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Research Article

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Stability Analysis of a System of Stochastic Difference Equations with Exponential Nonlinearity

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Abstract
A system of two difference equations with exponential nonlinearity in each equation is studied under stochastic perturbations. Conditions of the stability in probability of a positive equilibrium are studied by virtue of the general method of Lyapunov functionals construction and the method of linear matrix inequalities (LMIs). The obtained results are illustrated via examples and figures with numerical simulation of the solution of the system of stochastic difference equations. The proposed research method can be applied to nonlinear systems of higher dimension with an order of nonlinearity higher than one, both for stochastic difference equations and for stochastic differential equations with delay in various important applications, for example, in quantum physics, in population models and others.

Keywords: Nonlinear Difference Equations, Positive Equilibrium, Stochastic Perturbations, Asymptotic Mean Square Stability, Stability in Probability, Linear Matrix Inequality (LMI), Numerical Simulations, MATLAB
MSC: 39A30; 39A50.

1. Introduction

Systems of both difference and differential equations with different forms of exponential nonlinearities are very popular in research and various applications (see, for instance, [1–17] and references therein), in particular, the model from quantum physics [5], the model of Nicholson's blowflies [2] or Mosquito population equation [12].

Here, similarly to [14], the stability of the positive equilibrium of a system with exponential nonlinearity is investigated under stochastic perturbations via the general method of Lyapunov functionals construction [16,18–20] and the method of linear matrix inequalities (LMIs) [21–29]. However, unlike, for instance, [7,14], where the exponential nonlinearity in each equation depends on only one variable, here each equation exponentially depends on all variables of the system under consideration. The obtained results are illustrated via examples and figures with the equilibrium and numerical simulation of the solution of the considered system of difference equations. Numerical analysis of the considered LMIs is carried out using MATLAB.

Consider the system of two nonlinear difference equations

$$\begin{aligned} x_1(n+1) &= a_1 + b_1 x_1(n-1) + c_1 x_1(n-1) e^{-p_1 x_1(n) - q_1 x_2(n)}, \\ x_2(n+1) &= a_2 + b_2 x_2(n-1) + c_2 x_2(n-1) e^{-p_2 x_1(n) - q_2 x_2(n)}, \\ & n \in N = \{0, 1, \dots\}, \end{aligned} \tag{1}$$

with positive parameters, $b_i < 1$, and positive initial conditions $x_i(j) = \phi_i(j)$, $i = 1, 2, j \in N_0 = \{-1, 0\}$.

1.1 Equilibrium

It is clear that the equilibrium (x_1, x_2) of the system (1) is defined by the system of two algebraic equations

$$x_1 = a_1 + b_1 x_1 + c_1 x_1 e^{-p_1 x_1 - q_1 x_2},$$
$$x_2 = a_2 + b_2 x_2 + c_2 x_2 e^{-p_2 x_1 - q_2 x_2}. \tag{2}$$

Presenting the first equation (2) in the form

$$e^{q_1 x_2}\left(1 - b_1 - \frac{a_1}{x_1}\right) = c_1 e^{-p_1 x_1} \tag{3}$$

and calculating the logarithm, we get

$$x_2 = f_1(x_1) := \frac{1}{q_1}\left(\ln c_1 - p_1 x_1 - \ln\left(1 - b_1 - \frac{a_1}{x_1}\right)\right), \quad x_1 > \frac{a_1}{1 - b_1}. \tag{4}$$

Similarly, from the second equation (2) we have

$$e^{p_2 x_1}\left(1 - b_2 - \frac{a_2}{x_2}\right) = c_2 e^{-q_2 x_2} \tag{5}$$

and

$$x_1 = \frac{1}{p_2}\left(\ln c_2 - q_2 x_2 - \ln\left(1 - b_2 - \frac{a_2}{x_2}\right)\right), \quad x_2 > \frac{a_2}{1 - b_2}. \tag{6}$$

It is clear that the function $x_2 = f_1(x_1)$ given by (4) is defined and positive if $x_1 \in (x_{1\,\min}, x_{1\,\max})$, where $x_{1\,\min} = \frac{a_1}{1 - b_1}$ and $x_{1\,\max}$ is a unique root of the equation

$$1 - b_1 - \frac{a_1}{x_1} = c_1 e^{-p_1 x_1}, \tag{7}$$

which follows from (3) by $x_2 = 0$.

Calculating the derivative in (4)

$$x_2' = \frac{1}{q_1}\left(-p_1 - \frac{a_1}{((1 - b_1)x_1 - a_1)x_1}\right) < 0,$$

it is easy to see that $x_2 = f_1(x_1)$ is strictly decreasing function. Moreover, $\lim_{x_1 \to x_{1\,\min}} f_1(x_1) = +\infty$ and $f_1(x_{1\,\max}) = 0$.

Calculating the derivative of the function $x_2 = f_2(x_1)$, defined implicitly by (6), we have

$$p_2 = -\left(q_2 + \frac{a_2}{((1 - b_2)x_2 - a_2)x_2}\right) x_2' > 0,$$

i.e., $x'_2 < 0$. It means that $x_2 = f_2(x_1)$ is strictly decreasing function for $x_1 \geq 0$. Moreover, $\lim_{x_1 \to \infty} f_2(x_1) = \frac{a_2}{1 - b_2}$ and $x_2 = f_2(0)$ is a unique root of the equation

$$1 - b_2 - \frac{a_2}{x_2} = c_2 e^{-q_2 x_2}, \tag{8}$$

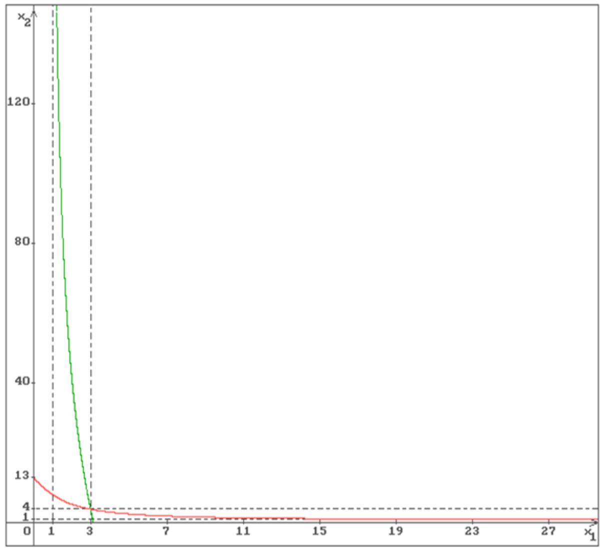

Figure 1: The Graphs of the Functions $x_2 = f_1(x_1)$ (green) and $x_2 = f_2(x_1)$ (red)

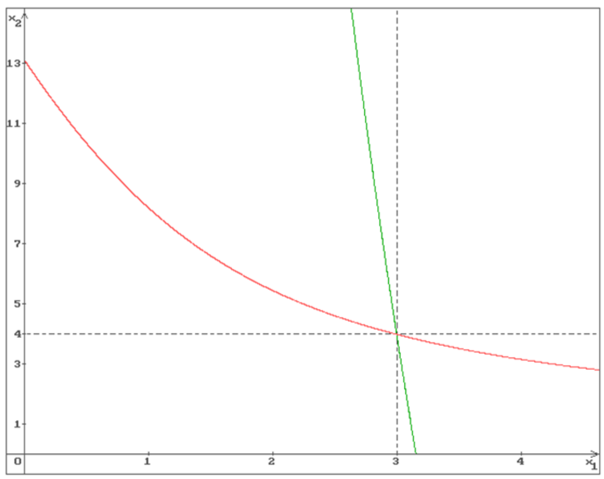

Figure 2: The Intersection Point of the Graphs of the Functions $x_2 = f_1(x_1)$ (green) and $x_2 = f_2(x_1)$ (red) is the Equilibrium (x_1^*, x_2^*) = (3,4)

which follows from (5) by $x_1 = 0$. It is easy to see that the root x_2 of this equation satisfies the condition $x_2 > \dfrac{a_2}{1-b_2}$.

It is clear that two strictly decreasing functions $x_2 = f_1(x_1)$ and $x_2 = f_2(x_1)$ have (see Figure 1 and Figure 2) one common point, which is a solution of the system (2) and is the unique equilibrium (x_1^*, x_2^*) of the system (1).

Remark 1.1 *The equilibrium* (x_1^*, x_2^*) *of the system* (1) *satisfies the conditions*

$$x_1^* \in \left(\frac{a_1}{1-b_1}, x_{1\,\max}\right), \quad x_2^* \in \left(\frac{a_2}{1-b_2}, x_{2\,\max}\right), \tag{9}$$

where $x_{1\,\max}$ and $x_{2\,\max}$ *are roots of the equations* (7) *and* (8) *respectively.*

Example 1.1 *Consider the system* (1) *with*

$$a_1 = 0.3, \quad a_2 = 0.4, \quad b_1 = 0.7, \quad b_2 = 0.6,$$
$$c_1 = 0.2810, \quad c_2 = 0.4215, \quad p_1 = p_2 = 0.1, \quad q_1 = q_2 = 0.01.$$

Then the solution of the system (2) is $(x_1^*, x_2^*) = (3,4)$, from (9) and (7), (8) it follows that $x_{1\min} = 1$, $x_{2\min} = 1$, $x_{1\max} = 3.156$, $x_{2\max} = 13.147$. In Figures 1 and 2 the graphs of the functions $x_2 = f_1(x_1)$ (green), $x_2 = f_2(x_1)$ (red) and the equilibrium (x_1^*, x_2^*) are shown.

In Figure 1 also the asymptotes $x_1 = \frac{a_1}{1-b_1} = 1$ and $x_2 = \frac{a_2}{1-b_2} = 1$ of the functions $f_1(x_1)$ and $f_2(x_1)$ respectively are shown.

2. Stochastic Perturbations and the System Transformation

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a basic probability space, $\mathfrak{F}_n \in \mathfrak{F}$, $n \in N$, be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, i.e., $\mathfrak{F}_{n_1} \subset \mathfrak{F}_{n_2}$ for $n_1 < n_2$, $\mathbf{E}$ be the mathematical expectation with respect to the measure $\mathbf{P}$, $\xi_1(n)$ and $\xi_2(n)$, $n \in N$, be two mutually independent sequences with $\mathfrak{F}_n$-adapted mutually independent random values such that [16]

$$\begin{gathered} \mathbf{E}\xi_i(n) = 0, \quad \mathbf{E}\xi_i^2(n) = 1, \\ \mathbf{E}\xi_i(n)\xi_j(m) = 0 \quad \text{if} \quad i \neq j \quad \text{or} \quad n \neq m, \quad i,j = 1,2. \end{gathered} \tag{10}$$

Let us assume that the system (1) is exposed to stochastic perturbations that are directly proportional to the deviation of the system state $(x_1(n), x_2(n))$ from the equilibrium (x_1^*, x_2^*). Then the system (1) takes the form

$$\begin{aligned} x_1(n+1) &= a_1 + b_1 x_1(n-1) + c_1 x_1(n-1) e^{-p_1 x_1(n) - q_1 x_2(n)} \\ &\quad + \sigma_1(x_1(n) - x_1^*)\xi_1(n+1), \\ x_2(n+1) &= a_2 + b_2 x_2(n-1) + c_2 x_2(n-1) e^{-p_2 x_1(n) - q_2 x_2(n)} \\ &\quad + \sigma_2(x_2(n) - x_2^*)\xi_2(n+1). \end{aligned} \tag{11}$$

Remark 2.1 Note that the such type of stochastic perturbations was firstly proposed in [30] for a system of Ito's stochastic delay differential equations and was later used in many other works for both differential and difference equations (see, for instance, [16,20] and references therein). With this type of stochastic perturbations, the equilibrium of the original deterministic system remains also a solution of the stochastically perturbed system.

Presenting the solution of the system (11) in the form $(x_1(n), x_2(n)) = (y_1(n) + x_1^*, y_2(n) + x_2^*)$, we get

$$\begin{aligned} y_1(n+1) + x_1^* &= a_1 + b_1(y_1(n-1) + x_1^*) \\ &\quad + c_1(y_1(n-1) + x_1^*) e^{-p_1 y_1(n) - q_1 y_2(n)} e^{-p_1 x_1^* - q_1 x_2^*} \\ &\quad + \sigma_1 y_1(n)\xi_1(n+1), \\ y_2(n+1) + x_2^* &= a_2 + b_2(y_2(n-1) + x_2^*) \\ &\quad + c_2(y_2(n-1) + x_2^*) e^{-p_2 y_1(n) - q_2 y_2(n)} e^{-p_2 x_1^* - q_2 x_2^*} \\ &\quad + \sigma_2 y_2(n)\xi_2(n+1). \end{aligned} \tag{12}$$

Using (2) with $(x_1, x_2) = (x_1^*, x_2^*)$, from the first equation (12) we have

$$\begin{aligned} y_1(n+1) &= a_1 + b_1 x_1^* - x_1^* + b_1 y_1(n-1) \\ &\quad + c_1(y_1(n-1) + x_1^*) e^{-p_1 y_1(n) - q_1 y_2(n)} e^{-p_1 x_1^* - q_1 x_2^*} \\ &\quad + \sigma_1 y_1(n)\xi_1(n+1) \\ &= b_1 y_1(n-1) - c_1 x_1^* e^{-p_1 x_1^* - q_1 x_2^*} \\ &\quad + c_1(y_1(n-1) + x_1^*) e^{-p_1 y_1(n) - q_1 y_2(n)} e^{-p_1 x_1^* - q_1 x_2^*} \\ &\quad + \sigma_1 y_1(n)\xi_1(n+1) \\ &= \left(b_1 + c_1 e^{-p_1 x_1^* - q_1 x_2^*} e^{-p_1 y_1(n) - q_1 y_2(n)}\right) y_1(n-1) \\ &\quad - c_1 x_1^* e^{-p_1 x_1^* - q_1 x_2^*} \left(1 - e^{-p_1 y_1(n) - q_1 y_2(n)}\right) \\ &\quad + \sigma_1 y_1(n)\xi_1(n+1). \end{aligned}$$

Similarly, for the second equation (12) we get

$$
\begin{aligned}
y_2(n+1) &= a_2 + b_2 x_2^* - x_2^* + b_2 y_2(n-1) \\
&\quad + c_2(y_2(n-1) + x_2^*) e^{-p_2 y_1(n) - q_2 y_2(n)} e^{-p_2 x_1^* - q_2 x_2^*} \\
&\quad + \sigma_2 y_2(n) \xi_2(n+1) \\
&= b_2 y_2(n-1) - c_2 y^* e^{-p_2 x_1^* - q_2 x_2^*} \\
&\quad + c_2(y_2(n-1) + x_2^*) e^{-p_2 y_1(n) - q_2 y_2(n)} e^{-p_2 x_1^* - q_2 x_2^*} \\
&\quad + \sigma_2 y_2(n) \xi_2(n+1) \\
&= \left(b_2 + c_2 e^{-p_2 x_1^* - q_2 x_2^*} e^{-p_2 y_1(n) - q_2 y_2(n)}\right) y_2(n-1) \\
&\quad - c_2 x_2^* e^{-p_2 x_1^* - q_2 x_2^*} \left(1 - e^{-p_2 y_1(n) - q_2 y_2(n)}\right) \\
&\quad + \sigma_2 y_2(n) \xi_2(n+1).
\end{aligned}
$$

As a result we obtain the nonlinear system with the zero solution:

$$
\begin{aligned}
y_1(n+1) &= \left(b_1 + c_1 e^{-p_1 x_1^* - q_1 x_2^*} e^{-p_1 y_1(n) - q_1 y_2(n)}\right) y_1(n-1) \\
&\quad - c_1 x_1^* e^{-p_1 x_1^* - q_1 x_2^*} \left(1 - e^{-p_1 y_1(n) - q_1 y_2(n)}\right) \\
&\quad + \sigma_1 y_1(n) \xi_1(n+1), \\
y_2(n+1) &= \left(b_2 + c_2 e^{-p_2 x_1^* - q_2 x_2^*} e^{-p_2 y_1(n) - q_2 y_2(n)}\right) y_2(n-1) \\
&\quad - c_2 x_2^* e^{-p_2 x_1^* - q_2 x_2^*} \left(1 - e^{-p_2 y_1(n) - q_2 y_2(n)}\right) \\
&\quad + \sigma_2 y_2(n) \xi_2(n+1),
\end{aligned}
\tag{13}
$$

Remark 2.2 *Note that stability of the zero solution of the system* (13) *is equivalent to stability of the equilibrium* (x_1^*, x_2^*) of the system (11),

Using (2) and the linear approximation $e^{-x} = 1 - x + o(x)$, where $\lim_{x \to 0} \frac{o(x)}{x} = 0$, we obtain the linear part of the system (13)

$$
\begin{aligned}
z_1(n+1) &= \left(1 - \frac{a_1}{x_1^*}\right) z_1(n-1) - ((1-b_1)x_1^* - a_1)(p_1 z_1(n) + q_1 z_2(n)) \\
&\quad + \sigma_1 z_1(n) \xi_1(n+1), \\
z_2(n+1) &= \left(1 - \frac{a_2}{x_2^*}\right) z_2(n-1) - ((1-b_2)x_2^* - a_2)(p_2 z_1(n) + q_2 z_2(n)) \\
&\quad + \sigma_2 z_2(n) \xi_2(n+1).
\end{aligned}
\tag{14}
$$

Representing the linear system (14) in the matrix form, we get

$$
z(n+1) = -Az(n) + Bz(n-1) + \sum_{i=1}^{2} C_i z(n) \xi_i(n+1), \tag{15}
$$

where

$$
\begin{gathered}
z(n) = \begin{bmatrix} z_1(n) \\ z_2(n) \end{bmatrix}, \quad A = \begin{bmatrix} \alpha_1 p_1 & \alpha_1 q_1 \\ \alpha_2 p_2 & \alpha_2 q_2 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}, \\
C_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \\
\alpha_i = (1-b_i)x_i^* - a_i, \quad \beta_i = 1 - \frac{a_i}{x_i^*}, \quad i - 1, 2.
\end{gathered}
\tag{16}
$$

3. Stability

3.1 Some Necessary Definitions and Statements

Let $'$ be the transposition sign. Put now

$$y(n) = (y_1(n), y_2(n))', \quad z(n) = (z_1(n), z_2(n))', \quad n \in N,$$
$$\phi(j) = (\phi_1(j), \phi_2(j))', \quad j \in N_0.$$

Definition 3.1 ([16]). *The zero solution of the system* (13) *is called stable in probability if for any* ε > 0 and ε_1 ϵ (0, 1) there exists *a* δ > 0 *such that the solution* $y(n) = y(n, \phi)$ *of the system* (13) *satisfies the inequality* $\mathbf{P}\{\sup_{n\in N} |y(n)| > \varepsilon\} < \varepsilon_1$ *for any initial function* $\phi(j)$ *such that* $\mathbf{P}\{\|\phi\|_0 < \delta\} = 1$, *where* $\|\phi\|_0 = \max_{j\in N_0} |\phi(j)|$.

Definition 3.2 ([16])**.** *The zero solution of the system* (14) *is called mean square stable if for each ε > 0 there exists a δ >* 0 *such that* $\mathbf{E}|z(n)|^2 < \varepsilon$ *, n ϵ N, for any initial function ϕ(j) such that* $\|\phi\|^2 = \max_{j\in N_0} \mathbf{E}|\phi(j)|^2 < \delta$*; asymptotically mean square stable if it is mean square stable and for each initial function ϕ(j) such that* $\|\phi\|^2 < \infty$ *the solution z(n) of the system* (14) *satisfies the condition* $\lim_{n\to\infty} \mathbf{E}|z(n)|^2 = 0$.

Let $\mathbf{E}_n = \mathbf{E}\{./\mathfrak{F}_n\}$ be the conditional expectation with respect to the σ-algebra $\mathfrak{F}_n$, $U_\varepsilon = \{y : |y| \le \varepsilon\}$, $\varepsilon > 0$, and $\Delta V(n) = V(n+1) - V(n)$.

Theorem 3.1 ([16]). *Let for the system* (13) *there exists a functional* $V(n) = V(n, y(-1), ..., y(n))$ *satisfying the conditions*

$$\begin{aligned} &V(n, y(-1), ..., y(n)) \ge c_0|y(n)|^2, \\ &V(0, \varphi(-1), \varphi(0)) \le c_1\|\varphi\|_0^2, \\ &\mathbf{E}_n\Delta V(n, y(-1), ..., y(n)) \le 0, \quad y(j) \in U_\varepsilon, \quad -1 \le j \le n, \quad n \in N, \end{aligned} \tag{17}$$

where ε > 0, $c_0 > 0$, $c_1 > 0$. *Then the zero solution of the system* (13) *is stable in probability*.

Theorem 3.2 ([16]). *Let for the system* (14) *there exists a nonnegative functional* $V(n) = V(n, z(-1),..., z(n))$ *satisfying the conditions*

$$\begin{aligned} &\mathbf{E}V(0, \phi(-1), \phi(0)) \le c_1\|\phi\|^2, \\ &\mathbf{E}\Delta V(n) \le -c_2\mathbf{E}|z(n)|^2, \quad n \in N, \end{aligned} \tag{18}$$

where $c_1 > 0$, $c_2 > 0$. *Then the zero solution of the system* (14) *is asymptotically mean square stable*.

Remark 3.1 *Note that the system* (13) *has an order of nonlinearity higher than one. It is known* [16] *that in this case sufficient conditions for asymptotic mean square stability of the zero solution of the linear system* (14) *are also sufficient conditions for stability in probability of the zero solution of the nonlinear system* (13).

3.2 Stability Conditions

Theorem 3.3 Let there exist positive definite 2×2-matrices P and R such that the following linear matrix inequality (LMI)

$$\begin{bmatrix} A'PA + S_0 + R - P & A'PB \\ B'PA & B'PB - R \end{bmatrix} < 0 \tag{19}$$

holds, where

$$S_0 = \sum_{i=1}^{2} C_i'PC_i = \begin{bmatrix} \sigma_1^2 p_{11} & 0 \\ 0 & \sigma_2^2 p_{22} \end{bmatrix}, \tag{20}$$

the matrices C_1, C_2 *are defined in* (16) *and* p_{11} , p_{22} *are diagonal elements of the matrix P. Then the equilibrium* (x_1^*, x_2^*) *of the system* (11) *is stable in probability*.

Proof : Following the general method of Lyapunov functionals construction [16,18–20], consider the functional $V(n)$ in the form $V(n) = V_1(n) + V_2(n)$, where $V_1(n) = z'(n)Pz(n)$, $P > 0$, $z(n)$ is defined in (16) and the additional functional $V_2(n)$ will be chosen below. For the functional $V_1(n)$ via (15) we have

$$\begin{aligned}\mathbf{E}\Delta V_1(n) =&\mathbf{E}[V_1(n+1)-V_1(n)]\\ =&\mathbf{E}[z'(n+1)Pz(n+1)-z'(n)Pz(n)]\\ =&\mathbf{E}\Big[\Big(-z'(n)A'+z'(n-1)B'+\sum_{i=1}^{2}z'(n)C_i'\xi_i(n+1)\Big)\\ &*P\Big(-Az(n)+Bz(n-1)+\sum_{i=1}^{2}C_iz(n)\xi_i(n+1)\Big)-z'(n)Pz(n)\Big].\end{aligned}$$

From here via (10) and (20) it follows that

$$\begin{aligned}\mathbf{E}\Delta V_1(n) =&\mathbf{E}[z'(n)(A'PA+S_0-P)z(n)+z'(n)A'PBz(n-1)\\ &+z'(n-1)B'PAz(n)+z'(n-1)B'PBz(n-1)]\end{aligned}$$

or in the matrix form

$$\mathbf{E}\Delta V_1(n)=\mathbf{E}\begin{bmatrix}z(n)\\ z(n-1)\end{bmatrix}'\begin{bmatrix}A'PA+S_0-P & A'PB\\ B'PA & B'PB\end{bmatrix}\begin{bmatrix}z(n)\\ z(n-1)\end{bmatrix}. \tag{21}$$

Using the additional functional $V_2(n) = z'(n-1)Rz(n-1)$, $R > 0$, with

$$\Delta V_2(n)=z'(n)Rz(n)-z'(n-1)Rz(n-1),$$

for the functional $V(n) = V_1(n) + V_2(n)$ from (21) we obtain

$$\mathbf{E}\Delta V(n)=\mathbf{E}\begin{bmatrix}z(n)\\ z(n-1)\end{bmatrix}'\begin{bmatrix}A'PA+S_0+R-P & A'PB\\ B'PA & B'PB-R\end{bmatrix}\begin{bmatrix}z(n)\\ z(n-1)\end{bmatrix}. \tag{22}$$

From (22) and the LMI (19) for some c > 0 we have $\mathbf{E}\Delta V(n)\le -c\mathbf{E}|z(n)|^2$, i.e., the constructed functional $V(n)$ satisfies the conditions of Theorem 3.2. Therefore, the zero solution of the linear equation (15) is asymptotically mean square stable. Via Remarks 3.1 and 2.2 it means that the equilibrium (x_1^*, x_2^*) of the system (11) is stable in probability. The proof is completed.

Remark 3.2 Note that instead of the LMI (19) for definition of stability some other LMIs also can be used. Using, for instance, the additional functional $V_2(n)$ in the form $V_2(n) = z'(n-1)(R + B'PB)z(n-1)$ instead of the LMI (19) we obtain the LMI

$$\begin{bmatrix}A'PA+B'PB+S_0+R-P & A'PB\\ B'PA & -R\end{bmatrix}<0. \tag{23}$$

If at least one from the LMIs (19) and (23) holds then the equilibrium (x_1^*, x_2^*) of the system (11) is stable in probability. Other ways to get appropriate LMIs are shown also in [14].

Example 3.1 Consider the system (11) with the values of the parameters given in Example 1.1. Via MATLAB the maximal values of $\sigma_1 = 0.232$ and $\sigma_2 = 0.384$ were obtained, by which the LMIs (19) and (23) hold respectively for the positive definite matrices

$$P=\begin{bmatrix}10240.838 & 914.037\\ 914.037 & 545.487\end{bmatrix},\qquad R=\begin{bmatrix}8820.374 & 827.335\\ 827.335 & 456.341\end{bmatrix},$$

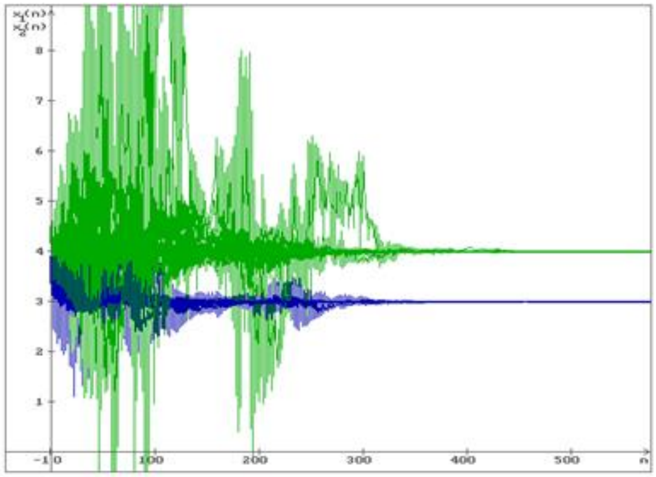

Figure 3: 50 Trajectories of the System (11) Solution: $x_1(n)$ (blue) and $x_2(n)$ (green). The Solution Converges to the Stable Equilibrium (x_1^*, x_2^*) = (3,4).

and

$$P = \begin{bmatrix} 7031.372 & 627.491 \\ 627.491 & 374.625 \end{bmatrix}, \qquad R = \begin{bmatrix} 361.556 & 59.703 \\ 59.703 & 9.976 \end{bmatrix}.$$

In Figure 3 50 trajectories of the solution of the system (11) are shown. All trajectories converge to the stable equilibrium (x_1^*, x_2^*) = (3,4).

4. Conclusion

Stability of a system of nonlinear difference equations under stochastic perturbations is investigated. The nonlinearity of exponential form in each equation depends on all variables of the system under consideration. The conditions of stability in probability for positive equilibrium of the considered system, obtained via the general method of Lyapunov functionals construction, are formulated in terms of linear matrix inequalities (LMIs) and are illustrated by numerical examples and figures. The method of stability investigation, used in the paper, can be applied to many other types of nonlinear systems with an order of nonlinearity higher than one for both difference and differential equations in various applications.

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