \frac{\partial E_p}{\partial m_p} + \frac{K_1}{dv} + \frac{K_2}{dv} + \frac{K_3}{dv} = 0, \quad (3.2\text{-}7)$$

$$\frac{\partial Y}{\partial m_n} = \frac{\partial E_p}{\partial m_n} + \frac{K_1}{dv} + \frac{K_2}{dv} + \frac{K_3}{dv} = 0, \quad (3.2\text{-}8)$$

$$\frac{\partial Y}{\partial K_1} = \int_0^{V_s^-} \rho_p\, dV_s + \int_0^{V_b^-} \rho_p\, dV_b - M_p^- = 0, \quad (3.2\text{-}9)$$

$$\frac{\partial Y}{\partial K_2} = \int_0^{V_s^-} \rho_n\, dV_s + \int_0^{V_b^-} \rho_n\, dV_b - M_n^- = 0, \quad (3.2\text{-}10)$$

$$\frac{\partial Y}{\partial K_3} = \int_0^{V_{neu}}\left(\rho_p + \rho_n\right)dV_{neu} - M_{neu} = 0, \quad (3.2\text{-}11)$$

Substituting (3.2-1) --- (3.2-5) into (3.2-6), subtracting (3.2-8) from (3.2-7), and adding (3.2-7) and (3.2-8), we have:

$$\frac{\partial E_p}{\partial m_p} = -\frac{\partial E_p}{\partial m_n} = 0, \quad (3.2\text{-}12)$$

And $K_1 + K_2 + K_3$ Since K_1, K_2 and K_3 are independent to each other, then, we have $K_1 = K_2 = K_3 = 0$. From (3.2-12), we have

$$\frac{\partial E_p}{\partial m_p} = \frac{\partial E_p}{\partial m_p}\left[\frac{\partial m_p}{\partial r_f}dr_f + \frac{\partial m_p}{\partial z_f}dz_f\right] =$$

$$\int_f^O\{\{\left[GM_p^-\left[H_f^{-1} - 3r_f^2\left(r_f^2 + z_f^2\right)^{1/2}H_f^{-2}\right] + \omega_c^2\right]dr_f + GM_p^-\left[H_f^{-1} -\right.$$

$$\left.3z_f^2\left(r_f^2 + z_f^2\right)^{1/2}H_f^{-2}\right]dz_f\}dr + \int_f^O\{GM_p^-[H_f^{-1} - 3r_f^2\left(r_f^2 + z_f^2\right)^{1/2}H_f^{-2}]$$

$$dr_f + \left\{GM_p^-\left[H_f^{-1} - 3z_f^2\left(r_f^2 + z_f^2\right)^{1/2}H_f^{-2}\right]dz_f\right\}dz = 0, \quad (3.2\text{-}13)$$

$$\frac{\partial E_p}{\partial m_n} = -\int_f^O\{\{\left[GM_n^-\left[H_f^{-1} - 3r_f^2\left(r_f^2 + z_f^2\right)^{1/2}H_f^{-2}\right] + \omega_c^2\right]dr_f +$$

$$GM_n^-\left[H_f^{-1} - 3z_f^2\left(r_f^2 + z_f^2\right)^{1/2}H_f^{-2}\right]dz_f\}\}dr + \int_f^O\{$$

$$\left\{GM_n^-\left[H_f^{-1} - 3r_f^2\left(r_f^2 + z_f^2\right)^{1/2}H_f^{-2}\right]dr_f + GM_n^-[H_f^{-1} - 3z_f^2\left(r_f^2 +\right.\right.$$

$$\left.\left.z_f^2\right)^{1/2}H_f^{-2}]dz_f\right\}\}dz = 0, \quad (3.2\text{-}14)$$

Since f (r_f, z_f) can be arbitrary chosen, by Newton-Leibniz formula, the integrand of (3.2-13) and (3.2-14) must be zero. Then, we have:

$$\left\{\left[GM_p^-\left[H_f^{-1}-3r_f^2\left(r_f^2+z_f^2\right)^{1/2}H_f^{-2}\right]+\omega_c^2\right]dr_f+GM_p^-\left[H_f^{-1}-3z_f^2\left(r_f^2+z_f^2\right)^{1/2}H_f^{-2}\right]dz_f\right\}=0, \quad (3.2\text{-}15)$$

$$\left\{GM_n^-\left[H_f^{-1}-3r_f^2\left(r_f^2+z_f^2\right)^{1/2}H_f^{-2}\right]+\omega_c^2]dr_f+GM_n^-[H_f^{-1}-3z_f^2\left(r_f^2+z_f^2\right)^{1/2}H_f^{-2}]dz_f\right\}=0, \quad (3.2\text{-}16)$$

Since dr_f and dz_f are independently chosen, from (3.2-15), we have

$$\left[H_f^{-1}-3z_f^2\left(r_f^2+z_f^2\right)^{1/2}H_f^{-2}\right]=0, \quad (3.2\text{-}17)$$

and

$$GM_p^-\left[H_f^{-1}-3r_f^2\left(r_f^2+z_f^2\right)^{1/2}H_f^{-2}\right]+\omega_c^2=0, \quad (3.2\text{-}18)$$

Eq. (3.2-17) gives:

$$r_0^2-2z_0^2=0, \quad (3.2\text{-}19)$$

Or

$$\tan\alpha_0=\frac{z_0}{r_0}=\frac{1}{\sqrt{2}}, \quad \alpha_0=35°15', \quad (3.2\text{-}20)$$

Eq. (3.2-18) gives:

$$M_p^-=\frac{\omega_c^2}{G}\frac{H_f^2}{H_f-3r_f^2\left(r_f^2+z_f^2\right)^{1/2}}, \quad (3.2\text{-}21)$$

From (3.2-16), we have the same (3.2-17) and (3.2-18), but M_p^- is replaced by M_n^-. That is

$$M_n^-=\frac{\omega_c^2}{G}\frac{H_f^2}{H_f-3r_f^2\left(r_f^2+z_f^2\right)^{1/2}}, \quad (3.2\text{-}22)$$

All necessary conditions are satisfied by (3.2-19) --- (3.2-22) and where the (r_f, z_f) is denoted by (r_0, z_0).

Substituting (3.2-19) into (3.2-21) and (3.2-22), we have

$$M_n^-=-3\sqrt{3}\frac{\omega_c^2}{G}z_0^3, \quad (3.2\text{-}23)$$

$$M_p^-=3\sqrt{3}\frac{\omega_c^2}{G}z_0^3=M_n^-, \quad (3.2\text{-}24)$$

Now, for a system with minimum potential energy, all necessary conditions shown in (3.2-7) --- (3.2-11) are satisfied by (3.2-19) --- (3.2-24).

3.3 The Determination of the boundary of NEU zone $\aleph_n$

In NEU zone $\aleph_n$, $m_p=m_n$, Eq. (3.2-19) or (3.2-20) must also be satisfied, we have:

$$\tan\alpha_0=z_{neu}/r_{neu}=1/\sqrt{2}, \quad (3.3\text{-}1)$$

Eq. (3.3-1) represents a straight line with apex at O (0,0) inclined angle α_0 = 35°15', revolving around the z-axis by hypotheses 2.

Since by (3.2-6), M_{neu} = 0. and

$$V_{neu}=0, \quad (3.3\text{-}2)$$

Eq (3.3-1) and (3.3-2) show that the boundary of $\aleph_n$ is a cone, apex at O (0,0), with cone angle 2β = 2(90° - 35°15') = 109°30'.

The sufficient conditions of a minimum potential energy of mantle density distribution is obvious and not to discuss here.

3.5 Equations of static mantle density distribution

Summary the above, the equations of static density mantle distribution

$$\int_0^{V_s^-}\rho_{ps}(r_f,z_f)\,dV_s+\int_0^{V_b^-}\rho_{pb}(r_f,z_f)dV_b=M_p^-=3\sqrt{3}\frac{\omega_c^2}{G}z_0^3, \quad (3.5\text{-}1)$$

Subjected to $r_f^2-2z_f^2=0.$ (3.5-2)

$$\int_0^{V_s^-}\rho_{ns}(r_f,z_f)\,dV_s+\int_0^{V_b^-}\rho_{nb}(r_f,z_f)dV_b=M_n^-=3\sqrt{3}\frac{\omega_c^2}{G}z_0^3, \quad (3.5\text{-}3)$$

Subjected to $r_f^2-2z_f^2=0.$

$$M_{mant}=M_p+M_n\approx M_p^-+M_n^-=6\sqrt{3}\frac{\omega_c^2}{G}z_0^3, \quad (3.5\text{-}4)$$

In the above equations, there are four unknown functions ρ_{ps} (r_f, z_f), ρ_{ns} (r_f, z_f), in SIN zone; ρ_{pb} (r_f, z_f), ρ_{nb} (r_f, z_f), in BUO zone, together with three constraints (3.5-1) --- (3.5-4). Complementary condition is needed for solving these equations.

3.6 Complementary equations

3.6.1 Complementary equations 1--- The integration of ρ_{ps} (r_f, z_f), ρ_{ns} (r_f, z_f), ρ_{pb} (r_f, z_f) and ρ_{nb} (r_f, z_f), (3.6.1-1) --- (3.6.1-7)

constant belongs to Fredholm type; while limit being a variable belongs to Volterra type . If limits being to both, then it belongs to Volterra/Fredhom type.

Since (3.5-2) relates two independent variables r_f and z_f, then the equations (3.5-1) , (3.5-3) are reduced to 1-D integral equations with one variable $z_f = z_0$. That is:

Eq. (3.5-1), for positive mass in SIN zone, the is: M_p^- is:

$$2\pi\int_{z_f}^{z_A}\left[\int_{r_f^2}^{r_C^2}\rho_p(r_f,z_f)dr_f^2\right]dz_f=M_{ps}^-, \quad (3.6.1\text{--}1)$$

Where

A (r_A, z_A) is the intersection point of line $r_f=\sqrt{2}z_f$ and boundary

equation (2.1-1) $r_f^2=R_e^2-\left(R_e^2/R_p^2\right)z_f^2;$

$$r_A = \sqrt{2} z_A, \qquad (3.6.1\text{-}2)$$

$$z_A = \frac{R_e R_p}{\sqrt{2R_p^2 + R_e^2}}, \qquad (3.6.1\text{-}3)$$

$f(r_f, z_f)$ is the intersection of horizontal line $z = z_f$ and vertical line $r = r_f$. Where $r_f = r_B + r_d,\ r_B = \sqrt{2} z_B,\ r_f^2 = (r_B + r_d)^2 = 2z_f^2 + 2\sqrt{2} z_f r_d + r_d^2,\ z_B = z_f,\ r_B \le r_d \le r_c$.

$C(r_c, z_c)$ is the intersection point of horizontal line at $z = z_f$ and the boundary equation (2.1-1), where $z_C = z_f$,

$$r_C = \sqrt{R_e^2 - \left(\frac{R_e^2}{R_p^2}\right) z_f^2}, \qquad (3.6.1\text{-}4)$$

The, M_{pb}^- for positive mass, in BUO zone of (3.5-1) is:

$$2\pi\left\{\int_0^{z_A} dz_f \int_{r_f^2}^{r_B^2} \rho_p(r_f, z_f)\, dr_f^2 + \int_{z_f}^{R_p} dz_f \int_0^{r_C^2} \rho_p(r_f, z_f)\, dr_f^2\right\} = M_{pb}^-, \qquad (3.6\text{-}1\text{-}5)$$

Where $B(r_B, z_B)$ is the intersection point of line $r_f = \sqrt{2} z_f$ and vertical line at r_f. Where $r_B = r_f$, $z_B = \frac{r_B}{\sqrt{2}} = z_f$.

Now, Eq. (3.5-1) becomes:

$$2\pi \int_{z_f}^{z_A}\left[\int_{r_f^2}^{r_C^2} \rho_p(r_f, z_f) dr_f^2\right] dz_f \ + 2\pi\left\{\int_0^{z_A} dz_f \int_{r_f^2}^{r_B^2} \rho_p(r_f, z_f)\, dr_f^2 + \int_{z_f}^{R_p} dz_f \int_0^{r_C^2} \rho_p(r_f, z_f)\, dr_f^2\right\} = M_{ps}^- + M_{pb}^- = M_p^- = 6\sqrt{3}\,\frac{\omega_c^2}{G} z_0^3, \qquad (3.6\text{-}1\text{-}6)$$

Eq. (3.5-3), for negative mass, similar to (3.6.1-6), we have:

$$2\pi \int_{z_f}^{z_A}\left[\int_{r_f^2}^{r_C^2} \rho_n(r_f, z_f) dr_f^2\right] dz_f \ + 2\pi\left\{\int_0^{z_A} dz_f \int_{r_f^2}^{r_B^2} \rho_n(r_f, z_f)\, dr_f^2 + \int_{z_f}^{R_p} dz_f \int_0^{r_C^2} \rho_n(r_f, z_f)\, dr_f^2\right\} = M_{ns}^- + M_{nb}^- = M_n^- = 3\sqrt{3}\,\frac{\omega_c^2}{G} z_0^3, \qquad (3.6\text{-}1\text{-}7)$$

3.6.2 Volume of V_s, V_b.

$$V_s = \int_0^{V_s} dV_s = 2\pi \int_0^{z_A} dz_f \int_{r_B^2}^{r_C^2} dr_f^2 = 2\pi\left\{R_e^2 z_A - \frac{1}{3}\left[2 + \frac{R_e^2}{R_p^2}\right] z_A^3\right\}, \qquad (3.6.2\text{-}1)$$

$$V_b = \int_0^{V_b} dV_b = 2\pi\left\{\int_0^{z_A} dz_f \int_0^{r_B^2} dr_f^2 + \int_{z_A}^{R_p} dz_f \int_0^{r_C^2} dr_f^2\right\} = 2\pi\left\{\frac{2}{3} z_A^3 + R_e^2\left[R_p - z_A\right] - \frac{1}{3}\left(\frac{R_e^2}{R_p^2}\right)\left[R_p^3 - z_A^3\right]\right\}, \qquad (3.6.2\text{-}2)$$

3.6.3 Volume of V_s^- and V_b^-

$$V_s^- = 2\pi\left\{\left[\int_0^{z_f^-} dz_f + \int_{z_f^+}^{z_A} dz_f\right]\left[\int_{r_B^2}^{r_f^{2-}} dr_f^2 + \int_{r_f^{2+}}^{r_C^2} dr_f^2\right]\right\}$$

$$= 2\pi \int_{z_f}^{z_A} dz_f \int_{r_f^2}^{r_C^2} dr_f^2 = 2\pi\left[(R_e^2 - r_d^2)(z_A - z_f) - \frac{1}{3}\left(1 + \frac{R_e^2}{R_p^2}\right)\left(z_A^3 - z_f^3\right) - \sqrt{2} r_d\left(z_A^2 - z_f^2\right)\right], \qquad (3.6.3\text{-}1)$$

Where

$$\int_0^{z_f^-} dz_f + \int_{z_f^+}^{z_A} dz_f = \int_{z_f}^{z_A} dz_f, \text{ and } z_f^+ = z_f - z_f^-.$$

$$\int_{r_B^2}^{r_f^{2-}} dr_f^2 + \int_{r_f^{2+}}^{r_C^2} dr_f^2 = \int_{r_f^2}^{r_C^2} dr_f^2, \text{ and } r_f^+ = r_f - r_f^-.$$

point $f(r_f, z_f)$ has not been included in the integral.

Similarly,

$$V_b^- = \int_0^{V_b^-} dV_b = 2\pi\left\{\int_0^{z_A} dz_f \int_0^{r_B^2} dr_f^2 + \int_{z_f}^{R_p} dz_f \int_{r_f^2}^{r_C^2} dr_f^2\right\} = 2\pi\left\{\frac{2}{3}\left(z_A^3 - z_f^3\right) + \left(R_e^2 - r_d^2\right)\left[R_p - z_A\right] - \frac{1}{3}\left(2 + \frac{R_e^2}{R_p^2}\right)\left[R_p^3 - z_f^3\right] - \sqrt{2} r_d\left(R_p^2 - z_f^2\right)\right\}, \qquad (3.6.3\text{-}2)$$

Where point $f(r_f, z_f)$ has not been included in the integral.

4. Method of match-trying

Since integration can be carried only for known functions, therefore one can not integrated an integral equation including unknown function directly. Usually, at first, assuming a given function, then, substituting it to the equation and get the outcome. If the outcome matches to the condition provided by the equation, then, it is the solution; otherwise, using the outcome as a given function, repeating the process, until the outcome is converged (if the iteration is convergence).

Here, we use the method of match-trying, a foolish method, to solve the problem. That is: guess a solution, to see whether it satisfies the provided conditions or not. If it does, then, it is the solution. If not, guess another function, trying again. We have tried many methods. Finally, only the following guess works.

Suppose that: in SIN zone, for positive mass,

$$\rho_{ps}(r_f, z_f) = C_1, \quad (C_1 = \text{const}), \qquad (4.4\text{-}1)$$

Substituting (4.4-1) into (3.6.1-1), we have:

$$2\pi \int_{z_f}^{z_A}\left[\int_{r_f^2}^{r_C^2} C_1\, dr_f^2\right] dz_f = 2\pi C_1 \int_{z_f}^{z_A}\left[R_e^2 - \left(1 + \frac{R_e^2}{R_p^2}\right) z_f^2\right] dz_f =$$

$$2\pi C_1\left[(R_e^2 - r_d^2)(z_A - z_f) - \frac{1}{3}\left(1 + \frac{R_e^2}{R_p^2}\right)\right.$$

$$\left(z_A^3 - z_f^3\right) - \sqrt{2}r_d\left(z_A^2 - z_f^2\right)\Big] = M_{ps}^-, \quad (4.4\text{-}2)$$

Suppose that: in BUO zone, for positive mass,

$\rho_{ps}(r_f, z_f) = C_2$, ($C_2$ = const), (4.4-3)

Substituting (4.4-3) into (3.6.1-5), we have:

$$2\pi C_2\left\{\int_0^{z_A} dz_f \int_0^{r_B^2} dr_f^2 + \int_{z_f}^{R_p} dz_f \int_{r_f^2}^{r_C^2} dr_f^2\right\} =$$

$$2\pi C_2\left\{\frac{2}{3}z_A^3 + \frac{1}{3}\left(1 + \frac{R_e^2}{R_p^2}\right)\left[R_p^3 - z_f^3\right] + \left(R_e^2 - r_d^2\right)\left(R_p - z_f\right) + \sqrt{2}r_d\left(R_p^2 - z_f^2\right)\right\} = M_{pb}^-,$$

Adding (4.4-4) and (4.4-2), we have: (4.4-4)

$$2\pi\left\{\left[\left(R_e^2 - r_d^2\right)(z_A - z_f) - \frac{1}{3}\left(1 + \frac{R_e^2}{R_p^2}\right)\left(z_A^3 - z_f^3\right) - \sqrt{2}r_d\left(z_A^2 - z_f^2\right)\right]C_1 + \left[\frac{2}{3}z_A^3 - \frac{1}{3}\left(1 + \frac{R_e^2}{R_p^2}\right)\left(R_p^3 - z_f^3\right) + \left(R_e^2 - r_d^2\right)\left(R_p - z_f\right) - \sqrt{2}r_d\left(R_p^2 - z_f^2\right)\right]C_2\right\} = M_p^- = 3\sqrt{3}\frac{\omega_c^2}{G}z_0^3, \quad (4.4\text{-}5)$$

Simplifying (4.4-5), we have:

$$A_1 z_f^3 + A_2 z_f^2 + A_3 z_f + A_4 = \frac{3\sqrt{3}}{2\pi}\frac{\omega_c^2}{G}z_0^3, \quad (4.4\text{-}6)$$

$$A_1 = \frac{1}{3}\left(1 + \frac{R_e^2}{R_p^2}\right)(C_1 + C_2), \quad (4.4\text{-}7)$$

$$A_2 = \sqrt{2}r_d(C_1 + C_2), \quad (4.4\text{-}8)$$

$$A_3 = -\left(R_e^2 - r_d^2\right)(C_1 + C_2), \quad (4.4\text{-}9)$$

$$A_4 = \left(R_e^2 - r_d^2\right)\left(z_A C_1 + R_p C_2\right) - \frac{1}{3}\left[\left(1 + \frac{R_e^2}{R_p^2}\right)\left(z_A^3 C_1 + R_p^3 C_2\right) - \sqrt{2}r_d\left(z_A^2 C_1 + R_p^2 C_2\right), \quad (4.4\text{-}10)$$

Similarly, for negative mass,, we have:

$$\rho_{ns}(r_f, z_f) = C_3, \quad (C_3 = \text{const}) \quad (4.4\text{-}11)$$

$$\rho_{nb}(r_f, z_f) = C_4, \quad (C_4 = \text{const}) \quad (4.4\text{-}12)$$

$$2\pi\left\{\left[\left(R_e^2 - r_d^2\right)(z_A - z_f) - \frac{1}{3}\left(1 + \frac{R_e^2}{R_p^2}\right)\left(z_A^3 - z_f^3\right) - \sqrt{2}r_d\left(z_A^2 - z_f^2\right)\right]C_3 + \left[\frac{2}{3}z_A^3 + \frac{1}{3}\left(1 + \frac{R_e^2}{R_p^2}\right)\left(R_p^3 - z_f^3\right) + \left(R_e^2 - r_d^2\right)\left(R_p - z_f\right) - \sqrt{2}r_d\left(R_p^2 - z_f^2\right)\right]C_4\right\} = M_n^- = -3\sqrt{3}\frac{\omega_c^2}{G}z_0^3, \quad (4.4\text{-}13)$$

Simplifying (4.4-13), we have:

$$A_5 z_f^3 + A_6 z_f^2 + A_7 z_f + A_8 = \frac{3\sqrt{3}}{2\pi}\frac{\omega_c^2}{G}z_0^3, \quad (4.4\text{-}14)$$

$$A_5 = \frac{1}{3}\left(1 + \frac{R_e^2}{R_p^2}\right)(C_3 + C_4), \quad (4.4\text{-}15)$$

$$A_6 = \sqrt{2}r_d(C_3 + C_4), \quad (4.4\text{-}16)$$

$$A_7 = -\left(R_e^2 - r_d^2\right)(C_3 + C_4), \quad (4.4\text{-}17)$$

$$A_8 = \left(R_e^2 - r_d^2\right)\left(z_A C_3 + R_p C_4\right) - \frac{1}{3}\left[\left(1 + \frac{R_e^2}{R_p^2}\right)\left(z_A^3 C_3 + R_p^3 C_4\right) - \sqrt{2}r_d\left(z_A^2 C_3 + R_p^2 C_4\right), \quad (4.4\text{-}18)$$

Substituting (4.4-5), (4.4-13) into (3.5-4), we have:

$$M_{mant} = (A_1 + A_5)z_f^3 + (A_2 + A_6)z_f^2 + (A_3 + A_7)z_f + (A_4 + A_8) = \frac{6\sqrt{3}}{2\pi}\frac{\omega_c^2}{G}z_0^3, \quad (4.4\text{-}19)$$

We need complementary boundary conditions for solving unknowns.

4.5 Complementary equations 2--- The boundary conditions

The boundary condition in SIN zone, is:

$\rho_n(r_b, z_b) = 0$, and $\rho_p\ (r_b, z_b) \neq 0$, (4.5-1)

where $\rho_n(r_b, z_b) = 0$ means no negative mass exists, while positive mass exists. The bounder is dividing positive mass and negative mass from mixture of $\rho_p(r_f, z_f)$ and $\rho_n(r_f, z_f)$ in SIN zone,

At the bounder, the forces acting on positive mass $\rho_p(r_b, z_b)$ and on negative mass $\rho_n(r_b, z_b)$ must be in equilibrium. That is:

$$\sum F_r = F_{rp} + F_{rn} = 0, \quad (4.5\text{-}2)$$

$$\sum F_z = F_{zp} + F_{zn} = 0, \quad (4.5\text{-}3)$$

$$F_{rp} + F_{rn} = \left\{m_p\left[GM_p^- H_b^{-1} + \omega_c^2\right]\right\}r_b - \left\{m_n\left[GM_n^- H_b^{-1} + \omega_c^2\right]\right\}r_b$$

$$= \left\{m_p\left[GM_p^- H_b^{-1} + \omega_c^2\right] - m_n\left[GM_n^- H_b^{-1} + \omega_c^2\right]\right\}r_b = 0, \quad (4.5\text{-}3)$$

$$F_{zp} + F_{zn} = \left\{\left[Gm_p M_p^- H_b^{-1}\right] - \left[Gm_n M_n^- H_b^{-1}\right]\right\}z_b = 0, \quad (4.5\text{-}4)$$

From (4.5-3), we have:

$r_b = 0$, (4.5-5)

Or $\left\{m_n\left[GM_n^- H_b^{-1} + \omega_c^2\right]\right\} = 0$, ($m_p = 0$, $m_n \neq 0$), i.e.,

$$H_b = \left(r_b^2 + z_b^2\right)^{3/2} = -\frac{\omega_c^2}{GM_p^-} = \frac{1}{3\sqrt{3}}z_0^3 \quad (4.5\text{-}6)$$

From (4.5-4), we have:

$z_b = 0$, (4.5-7)

Or $\left\{\left[Gm_p M_p^- H_b^{-1}\right] - \left[Gm_n M_n^- H_b^{-1}\right]\right\} = 0$, ($m_p = 0, m_n \neq 0$), i.e.,

$H_b^{-1} = 0$, i. e., $H_b \to \infty$, (4.5-8)

Since (4.5-6) conflicts to (4.5-8), therefore, the only choice is (4.5-5) and (4.5-7). They can satisfy both requirements of (4.5-3) and (4.5-4).

$$\tan\alpha_b = z_b/r_b = 0/0 = 1/\sqrt{2}, \quad \alpha_b = 35°15', \quad (4.5\text{-}9)$$

This means that points on inclined line $r_b = \sqrt{2}z_b$ do not exist negative mass, while positive mass exists.

The boundary condition in BUO zone is:

$\rho_p(r_a, z_a) = 0$, and $\rho_n\ (r_a, z_a) \neq 0$, (4.5-10)

where no positive mass exists, while negative mass exists. Similarly as calculation as above, we have:

$$H_a = (r_a^2 + z_a^2)^{3/2} = \frac{1}{3\sqrt{3}}z_0^3, \quad (4.5\text{-}11)$$

$$z_a = 0, \quad (4.5\text{-}12)$$

$$H_a = (r_a^2 + 0)^{3/2} = r_a^3 = \frac{1}{3\sqrt{3}}z_0^3, \quad (4.5\text{-}13)$$

$$\tan\alpha_a = z_a/r_a = 0, \quad \alpha_a = \pi/2, \quad (4.5\text{-}14)$$

This means that points on vertical line, the z-axis, do not exist positive mass, while negative mass exists.

4.6 Determination of four constants C_1, C_2, C_3, C_4.

Now, we can use the boundary conditions to final determine constants

$\rho_{ps}(r_f, z_f) = C_1$, $\rho_{pb}(r_f, z_f) = C_2$, $\rho_{ns}(r_f, z_f) = C_3$, $\rho_{nb}(r_f, z_f) = C_4$.

By (4.5-9), we have

$$\rho_{ns}(r_f, z_f) = C_3 = 0, \quad (0 < \alpha_b < 35°15'), \quad (4.6\text{-}1)$$

By (4.5-11), we have

$$\rho_{pb}(r_f, z_f) = C_2 = 0, \quad (35°15' < \alpha_a < \tfrac{\pi}{2}), \quad (4.6\text{-}2)$$

Now, remainders C_1, C_4 need to be determined.

Rewritten (4.4-8), hidden the right hand side $\frac{3\sqrt{3}}{\pi}\frac{\omega_c^2}{G}z_0^3$

temperately,

$$A_1 z_f^3 + A_2 z_f^2 + A_3 z_f + A_4 = 0, \quad (4.6\text{-}3)$$

(4.6-3) is a three order algebraic equation. It can be solved by Cardan formula [9]. Suppose that the real solution is z_{real1} (at least one real solution). Then

$$C_1 z_{real1}^3 = \frac{3\sqrt{3}}{2\pi}\frac{\omega_c^2}{G}z_0^3, \quad (4.6\text{-}4)$$

C_4 is determined by (4.6-5).

Now, all four unknowns $\rho_{ps}(r_f, z_f) = C_1$, $\rho_{pb}(r_f, z_f) = C_2$, $\rho_{ns}(r_f, z_f) = C_3$, $\rho_{nb}(r_f, z_f) = C_4$ have been calculated.

5. Discussion

What is the difference between M_p and M_p^-, or V_s and V_S^-?

Answer: $V_s^- = V_s - f(r_f, z_f)$, $M_p^- = \int_0^{V_{s+b}^-} \rho_p\, dV_{s+b}^- = M_p - f(r_f, z_f)$.

That is: $V\bar{s}$ is equal to V_s minus a point f (r_f, z_f); M_p^- equals M_p minus a point f (r_f, z_f).

Why we need to introduce the concept of M_p^- and V_s^-?

Answer: Although the difference between M_p^- and M_p, V_s^- or and V_S is small but introducing the concept of M_p^- and V_s^- is very important. If the left hand side of equation (3.5-1) is:

$$\int_0^{V_s} \rho_p(r_f, z_f)\, dV_s + \int_0^{V_b} \rho_p(r_f, z_f)\, dV_b = M_p = 3\sqrt{3}\frac{\omega_c^2}{G}z_0^3, \quad (5\text{-}1)$$

then, the left hand side is a fixed value, because all the limits of the two integrals are fixed, the equation is a Fredholm equation, so that the left hand side is a constant, no matter what $\rho_p(r_f, z_f)$ is. But the right hand side is a variable z_0. That means (5-1) is unable hold. Or (5-1) has no solution.

Therefore, the outcome of left hand side must be a variable, if we hope that (5-1) holds. So that we introduce M_p^- and V_s^-, in order to change some of the limit to be variable. This can be achieved by moving out the point f (r_f, z_f) from the integration (this can be seen e.g., from comparing (4.2-1) with (4.3-1), where V_s is a constant, while V_s^- is a variable). Thus, the equation becomes to a Fredlom/Volterra type and it can have solution.

6 Results and conclusion

Using Archimedes Principle of Sink or Buoyancy (APSB), Newton's universal gravity, buoyancy, lateral buoyancy, centrifugal force and Principle of Minimum Potential Energy (PMPE), this paper improves the derivation of equation of static mantle density distribution. It is a set of mutual related 2-D integral equations of Volterra/Fredholm type.

Using method of match-trying, we find the solution by assumption of constant density distribution. Then, the integration can be carried out and translated the integral equation to algebraic equation and solved by Cardan formula.

Some new results are: (1) The mantle is divided into sink zone, neural zone and buoyed zone. The sink zone is located in a region with boundaries of a inclined line, angle $\alpha_o = 35°\ 15'$, with apex at O(0,0) revolving around the z-axis, inside the crust involving the equator. Where no negative mass exists, while positive mass is uniformly distributed. The buoyed zone is located in the remainder part, inside the crust involving poles. Where no positive mass exists, while negative mass is uniformly distributed. The neural zone is the boundary between the buoyed and sink zones. The shape of core (in sink zone) is not a sphere. (2) The total positive mass is equal to the total negative mass. (3) The volume of BUO zone is near twice the volume of SIN zone.(4) No positive mass exists in an imagine tunnel passing through poles (the z-axis).

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