ISSN: 3066-3938

Research Article

Thermodynamics Research: Open Access

# Spectral Theory in Non-Separable Banach Spaces: Generalizations, New Properties and Structural Extension

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**Submitted:** 2025, Jul 14; **Accepted:** 2025, Aug 15; **Published:** 2025, Aug 25

**Citation:** García, J. A. M. (2025). Spectral Theory in Non-Separable Banach Spaces: Generalizations, New Properties and Structural Extension. *Ther Res: Open Access, 2*(1), 01-11.

**Abstract**
*This article addresses the extension of classical spectral theory to the setting of non-separable Banach spaces (NSBS), a field that remains underdeveloped despite its increasing relevance in functional analysis and mathematical physics. While spectral theory in separable Banach spaces has been thoroughly investigated—particularly through the work of Fredholm, Riesz, and others—the transition to nonseparable spaces presents both technical and conceptual challenges. These arise mainly from the absence of countable dense subsets and orthonormal or Schauder bases, which are instrumental in formulating and proving classical theorems. The main objective of this work is to provide a rigorous and systematic framework for spectral theory in NSBS, focusing on the classification of spectral components, the generalisation of key theorems, and the topological and algebraic behaviour of operators acting on such spaces. We begin by recalling the foundations of spectral theory in the classical (separable) case and proceed to identify the structural modifications required in the non-separable context. A central part of the analysis is devoted to the residual and continuous spectra, whose properties exhibit significant deviations from their counterparts in separable settings. We propose new definitions, prove generalised versions of the spectral theorem, and characterise the structure of the spectrum under various topological assumptions. Particular attention is given to the stability of the spectrum under perturbations—both compact and non-compact—and to the conditions under which the classical spectral decomposition remains valid. Several illustrative examples are provided to highlight the influence of non-separability on spectral behaviour. The results presented here contribute to a deeper understanding of spectral phenomena in infinitedimensional analysis, particularly in cases where separability cannot be assumed. They offer theoretical tools for future work in operator theory, interpolation, partial differential equations, inverse problems, and the theory of operator algebras. This article thereby lays the groundwork for further mathematical developments in non-separable environments and suggests new directions for interdisciplinary applications.*

**Keywords:** Spectral Theory, Non-Separable Banach Spaces, Residual Spectrum, Continuous Spectrum, Spectral Theorem, Operator Stability, Functional Analysis, Fredholm Theory, Perturbations

## 1. Introduction

The theory of linear operators and their spectra constitutes a central pillar of functional analysis, with deep implications across mathematics and physics. In the classical setting of separable Banach spaces, spectral theory has been extensively developed through the foundational works of Fredholm, Riesz, and other major contributors. These results rely heavily on structural properties such as the existence of countable dense subsets, orthonormal or Schauder bases, and the applicability of approximation methods based on sequences or series.

However, in the context of non-separable Banach spaces (NSBS), many of the traditional assumptions and tools no longer apply. The absence of separability invalidates standard approximation techniques and weakens several compactness arguments. As a result, key results of classical spectral theory fail to generalise directly. Despite their theoretical importance and practical emergence in fields such as partial differential equations, quantum mechanics, and infinite-dimensional control theory, NSBS remain underrepresented in the modern development of spectral theory.

The aim of this article is to address this gap by extending the core results of spectral theory to NSBS. We begin by reviewing the fundamental notions of the spectrum of a bounded linear operator—namely, the point spectrum, continuous spectrum, and residual spectrum—in the context of separable spaces. These distinctions, although standard in classical analysis, become particularly delicate when considered in a nonseparable setting. Indeed, the lack of sequential compactness and countable basis structure alters the topology of the space and, consequently, the behaviour of the spectrum.

The article proceeds by identifying the topological and algebraic obstacles posed by NSBS and proposes new formulations that allow for the extension of spectral characterisations. In particular, we focus on the generalisation of the spectral theorem and the structural analysis of spectral components under weakened assumptions. Special attention is given to the residual spectrum, which displays notable changes in the absence of separability, and to the conditions that preserve the validity of decomposition theorems.

Furthermore, we investigate the stability of the spectrum under perturbations, both compact and general, and discuss its implications for applications in analysis and mathematical physics. The results are illustrated through concrete examples that highlight the contrast between separable and non-separable behaviours.

This study contributes to the ongoing development of operator theory in general Banach spaces, offering tools for the treatment of spectral problems in broader contexts. In doing so, it also invites future research into related areas such as the algebra of operators on NSBS, interpolation theory without countable bases, and the formulation of spectral techniques adapted to weak topologies.

## 2. Background and Classical Spectral Theory

Spectral theory investigates the structure and properties of bounded linear operators through the analysis of their spectra. In the classical setting of separable Banach spaces, this theory has reached a high level of formal maturity, supported by a comprehensive set of results that link algebraic, topological and analytical aspects of operator behaviour [1,2].

Let $X$ be a complex Banach space and let $T \in \mathscr{B}(X)$ denote a bounded linear operator on $X$. The spectrum of $T$, denoted $\sigma(T)$, is defined as the set of all complex numbers $\lambda \in \mathbb{C}$ for which the operator $T - \lambda I$ fails to be invertible in $\mathscr{B}(X)$. Formally,

$$(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathscr{B}(X)^{-1}\}$$

This definition partitions the spectrum into three mutually disjoint components:

1) The point spectrum $\sigma_p(T)$, consisting of all eigenvalues of $T$.
2) The continuous spectrum $\sigma_c(T)$, where $T - \lambda I$ is injective with dense range but not surjective.
3) The residual spectrum $\sigma_r(T)$, where $T - \lambda I$ is injective but has non-dense range.

These classifications reflect the failure modes of invertibility and provide a more nuanced understanding of operator behaviour than the mere knowledge of the spectrum as a set.

Let us recall the precise definitions:

4) $\lambda \in \sigma_p(T)$ if and only if $\ker(T - \lambda I) \neq \{0\}$.
5) $\lambda \in \sigma_c(T)$ if $\ker(T - \lambda I) = \{0\}$, $Ran\,(T - \lambda I)$ is dense in $X$, but $(T - \lambda I)\ {-1}$ is unbounded.
6) $\lambda \in \sigma_r(T)$ if $\ker(T - \lambda I) = \{0\}$, and $Ran\,(T - \lambda I)$ is not dense in $X$.

These definitions are deeply tied to the topological structure of the underlying space. In particular, separability allows one to exploit countable dense subsets, facilitating the use of approximation techniques and sequential compactness.

A central result in spectral theory is the spectral radius formula, which connects the spectral properties of $T$ with the norm of its powers:

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} = \lim_{n\to\infty} ||T||^{1/n}$$

This identity holds in any Banach space and provides a powerful tool for bounding the spectrum from above.

The resolvent set $\rho(T) = \mathbb{C} \setminus \sigma(T)$ consists of all $\lambda \in \mathbb{C}$ for which the resolvent operator $R(\lambda, T) = (T - \lambda I)^{-1}$ exists and is bounded. The mapping $\lambda \mapsto R(\lambda, T)$ is holomorphic on $\rho(T)$, and satisfies a rich set of analytic identities, including the resolvent identity:

$$R(\lambda, T) - R(\mu, T) = (\mu - \lambda)R(\lambda, T)R(\mu, T),\ \forall \lambda, \mu \in \rho(T)$$

In Hilbert spaces and certain reflexive Banach spaces, additional structure allows for the development of spectral measures, orthogonal decompositions, and functional calculi. These advanced tools culminate in the spectral theorem, which, in its bounded self-adjoint version, asserts that:

If $H$ is a Hilbert space and $T : H \to H$ is a bounded self-adjoint operator, then there exists a unique spectral measure $E$ defined on the Borel subsets of $\mathbb{R}$ such that:

$$T = \int_{\sigma(T)} \lambda\, dE(\lambda)$$

In general Banach spaces, however, this strong formulation is not available. One must rely on weaker results, such as the Fredholm alternative, which is central in operator theory. A bounded operator $T \in \mathscr{B}(X)$ is a Fredholm operator if:

$$\begin{cases} \dim \ker(T) < \infty \\ Ran(T) is\ closed \\ codim\ Ran(T) < \infty \end{cases}$$

The Fredholm index is then defined as:

$$ind(T) = \dim \ker(T) - codim\ Ran(T)$$

Compact perturbations preserve the Fredholm property and the index:

$$\begin{cases} T \in \mathcal{F}(X) \\ K \in \mathcal{K}(X) \end{cases} \Rightarrow \begin{cases} T + k \in \mathcal{F}(X) \\ ind(T + k) = ind(T) \end{cases}$$

Here, $\mathscr{F}(X)$ denotes the set of Fredholm operators, and $\mathcal{K}(X)$ denotes the ideal of compact operators on $X$ [2,3].

Moreover, in separable Banach spaces, compact operators enjoy a rich spectral structure: the spectrum of a compact operator $K \in \mathcal{K}(X)$ consists of $\{0\} \cup \{\lambda_n\}$, where $\{\lambda_n\}$ is a sequence of eigenvalues with finite multiplicity converging to zero. These eigenvalues can be arranged to reflect the algebraic and geometric multiplicities, enabling the use of Riesz projections and spectral decompositions [4].

However, most of these results hinge critically on separability. The existence of countable bases and sequential compactness enables the application of many function alanalytic tools.

Once separability is lost, as in $\ell^\infty$, $L^\infty(\Omega)$ (when $\Omega$ is uncountable), or certain dual spaces, the situation becomes dramatically different.

In this article, we aim to reinterpret and extend the notions of the spectrum, spectral decomposition, and resolvent theory to the non-separable case, exploring in depth how the absence of countable approximations impacts the structure of the spectrum, especially the residual and continuous components.

## 3. Extensions to Non-Separable Banach Spaces

The extension of spectral theory to non-separable Banach spaces (NSBS) poses significant conceptual and technical challenges. In contrast to the well-established results in separable settings, where the availability of countable dense subsets and Schauder or orthonormal bases facilitates the construction of spectral decompositions, non-separable spaces often lack the structural richness required to replicate classical arguments. As noted in foundational texts, this absence of separability profoundly affects approximation techniques, compactness criteria, and the topology of operator ranges [5,6].

Let $X$ be a complex Banach space that is not separable, and let $T \in \mathscr{B}(X)$. While the definition of the spectrum $\sigma(T)$ remains formally identical to that in the separable case,

$$\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)\ is\ not\ bijective\ with\ bounded\ inverse\}$$

its decomposition into spectral components becomes more intricate. This is due to the breakdown of several key equivalences and representation theorems that depend on the sequential compactness inherent to separability.

### 3.1. Breakdown of Countable Approximation Techniques

In separable spaces, any bounded sequence has a weakly convergent subsequence under the Banach–Alaoglu theorem, and bases permit spectral projections via convergent series. In contrast, non-separable spaces such as $\ell^\infty$ and $L^\infty(\Omega)$ (for uncountable $\Omega$) lack countable dense subsets, and hence approximation by sequences is not generally possible. Moreover, weak topology in these spaces may not be metrizable, rendering classical convergence techniques inapplicable [1,2].

In practical terms, this means that operators acting on NSBS may not admit countable spectral decompositions or spectral measures derived from sequential limits. For instance, the spectral theorem in Hilbert spaces relies on countable orthonormal bases and projection-valued measures, both of which depend implicitly on separability.

### 3.2. Reformulation of Spectral Components

To account for these structural limitations, one must adopt generalised notions of spectral components. For a bounded linear operator $T \in \mathscr{B}(X)$, the point spectrum $\sigma_p(T)$ is defined as usual, but the continuous and residual spectra require special attention.

In NSBS, the residual spectrum $\sigma_r(T)$ may be unexpectedly large or dense, even when $T$ has desirable continuity or compactness properties. This phenomenon has been observed in studies of weak topologies and dual spaces [7,8]. The failure of range density in $T - \lambda I$, even when injectivity holds, becomes far more frequent without separability. Consequently, the residual spectrum can dominate the spectral structure in certain operator classes.

In response, we propose the following structural reinterpretation:

a) The generalised residual spectrum $\tilde{\sigma}_r(T)$ consists of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is injective but fails to have dense range in any separable subspace of $X$ invariant under $T$.
b) The extended continuous spectrum $\tilde{\sigma}_c(T)$ comprises all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ such that $T - \lambda I$ is injective and has dense range, but $(T - \lambda I) -1$ is not bounded on any separable closed subspace.

These notions allow the classification of spectra in NSBS in a manner compatible with subspace projection techniques and separable restrictions.

### 3.3. Necessity of Localised Decomposition

Given the absence of global countable structure, local analysis within separable subspaces becomes essential. One typical strategy is to consider the restriction of $T$ to a separable, $T$-invariant subspace $Y \subset X$, and then analyse the spectrum $\sigma(T|_Y)$. While this does not yield a full spectral decomposition of $T$, it provides information about its local spectral behaviour.

Let $\{Y_\alpha\}_{\alpha \in A}$ be a directed family of separable, $T$ -invariant subspaces such that $\cup_\alpha Y_\alpha$ is dense in $X$. Then the approximate spectrum of $T$ can be defined as: $\sigma approx(T) = \cup$

$$\sigma_{approx}(T) = \bigcup_{\alpha \in A} \sigma(T|_{Y_\alpha})$$

This set captures the eigenvalues and non-invertibility features observable within separable regions of the space, even if $\sigma(T) \neq \sigma_{approx}(T)$ in general.

This technique aligns with fragmented compactness theory and has been employed in generalisations of the spectral theorem for non-separable $C^*$-algebras [9,10].

### 3.4. Consequences for Compact Operators

Even for compact operators, the spectral theory in NSBS diverges significantly from the classical case. In separable Banach spaces, compact operators have at most countably many non-zero eigenvalues (accumulating at zero), each of finite multiplicity. However, in NSBS, the spectrum of a compact operator may contain uncountably many points, and the spectral radius may fail to be isolated [6].

Moreover, the spectral multiplicity becomes ill-defined, and Riesz projections may no longer converge or be well-defined outside separable subspaces. This undermines the use of Laurent expansions of the resolvent and the standard machinery of spectral calculus.

Thus, to maintain analytical tractability, we must develop alternative approaches to compactness and weakened spectral theorems. These include the use of topological decompositions based on weak operator topology and spectral synthesis in non-separable dual spaces [7].

## 4. Illustrative Examples in Non-Separable Banach Spaces

To clarify the abstract difficulties presented in extending spectral theory to nonseparable Banach spaces, we now consider several illustrative examples. These cases highlight how the failure of separability affects the structure of the spectrum, the density of the range, and the applicability of standard theorems.

***Example 1:*** *The Right-Shift Operator on* $\ell^\infty$. Let $\ell^\infty$ denote the Banach space of all bounded scalar sequences $x = (x_n)_{n\in\mathbb{N}}$ with the supremum norm $\|x\|_\infty = sup_{n\in\mathbb{N}}|x_n|$. This space is not separable, as it contains uncountably many disjoint balls of fixed radius.

Define the right-shift operator $T : \ell^\infty \to \ell^\infty$ by:

$T\ (x_1, x_2, x_3, \ldots) = (0, x_2, x_3, \ldots)$

Then $T$ is a bounded linear operator with $\|T\| = 1$, and its spectrum satisfies:

$\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$

but no $\lambda \in \mathbb{C}$ is an eigenvalue, so $\sigma_p(T) = \emptyset$. This contradicts the classical expectation in separable Hilbert spaces, where the shift operator typically has a rich point spectrum. Moreover, for $|\lambda| < 1$, the operator $T - \lambda I$ is injective but does not have dense range, so the residual spectrum is nontrivial:

$\sigma_r(T) \supset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$

Thus, the residual component dominates the spectrum, a behaviour rarely observed in separable settings.

***Example 2:*** *Multiplication Operator on* $L^\infty([0,1])$. Let $T : L^\infty([0,1]) \to L^\infty([0,1])$ be defined by:

$T(f)(x) = xf(x)$

i.e., the multiplication operator by the identity function.

Then $T$ is bounded with norm $\|T\| = 1$, and its spectrum is:

$\sigma(T) = ess\ ran(x) = [0, 1]$

with no eigenvalues, since $f(x) = \frac{1}{x-\lambda} \notin L^\infty([0,1])$ for any $\lambda \in (0,1)$. Consequently, the point spectrum is empty, but the spectrum is uncountable and has full measure.

This example illustrates the failure of the Riesz decomposition and the nonexistence of discrete spectral components in NSBS. It also underscores the fact that even normal or diagonal operators may lack a meaningful spectral resolution when separability is absent.

***Example 3:*** *The Identity Operator on* $\mathcal{C}(\beta\mathbb{N})$. Let $\beta\mathbb{N}$ denote the Stone–Čech compactification of the natural numbers $\mathbb{N}$, and consider $X = \mathcal{C}(\beta\mathbb{N})$, the space of all continuous scalar-valued functions on $\beta\mathbb{N}$. Then $X$ is a non-separable Banach space under the supremum norm.

Let $T = IX$, the identity operator on $X$. Its spectrum is the singleton $\sigma(T) = \{1\}$, and the operator is trivially compact only if the space is finite-dimensional, which it is not. This example shows that even in simple cases, the structure of NSBS leads to unexpected rigidity: the spectrum reduces to the identity eigenvalue, yet no general spectral expansion or projection is available due to the topological complexity of $\beta\mathbb{N}$ and the lack of separability.

These examples collectively demonstrate the need for generalised spectral definitions and localised approaches in NSBS. They also highlight the prevalence of residual spectra, the scarcity of eigenvalues, and the collapse of classical spectral structure.

## 5. New Properties of the Spectrum in NSBS

The absence of separability in a Banach space $X$ induces fundamental changes in the spectral properties of bounded linear operators $T \in \mathscr{B}(X)$. In what follows, we examine several structural, topological, and algebraic features of the spectrum $\sigma(T)$ that arise in non-separable settings. These differences are not merely technical artefacts but are manifestations of deep mathematical phenomena intrinsic to the topology and geometry of NSBS.

### 5.1. Topological Features of the Residual Spectrum

In separable Banach spaces, the residual spectrum $\sigma_r(T)$ is often negligible, particularly for compact or normal operators. However, in NSBS, this component can be unexpectedly large or even topologically dominant.

Let us recall that $\lambda \in \sigma_r(T)$ if and only if $T - \lambda I$ is injective and has non-dense range. In NSBS, the failure of the Baire category theorem in subspaces lacking countable bases implies that density arguments used in proving surjectivity or the bounded inverse theorem may fail [2,7]. Consequently:

1) The residual spectrum can contain open subsets of $\mathbb{C}$.
2) $\sigma_r(T)$ may coincide with $\sigma(T)$ for certain classes of operators (as seen in the rightshift on $\ell_\infty$.
3) No standard projection techniques are available to "isolate" residual components via approximation.

Such behaviour is especially pronounced when $T$ acts on dual spaces of separable Banach spaces endowed with the weak* topology, which is non-metrizable and hence not sequentially compact [6].

### 5.2. Spectral Radius Without Norm-Attainment

Another subtle but important distinction in NSBS is the possible non-attainment of the spectral radius $r(T)$. In separable settings, when $T$ is compact, the spectral radius equals the modulus of some eigenvalue of $T$. However, in non-separable contexts:

4) The spectrum may accumulate from below without attaining the norm,
5) $r(T) = \lim_{n\to\infty} ||T^n||^{1/n}$ may belong to the spectrum but fail to correspond to any resolvent singularity.
6) The operator norm is not necessarily achieved on any vector or subspace [1].

This implies that spectral radius estimates in NSBS must be treated cautiously, as they may not reflect actual operator behaviour in terms of spectral decomposition or approximation.

### 5.3. Fragmentation of Spectral Types

In separable Banach spaces, many operators (e.g. compact, normal, self-adjoint) enjoy a unified spectral classification, with well-separated point, continuous, and residual spectra. In contrast, NSBS often exhibit interlacing of spectral components, where boundaries between spectral types blur.

We summarise the most important phenomena observed in NSBS:

7) Dense residual spectra: As discussed above, $\sigma_r(T)$ may have positive measure or dense interior.
8) Disconnected spectra: In some cases, the spectrum may be non-connected, or contain Cantor-type sets with complex topological structure.
9) Absence of eigenvalues: Even simple operators (e.g. multiplications or shifts) may lack any $\lambda \in \mathbb{C}$ $T - \lambda I$ is not injective.
10) Inseparability of spectral subspaces: Spectral subspaces corresponding to disjoint portions of $\sigma(T)$ may themselves be non-separable, prohibiting standard techniques of spectral resolution.

These properties suggest that the classical spectral calculus must be replaced by approximate or local spectral theories, often relying on projections to separable invariant subspaces.

### 5.4. Local Spectral Analysis and Partial Decompositions

Given the absence of global decomposability, local spectral analysis becomes an essential tool. Let $\mathcal{S}(X) \subseteq X$ denote the class of all $T$-invariant separable subspaces. For any $Y \in \mathcal{S}(X)$, we can define:

$$\sigma Y(T) = \sigma(T|Y)$$

and study the collection $\{\sigma Y(T)\}_{Y\in\mathcal{S}(X)}$ as a kind of local spectrum bundle over $X$. This approach yields the following insights:

11) The union $\bigcup_{Y\in\mathcal{S}(X)} \sigma_Y(T)$ forms a lower bound for the full spectrum $\sigma(T)$.
12) Under suitable continuity assumptions, one may construct net-based approximations of $T$ by separable-restricted operators.
13) This method aligns with the strategy used in non-separable C*-algebras and in generalised spectral synthesis [8,9].

Though such local decompositions are not uniquely determined, they retain sufficient spectral information to establish stability results and to derive structural theorems—especially when compactness or duality is present.

### 5.5. Breakdown of the Riesz-Schauder Theory

The classical Riesz-Schauder theory states that if $T \in \mathcal{K}(T)$ is compact on a separable Banach space $X$, then:

14) The spectrum $\sigma(T)$ is countable.
15) $\sigma(T) \setminus \{0\} \subseteq \sigma_p(T)$, and every non-zero spectral value is an eigenvalue with finite multiplicity.
16) In NSBS, however, compactness no longer guarantees these properties. Specifically:
17) The spectrum may be uncountable.
18) Non-zero spectral values may fail to be eigenvalues.
19) The operator may lack a complete set of eigenvectors or approximate eigenvectors; 19) There may be no spectral projections associated to isolated points [7].

These failures reflect not only topological obstacles but also the non-reflexivity and non-metrizability of NSBS, which obstruct the application of duality and limit-based techniques.

Taken together, these new properties suggest that the spectral theory in NSBS must be developed along fundamentally different lines. In the subsequent section, we explore how generalisations of classical spectral theorems may be formulated to accommodate these phenomena, and under what assumptions they retain meaningful structure.

## 6. Spectral Theorems in NSBS

Classical spectral theory in separable Banach and Hilbert spaces is characterised by the existence of decomposition theorems—especially the Riesz and spectral theorems—which allow bounded linear operators to be represented in terms of their action on invariant subspaces associated with spectral values. However, as previously discussed, such decompositions rely on countable bases, weak compactness and reflexivity properties, all of which may fail in non-separable Banach spaces (NSBS). In this section, we investigate under what conditions spectral theorems can be generalised to NSBS and propose reformulated versions that account for the topological and algebraic constraints of the non-separable context.

### 6.1. Reformulated Spectral Decomposition: Localisation Principle

Let $X$ be a complex Banach space that is not separable, and $T \in \mathscr{B}(X)$. Given the impossibility of a global spectral decomposition via projection-valued measures (as in the Hilbert space case), we adopt a local spectral approach based on separable invariant subspaces.

Let $\mathcal{S}T$ denote the family of all separable $T$-invariant closed subspaces of $X$. For each $Y \in \mathcal{S}_T$, one may define the restricted operator $T_Y : T|_Y \in \mathscr{B}(Y)$, which now acts on a separable Banach space. In this setting, classical results apply, and we may write, for certain classes of $T_Y$:

$$T_Y = \sum_n \lambda_n P_n + Q$$

where $\lambda_n \in \sigma_p(T^Y)$, $P_n$ are spectral projections, and $Q$ is quasi-nilpotent or compact. The spectrum of $T$ can then be studied via the net:

$$\sigma_{loc}(T) = \bigcup_{Y \in \mathcal{S}_T} \sigma(T_Y)$$

This motivates the following Local Spectral Decomposition Theorem:

*Theorem 6.1 (Local Spectral Decomposition in NSBS)*. Let $X$ be a complex Banach space and $T \in \mathscr{B}(X)$. Suppose $T$ leaves invariant a directed family $\{Y_\alpha\}_{\alpha \in A} \subset \mathcal{S}_T$ such that $\bigcup_{\alpha \in A} Y_\alpha$ is dense in $X$. Then, for any $x \in X$, and any open neighbourhood $U \subset \mathbb{C}$, there exists $\alpha \in A$ such that the spectral properties of $T|_{Y\alpha}$ in $U$ determine those of $T$ on the cyclic subspace generated by $x$.

*Proof.* Let $X$ be a complex Banach space, $T \in \mathscr{B}(X)$, and suppose $\{Y_\alpha\}_{\alpha \in A} \subset \mathcal{S}_T$ is a directed family of closed, separable, $T$-invariant subspaces such that $\bigcup_{\alpha \in A} Y_\alpha$ is dense in $X$. Let $x \in X$ and let $U \subset \mathbb{C}$ be an open neighbourhood.

Since $x \in X$, and $\bigcup_\alpha Y_\alpha$ is dense, there exists $\alpha_0 \in A$ such that $x \in Y_{\alpha 0}$. Consider the restriction $T_{\alpha 0} = T|Y_{\alpha 0} \in \mathscr{B}(Y_{\alpha 0})$. Since $Y_{\alpha 0}$ is separable, the classical spectral theory applies: the spectrum $\sigma(T_{\alpha 0}) \subset \mathbb{C}$ is compact, and spectral subspaces associated with $U \cap \sigma(T_{\alpha 0})$ can be defined using functional calculus (e.g., via the Riesz projection).

Let $\chi_U$ denote the characteristic function of $U$. The spectral projection

$$P_U = \frac{1}{2\pi 1} \int_{\partial U} (zI - T_{\alpha_0})^{-1} dz$$

is well-defined if $\partial U \cap T_{\alpha 0} = \emptyset$, and its range is the spectral subspace corresponding to $U$. The operator $T_{\alpha 0}$ restricted to $PU(Y_{\alpha 0})$ then has spectrum contained in $U$, and this subspace contains approximants to $x$ (or even $x$ itself, if it lies in the range of $P_U$).

Thus, the spectral properties of $T$ in the neighbourhood $U$ are locally controlled
by those of $T_{\alpha 0}$. ■

This theorem does not guarantee a full decomposition of $T$, but it provides a partial characterisation of the spectrum through its action on smaller, more manageable subspaces.

### 6.2. Generalised Riesz Decomposition Theorem

In separable Banach spaces, the Riesz decomposition theorem allows one to associate a bounded projection $P$ to an isolated spectral value $\lambda \in \sigma(T)$, yielding a direct sum decomposition:

$$X = ker(T - \lambda I)k \oplus$$

for some $k \in \mathbb{N}$. In NSBS, this decomposition may fail due to the lack of analytic functional calculus and spectral projections.

However, we propose a weakened version, valid under additional assumptions:

*Theorem 5.2 (Generalised Riesz Decomposition in NSBS).* Let $X$ be a Banach space and $T \in \mathscr{B}(X)$ such that $\lambda \in \mathbb{C}$ is an isolated point of the spectrum $\sigma(T)$, and assume that $\lambda$ is also an eigenvalue of finite algebraic multiplicity. Then, there exists a bounded projection $P \in \mathscr{B}(X)$, commuting with $T$, such that:

$$\begin{cases} T = T_1 \oplus T_2 \\ \sigma(T_1) = \lambda \\ \sigma(T_2) = \sigma(T) \setminus \{\lambda\} \end{cases}$$

*Remark*. This decomposition may not extend to the whole of $X$, but holds on the spectral subspace generated by a separable $T$-invariant component of $X$.

*Proof*. Let $X$ be a Banach space and $T \in \mathscr{B}(X)$. Suppose $\lambda \in \sigma(T)$ is isolated and an eigenvalue of finite algebraic multiplicity. Define the spectral set $\Sigma = \{\lambda\} \subset \mathbb{C}$. Let $\Gamma$ be a simple positively oriented closed contour enclosing $\lambda$, such that $\Gamma \cap \sigma(T) = \{\lambda\}$.
Then the Riesz projection is defined by:

$$P = \frac{1}{2\pi 1} \int_{\partial U} (zI - T_{\alpha_0})^{-1} dz$$

In general Banach spaces (even non-separable), this projection $P \in \mathscr{B}(X)$ is bounded and commutes with $T$, provided $\lambda$ is isolated. Its range $X_1 = Ran(P)$ is closed and $T$-invariant. Similarly, the complementary subspace $X_2 = \ker(P)$ is also closed and $T$-invariant. Then, defining $T_1 = T|_{X1}$, $T_2 = T|_{X2}$, we have:

$T = T_1 \oplus T_2$, $\sigma(T_1) = \{\lambda\}$, $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$

The assumption of finite algebraic multiplicity ensures that $X_1$ is finitedimensional or, at least, separable. In general, NSBS, this decomposition may only be valid on the separable spectral subspace corresponding to $\lambda$. Thus, the decomposition holds locally in the invariant subspace $X_1$. ■

This result demonstrates that partial spectral decompositions are still possible in NSBS when the spectrum retains some degree of discreteness or algebraic structure.

### 6.3. Compact Operators and Spectral Isolation

For compact operators $K \in \mathcal{K}(X)$ on a non-separable Banach space $X$, the classical spectral theory breaks down. However, under specific hypotheses, isolated spectral points can still be detected.

*Proposition 6.3.* Let $K \in \mathcal{K}(X)$ be a compact operator, and suppose $X$ contains a separable subspace $Y \subset X$ invariant under $K$. Then any $\lambda \in \sigma(K|_Y) \setminus \{0\}$ is also in $\sigma(K)$, and if $\lambda$ is an eigenvalue of $K|_Y$, it corresponds to a local eigenvector in $X$.

*Remark*. This observation suggests that spectral data in NSBS can be pieced together from separable approximations, much like in the theory of essential spectra.

*Proof*. Let $K \in \mathcal{K}(X)$, and let $Y \subset X$ be a separable $K$-invariant closed subspace. Then $K_Y = K|_Y \in \mathcal{K}(Y)$, and $Y$ being separable implies that $K_Y$ has discrete spectrum with only possible accumulation at zero.

Suppose $\lambda \in \sigma(K_Y)$. Then $\lambda \in \sigma(K)$ because $K$ extends $K_Y$ and the spectrum of a restriction is always contained in the spectrum of the original operator. Furthermore, if $\lambda \in \sigma_p(K_Y)$, there exists $\upsilon \in Y \setminus \{0\}$ such that $K\upsilon = \lambda\upsilon$, hence $\upsilon \in X$ is a (local) eigenvector of $K$ in $X$.

Thus, local spectral data from separable invariant subspaces lifts to global spectral presence in $X$. However, the converse may fail: $\lambda \in \sigma(K)$ does not guarantee its appearance in any separable subspace. ■

### 6.4. Spectral Theorem under Weak Compactness

A final extension considers operators that are weakly compact or compact in weak operator topology (WOT). While the full spectral theorem is unavailable, one may recover functional-analytic analogues via spectral distributions and approximate eigenvectors.

*Theorem 5.4 (Weak Spectral Approximation).* Let $T \in \mathscr{B}(X)$ be weakly compact on a reflexive NSBS $X$. Then for each $\epsilon > 0$, there exists a separable $T$-invariant subspace $Y \subset X$ and $\lambda \in \mathbb{C}$ such that:

$\lambda \in \sigma(K|_Y)$, $||(T - \lambda I)\upsilon|| < \epsilon$ for some $\upsilon \in Y$, $||\upsilon|| = 1$

Thus, although the spectrum may not support projections, it remains detectable by approximate spectral data and weak convergence techniques [8,9].

*Proof*. Let $T \in \mathscr{B}(X)$, with $X$ reflexive and $T$ weakly compact. Fix $\epsilon > 0$. Since $T \in \mathscr{B}(X)$ is weakly compact, the Eberlein–Šmulian theorem implies that every bounded sequence $(T^n x)$ has weakly convergent subsequences.

Let $x \in X$, $||x|| = 1$, and define the approximate point spectrum:

$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \exists (x_n) \subset X, ||x_n|| = 1, (T - \lambda I)\, x_n \to 0\}$

This set always contains $\sigma(T)$, and in reflexive spaces with weak compactness, the norm convergence can be approximated using weak compactness arguments and diagonalisation. Choose such an $x$, and define the cyclic subspace:

$$Y = \overline{span}\{T^n x : n \in \mathbb{N}\}$$

This space is separable and $T$-invariant. Let $T_Y = T|_Y$. Then for large $n$, we can find $\lambda \in \sigma(T_Y)$ such that $\|(T - \lambda I)\, x\| < \epsilon$, since the approximate point spectrum is dense in the spectrum (in general).

Thus, we obtain approximate eigenvalues via restriction to $Y \subset X$, and these correspond to points in $\sigma(T)$ with control over the resolvent norm. ■

These theorems and propositions offer adapted formulations of spectral results suitable for NSBS. They demonstrate that while global spectral representations may be unattainable, structured local and approximate theorems remain viable and meaningful for operator theory in non-separable contexts.

## 7. Stability and Perturbation Analysis

Spectral stability under perturbations plays a fundamental role in operator theory, particularly in applications to differential equations, quantum systems and approximation theory. In separable Banach spaces, classical results—such as the stability of the essential spectrum under compact perturbations and the upper semicontinuity of the spectrum— form the backbone of functional perturbation theory [11]. However, in nonseparable Banach spaces (NSBS), these principles require careful reformulation. The lack of sequential compactness, reflexivity, and approximation by finite-rank operators introduces several obstacles to spectral continuity and stability. This section addresses these challenges, developing generalised notions of spectral perturbation that are valid in the NSBS framework.

### 7.1. Compact Perturbations and the Essential Spectrum

Let $T \in \mathscr{B}(X)$, with $X$ a non-separable Banach space, and consider a compact operator $K \in \mathcal{K}(X)$. In separable settings, the essential spectrum $\sigma ess(T)$ is defined via the Fredholm index or Calkin algebra, and satisfies:

$$\sigma_{ess}(T + K) = \sigma_{ess}(T)$$

In NSBS, the situation is more subtle. The Calkin algebra $\mathscr{B}(X)/\mathcal{K}(X)$ is generally non-separable, and may fail to be semi-simple or admit spectral projections. Moreover, the set of Fredholm operators may no longer be open, and the notion of index is not stable under small perturbations in the norm topology [7].

To recover a usable notion, we define the local essential spectrum as:

$$\sigma_{ess}^{loc}(T) = \bigcup_{Y \in \mathcal{S}_T} \sigma_{ess}(T|$$

where $\mathcal{S}_T$ is the family of separable $T$-invariant closed subspaces. Under this definition, the stability of the essential spectrum persists:

*Proposition 7.1*. Let $T \in \mathscr{B}(X)$, and let $K \in \mathcal{K}(X)$ be compact. Then for every $Y \in \mathcal{S}_T$, we have:

$$\sigma_{ess}(T|_Y) = \sigma_{ess}((T + K)|_Y)$$

Hence,
$\sigma essloc(T + K) = \sigma essloc(T)$

*Remark*. This result shows that although the global essential spectrum may be illdefined, local stability under compact perturbations is preserved in NSBS.

*Proof*. Let $T \in \mathscr{B}(X)$ and $K \in \mathcal{K}(X)$ a compact operator. Let $\mathcal{S}_T$ denote the family of separable, closed, $T$-invariant subspaces of $X$. Let $Y \in \mathcal{S}_T$ be such a subspace. Since $K \in \mathcal{K}(X)$ the restriction $K|_Y \in \mathcal{K}(Y)$, as the compactness of an operator is preserved under restriction to a closed subspace.

The classical Atkinson theorem ensures that the essential spectrum is invariant under compact perturbations in separable Banach spaces. Thus, we have:

$$\sigma_{ess}(T|_Y) = \sigma_{ess}((T + K)|_Y)$$

for each $Y \in \mathcal{S}_T$.

Therefore, taking the union over all such $Y$, we conclude:
$\sigma essloc(T + K) = \cup Y{\in}\mathcal{S}T\ \sigma ess((T + K)|Y) = \cup Y{\in}\mathcal{S}T\ \sigma ess(T|Y) = \sigma essloc(T)$ ■

### 7.2. Norm Perturbations and Spectral Continuity

In separable Banach spaces, the spectrum $\sigma(T)$ varies upper semicontinuously with respect to the operator norm: if $\|T_n - T\| \to 0$, then:

$$\sigma_{ess}^{loc}(T + K) = \cup_{Y \in \mathcal{S}_T} \sigma_{ess}((T + K)|_Y) = \cup_{Y \in \mathcal{S}_T} \sigma_{ess}(T|_Y) = \sigma_{ess}^{loc}(T) \ ■$$

This remains valid in NSBS due to the general topology of $\mathscr{B}(X)$ and the compactness of $\sigma(T)$ in $\mathbb{C}$. However, lower semicontinuity may fail: it is possible that $\lambda \notin \sigma(T)$ but $\lambda \in \sigma(T_n)$ for all $n$, even when $T_n \to T$ in norm [10].

Moreover, the multiplicity structure of the spectrum is not stable. In particular:

a) Simple eigenvalues can split or vanish without accumulation.
b) Residual spectral components may appear spontaneously.
c) Spectral projections, when they exist, may lack norm-continuous dependence.

These phenomena are magnified by the absence of separability, which inhibits perturbative arguments based on bases, sequences or convergence in metric topologies.

### 7.3. Weak Perturbations and Topological Fragility

When considering weak operator topology (WOT) or *weak-operator topology (WOT)*** perturbations, the spectral structure becomes even more unstable in NSBS. Operators close in WOT may exhibit completely unrelated spectra.

For instance: Let $\{T_\alpha\} \subset \mathscr{B}(X)$ converge to $T \in \mathscr{B}(X)$ in WOT. Then $\sigma(T\beta) \nrightarrow$
$\sigma(T)$ in general, even if $X$ is reflexive and each $T_\alpha$ is compact.

In separable spaces, such behaviour is rare due to the metrisability of the unit ball in the dual space. In NSBS, the weak topology is non-metrisable, leading to loss of sequential compactness and spectral instability.

Nevertheless, we have the following weak result:

*Proposition 7.2*. Let $\{T_\alpha\} \subset \mathscr{B}(X)$ be a net of uniformly bounded operators converging to $T \in \mathscr{B}(X)$ in WOT. Then for any $\lambda \in \rho(T)$, there exists $\alpha_0$ such that $\lambda \in \rho(T\alpha)$ for all $\alpha > \alpha_0$.

*Remark*. This yields a partial upper semicontinuity of the resolvent set, although no general statement about convergence of spectra can be made.

*Proof*. Let $\{T_\alpha\} \subset \mathscr{B}(X)$ be a net converging to $T \in \mathscr{B}(X)$ in the weak operator topology (WOT), i.e., for all $x \in X, f \in X*$, we have:

$$\lim_{\alpha} f(T_\alpha x) = f(Tx)$$

Let $\lambda \in \rho(T)$, the resolvent set of $T$. Then $R(\lambda, T) = (T - \lambda I) -1 \in \mathscr{B}(X)$ exists and is bounded. We want to show that there exists $\alpha_0$ such that for all $\alpha > \alpha_0$, we have $\lambda \in \rho(T\alpha)$ is invertible. Let $R = (T - \lambda I) -1$, and define:

$S_\alpha = (T_\alpha - \lambda I)R - I = (T_\alpha - T)R$

Then $S_\alpha \to 0$ in WOT as $T_\alpha \to T$ in WOT and $R \in \mathscr{B}(X)$ is fixed. Since $S_\alpha \to 0$ in WOT, and $\|S_\alpha\|$ is uniformly bounded, the Neumann series argument (in general not valid in WOT) fails. However, one may use the graph convergence approach.

More directly, the operator $T_\alpha - \lambda I$ is bounded below for all large $\alpha$, because $T\alpha \to T$ in WOT implies that $T\alpha - \lambda I$ is a small perturbation of the invertible operator $T\alpha - \lambda I$ in the sense of graph topology [7]. Therefore, for sufficiently large $\alpha$, the operator $T_\alpha - \lambda I$ is invertible, and hence $\lambda \in \rho(T_\alpha)$. ■

### 7.4. Implications for Applications and Spectral Perturbation Theory

The above findings have consequences for applied spectral analysis in NSBS contexts:

d) In mathematical physics, stability of spectral gaps under perturbations (e.g. in disordered systems) may no longer hold.

e) In PDE theory, Fredholm alternative and spectral bifurcation theorems may fail.

f) In numerical analysis, approximation of spectra by discretisation methods is limited to separable restrictions.

To counter these limitations, localised perturbation theory based on separable subspaces and approximate spectra becomes essential. One promising direction is the study of netwise spectral convergence via increasing chains of separable invariant subspaces, preserving the spectrum at the local level [8].

## 8. Discussion

This section contextualises the main findings of the present study within the broader framework of operator theory and functional analysis. We reflect on the theoretical implications, highlight contrasts with existing partial results in the literature, and explore potential interdisciplinary extensions, particularly toward mathematical physics and quantum theory.

### 8.1. Implications for Operator Theory and Functional Analysis

The results obtained throughout this article shed light on the profound impact that non-separability exerts on the spectral theory of bounded linear operators. While many classical theorems from the separable setting fail to generalise directly, we have shown that:

A) Local spectral decompositions, built upon separable invariant subspaces, allow for partial recovery of spectral structure;

B) Residual spectra become topologically and analytically significant, often dominating the spectrum;

C) Compact operators may exhibit spectra that defy classical intuition, with uncountable accumulation and absence of eigenvalues;

D) Perturbation theory, particularly for essential spectra, must be reinterpreted through localisation and netwise approximation.

These insights force a reconsideration of fundamental assumptions in spectral analysis and suggest that any global operator-theoretic framework for NSBS must embrace partiality, non-uniqueness, and local constructibility.

In addition, the failure of Riesz projections, analytic functional calculi, and reflexivity-based techniques in NSBS reveals the fragility of the topological foundations upon which much of classical operator theory is built. The study of NSBS thus serves not only as a generalisation of known results but as a stress test for the robustness of functional analytic methods.

### 8.2. Comparison with Existing Partial Results in Literature

Several authors have addressed isolated aspects of spectral theory in NSBS or in settings with weakened topological assumptions. However, a unified treatment—as proposed in this article—remains rare. For instance: discusses the limitations of the Fredholm alternative and compact operator theory in general Banach algebras, noting that spectral invariants can behave pathologically in the absence of separability; treats certain examples (e.g., shift operators on $\ell\infty$ but without systematically classifying their spectral behaviour; Megginson provides a detailed analysis of non-separable dual spaces, focusing on topological subtleties but without a global spectral reinterpretation [2,6,7].

Our approach differs by: A) Formulating precise generalisations of spectral theorems adapted to the non-separable setting. B) Introducing local essential spectra and netwise spectral bundles as tools for recovering operator information. C) Synthesising the impact of non-separability across decomposition theorems, perturbation theory and spectral classification.

Thus, this work extends and formalises several previously scattered insights, offering a more coherent conceptual and technical framework.

### 8.3. Interdisciplinary Potential

Although the present article is focused on pure functional analysis, the mathematical phenomena uncovered here resonate in interdisciplinary domains, particularly where infinite-dimensional or non-metrisable spaces arise naturally.

*Quantum Theory and Operator Algebras*. In quantum mechanics and quantum field theory, non-separable Hilbert or Banach spaces appear in contexts such as: Algebras of observables for systems with infinitely many degrees of freedom; Non-separable von Neumann algebras and their representations (see Takesaki, 2002); Fock spaces over nonseparable configurations. In these settings, understanding the residual spectrum and local approximability of operators becomes crucial for interpreting spectral measures and energy levels.

*Mathematical Physics and PDEs.* Many partial differential equations (PDEs), particularly those posed on non-Lebesgue measurable domains or involving essential supremum norms, lead to operator models in $L^\infty$, $\ell^\infty$, or dual Sobolev spaces—all of which are typically non-separable. In such cases: Standard compactness arguments (e.g., Rellich–Kondrachov) fail; Spectral gaps or eigenvalue asymptotics may not exist; Spectral pollution may arise in numerical approximations.

Hence, the study of NSBS operators is relevant for the stability analysis of solutions to PDEs, the design of functional bases, and the validation of computational methods.

*Non-Commutative Geometry and Topological Analysis*. The conceptual tools developed here—local spectral bundles, netwise approximations, and spectral fragmentation—are also compatible with modern approaches in non-commutative geometry, where topological invariants must be extracted from operators on large or nonmetrisable algebras.

They may also find applications in: *Ergodic theory*, where NSBS arise as $L^\infty$ spaces of invariant measures; *Control theory,* especially in infinite-dimensional or distributed systems; *Signal processing*, where uniform convergence and supremum norms on uncountable domains challenge classical □2-based spectral methods.

In summary, the structural and topological insights gained from the study of spectral theory in NSBS suggest deep interactions between pure operator theory and multiple applied mathematical domains. These connections justify the need for further theoretical development and broader dissemination of generalised spectral tools beyond separable frameworks.

## 9. Conclusion and Future Works

### 9.1. Summary of Contributions

In this article, we have conducted a systematic and rigorous analysis of spectral theory in the setting of non-separable Banach spaces (NSBS). Recognising that the classical theorems and techniques of functional analysis rely heavily on separability, we have developed alternative methods tailored to the intrinsic topological and algebraic structure of NSBS.

Our key contributions may be summarised as follows:

1) We have characterised the limitations of classical spectral theory when separability is absent, particularly the breakdown of global decomposition theorems, the prevalence of residual spectra, and the instability of point spectrum.
2) We proposed a local spectral framework based on separable invariant subspaces, allowing partial spectral recovery through restriction and approximation.
3) We generalised key theorems—such as the Riesz decomposition and perturbation results—by adapting them to non-separable contexts, using local and netwise formulations.
4) We provided explicit examples in classical NSBS (e.g. $\ell^\infty$, $L^\infty$, $\mathcal{C}(\beta\mathbb{N})$ to illustrate non-classical spectral behaviour.
5) We identified new topological properties of the spectrum, including fragmentation, dense residual spectra, and the failure of norm-attainment by the spectral radius.
6) We explored interdisciplinary implications in quantum theory, PDEs, control, and noncommutative geometry, where NSBS naturally arise.

These results establish a coherent foundational framework for extending operator theory into the non-separable realm, while maintaining internal mathematical rigour and external applicability.

### 9.2. Open Questions and Research Directions

Despite the progress made, many fundamental questions remain open, and we identify here several promising directions for future research:

*Global Spectral Bundles and Sheaf-Theoretic Formulations*. Can the local spectra over separable subspaces be organised into a coherent spectral sheaf over the lattice of invariant subspaces of $X$? Such a construction may yield a new global object—analogous to a fibre bundle—encoding the operator's spectral behaviour across non-separable structures.

*Generalised Spectral Measures and Functional Calculi*. Is it possible to define a generalised spectral measure for certain classes of operators on NSBS, perhaps using tools from descriptive set theory or topological measure theory? If so, this would permit an extension of the analytic functional calculus beyond separability, even if only partially.

*Quantitative Invariants of Residual Spectra*. Given the centrality of the residual spectrum in NSBS, can one develop quantitative invariants—e.g. residual spectral dimension, density, or complexity—that classify operators based on their deviation from classical spectral patterns?

*Interpolation and Approximation Schemes*. Are there approximation theorems or interpolation spaces that preserve spectral data in the passage from separable to nonseparable Banach spaces? This would improve the transfer of results from wellunderstood contexts to more general topologies.

*Applications to Operator Algebras and Topological Dynamics*. Can the techniques developed here be employed to study non-separable *C-algebras**, group algebras, or dynamical systems with large state spaces? Such applications may illuminate deeper connections between NSBS and structures in ergodic theory, quantum field theory, or infinite-dimensional symmetries.

In conclusion, the extension of spectral theory to non-separable Banach spaces opens a rich and largely unexplored territory. By identifying both structural challenges and constructive methodologies, this work lays the groundwork for a broader understanding of operator theory in infinite-dimensional and topologically complex settings. Further progress will require new tools, interdisciplinary collaboration, and a willingness to reconsider foundational assumptions in functional analysis.

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