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Special-relativistic Mechanics for Pedestrians, Including a Generalization of Newton's Axiomatic and A Dynnnic Derivation of the Poincare Transformation Zritbout Evoking the Principle of Relativity and Any Non-Mechanical Quantities

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Abstract
The special-relativistic equation of motion is derived from first principles by generalizing Newton's steps from the change of stationary states to the equation of motion. Most important, the change of the mass of a particle is allowed to intrinsically depend on its velocity, while a dependence on time, position, acceleration etc. is nonphysical. (An extrinsic dependence would occur when the external force acting upon it would depend on the velocity.) Generalizations of Newton's definition of momentum and axioms are proposed which enable that. If the mass depends on the velocity and the space is still flat, the Minkowski spacetime inevitably emerges.

When compared with kinematic derivations, this treatment exhibits the advantage to show the applicability of the Poincare transformation to the case of a particle subject to an external force. Even more important, not special relativity needs an additional assumption for introducing the — when compared with classical mechanics — new parameter c. On the contrary, classical mechanics (often implicitly and unconsciously) makes one or more additional assumptions which suppress the appearance of c.

Last but not least, we are not aware of another related text which needs NEITHER the principle of relativity NOR the properties of light, NOR using any non-mechanical quantity.

1. Introduction

One characteristic feature of special relativity consists in that a speed-dependent mass $m(v)$ occurs. Surprisingly enough, already various considerations in Newton's 'Principia' [69] do allow for a generalization towards a velocity-dependent mass. Admittedly, a much more general and powerful treatment is Dirac's 'Forms of Relativistic Dynamics' [14]. However, he uses the principle of relativity, and his transition from Galilean to Lorentz invariance is rather a heuristic one. It suggests c to be added to non-relativistic mechanics to obtain special-relativistic mechanics. We will show that the opposite holds, riz., that within non-relativistic mechanics, assumptions are (implicitly and/or unconsciously) made which suppress the appearance of c. We thus consider this contribution to be a justification of Dirac's approach, too.

Another major advantage consists in that we will need NEITHER the principle of relativity (for certain subtleties, see, e.g. [43] and refs. therein) NOR the properties of light, NOR non-mechanical quantities. Therefore, as first attempted by Ignatowsky in 1910 [52] and

Frank & Rothe [40] one year later, this approach does not rely on Einstein's second postulate about the independence of the speed of light of the velocity of the observer ([16] pp. 900f.). While there is a huge amount of approaches avoiding Einstein's second postulate (see, e.g. [13][63][74][77] and references therein), we are not aware of an approach avoiding the principle of relativity. Cester [9] also has argued that Newton's second axiom allows the mass to be velocity-dependent. This argument is formally correct, though it contradicts the explanation to Definition 2 in the 'Principia' according to which m = const., see Subsection 2.2. For this, a generalization of Newton's Definition 2 will be proposed which frees us from the limitation m = const.

Thus, in Section 2, we will begin with a generalization of Newton's notions of space and time, Definition 2, and the "Axioms or the Laws of Motion". Special attention will be paid to the possibility of a dependence of the mass of a body on its velocity. In Section 3, we will generalize Newton's equation of change of (stationary) states and calculate the velocity-dependence of the mass in a rigorous manner but the numerical value of the reference speed to be used. Section 4 will sketch special-relativistic mechanics, basing on the Poincar"e transformation (Subsection 4.1) and the corresponding generalization of d'Alembert's force of inertia (Subsection 4.2). Moreover, Einstein's famous equation $E = mc^2$ will be derived without non-mechanical quantities (Subsection 4.3), while the Subsections 4.4 f. will deal with the velocity-dependence of the mass, again, and its fatal consequence for Heaviside's gravito-electromagnetic equations, for which we will give a short heuristic derivation. Finally, Section 5 will summarize and conclude this article. It represents a major extension and completion of our earlier treatments [30][32][33][34][82][83]. More important, this fully spatially three-dimensional treatment, (i), is much less involved than fully spatially three-dimensional kinematic ones and, (ii), avoids certain problems occurring in one-dimensional treatments like that noted by Feigenbaum in [39] p. 3.
Altogether, a treatment "carefully excising all that is superfluous and redundant" ([39] p. 1) is striven for.

2. Generalization of Newton's Definition of Momentum and Axioms

"Since the Principia is one of those works everyone talks of but no one reads, anything said about it other than the usual honey-sauced eulogy must stand up against righteous indignation from all sides. But it is a work of science, not a bible. It should be studied and weighed — admired, indeed, but not sworn upon. It has its novelties and its repetitions, its elegant perfections and its errors, its lightning abbreviations and its needless detours, its extraordinary standards of rigor and its logical gaps, its elimination of stated hypotheses and its introduction of unstated ones." (Clifford Ambrose Truesdell [85])

In contrast to Newton's Law (Axiom) 2, his Definition 2 prescribes the mass of a body to equal its constant rest mass, $m = m_0$. This inevitably leads to d'Alembert's [1] Galileo-covariant force of inertia, therefore, to non-relativistic mechanics. The simplest possibility to free us from that constraint consists in omitting the explanation to Definition 2. However, we exploit this opportunity to endeavor a broader generalization of Newton's axiomatic in his Definition 2 and Laws (Axioms). In contrast to Born [7] and Cester et al. [9][10], not any non-mechanical quantity will be used.

"There is an experimental adage to the effect that one should "only adjust one parameter at a time" — the theorist's equivalent is that one should "only adjust one theoretical assumption at a time". Con-trolled restraint in relaxing one's input assumptions is essential if one is to isolate exactly which theoretical aspect is critical to which phenomenological result." (Baccetti, Tate & Visser 2012 [3]).

2.1. Space and Time

For later use, we abandon Newton's discrimination between absolute and relative times ([69] Definitions, Scholium) and add coordinate systems.

Prerequisite 2.1.1 There are continuous and unlimited three-dimensional space and one-dimensional time.

This prerequisite is taken for granted in all branches of physics which use con-tinuous coordinate systems. It does not yet mean that the metric is Euclidean.

Corollary 2.1.1 If there are no matter and fields, then, by the principle of sufficient reason, space and time are homogeneous and isotropic; space is Euclidean. If there is just one particle but still no interaction, the space is homogeneous and isotropic w.r.t. a motion of that particle, its metric is still Euclidean.

Prerequisite 2.1.2 In that cases, space and time may be equipped with arbitrarily many conveyable Cartesian three- (x, y, z) and one-dimensional (t) coordinate systems, respectively.

Corollary 2.1.2 Those coordinate systems can arbitrarily move against each other.

Definition 2.1.1 The coordinate systems, in which the mechanical laws assume their simplest form, are called *inertial*, or *Galilean* coordinate systems *aka* reference frames.

2.2. Newton's Original Definition 2 [geändert]

The original formulation of Newton's Definition 2 is this [69]:

Definition 2 *Quantity of motion* [nowadays 'momentum'] *is a measure of motion that arises from the re/ocity and the quantity of matter jointly.*

. . . if a body is twice as large as another and has equal velocity there is twice as much motion, and [moreover] if it has twice the velocity there is four times as much motion.

(i), the first half of the last sentence means that the momentum is $\vec{p} = m_0\hat{g}(\vec{v})\cdot\vec{v},$ where m_0 is the rest mass of a body and $\hat{g}(\vec{v})$a function yet to be specified. According to Law (Axiom) 2, the momentum is a vector, see Subsection 2.6 below. (ii), there are two possibilities to understand its second half.
1. The body is as large as before. Then, $\vec{p} = m_0\frac{v}{V}\vec{v}$, where V denotes some reference speed. However, inserting that into Newton's Law (Axiom)
2, one obtains $\frac{m_0}{V}\left(\frac{\mathrm{d}v}{\mathrm{d}t}\vec{v} + v\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right) = \vec{F}_{\mathrm{ext}}$, i.e., $v \propto \sqrt{t}$. The hodograph $\vec{v}(t)$ would not be smooth because lim $\lim_{t\to 0}\frac{\mathrm{d}\vec{v}}{\mathrm{d}(t)}(t) = \infty$. Moreover, it would contradict Lemma 10 in Book 1 of the 'Principia' [70], according to which, at the starting point $v = 0$, the non-relativistic equations apply (see also [72] p. 137).
2. The mass of the body is twice as large as before. Then, $\vec{p} = \vec{p}_0 := m_0\vec{v}$, $\hat{g}(\vec{v}) \equiv \hat{I}$.

That and Corollary 1 to the axioms, which holds only if the velocities vectorially add like the forces do, show that the mass ("quantity of matter") is supposed to be independent of the velocity.

Notice that Newton has paid special attention to the case that the body begins to move from the state at rest [70]. Metaphysically, or heuristically, one could argue that there is no influence of the velocity if there is no velocity.

Newton himself has not explored the possibility of a variable mass, at least not in the 'Principia'. His concrete results correspond to a constant mass.

1. A body moves along an ellipsis, if it is subject to a centripetal force ∝ r being located in its centre. (Book I, Sect. 2, Prop. 10 — three-dimensional harmonic oscillator).
2. A body moves along an ellipsis, if it is subject to a centripetal force $\propto 1/r^2$ being located in one of its focuses. (Book I, Sect. 3, Prop. 11 — Kepler ellipsis).

2.3. New Definition 2.

Nonlinear function $\vec{p}(\vec{v})$, or $m = m(v)$

In our generalization of Newton's original Definition 2 above, we will allow for a *nonlinear* relation between momentum $\vec{p}$ and velocity $\vec{v}$, $\vec{p}(\vec{v}) = \hat{g}(\vec{v})\cdot\vec{v}$. This usually treated as an intrinsic, or explicit dependence of the mass of a particle on its velocity; an extrinsic dependence would be caused by a velocity-dependence of the external force.

- There is *no* intrinsic, or explicit dependence of the mass on position and time, because, otherwise, force-free motion would not be uniform.
- A dependence on the momentum, $m = m(\vec{p})$, would be an implicit function $m(m\vec{v})$ of the velocity.
- A dependence on the acceleration or higher-order time derivatives of $\vec{x}(t)$ w.r.t. would contradict Definition 2 in the foregoing subsection and its minimum modification below.
- By the isotropy of space (see Corollary 2.1.2), the possible variation of the mass of an isotropic body is independent of the directions of its motion and external force.

Altogether, the mass m of a body is a scalar quantity which at most may depend on its speed v, i.e., $m(\vec{v}) = m(v)$, $\hat{g}(\vec{v}) = g(v)\hat{I}$ ($\hat{I}$ being a unit matrix).

New Definition 2 *Quantity of motion, or momentum* $\vec{p}$ *is a measure of motion that is the product of the re/ocity* $\vec{v}$ *and the mass m,* $\vec{p}$ *= m* $\vec{v}$*.*

The mass of a body may depend on its speed v.
We will see in Subsection 2.6 below that Newton's original formulation of Law (Axiom) 2, i.e. the corresponding equation of state change (4) allows for a variable mass as

$$\vec{F}_{\mathrm{ext}} = \frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}t}\vec{v} + m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \tag{1}$$

([9] p. 21, eqs. (1.3), (2.1)). Long before, this has been exploited in rocket theory [68]. However, the mass of a rocket changes during its acceleration not by an intrinsic dependence, but by burning the fuel. In his subsequent derivation of special-relativistic mechanics, Cester [9] uses, (i), inspired by Born [7], the radiation pressure and, (ii), the optical Doppler eflect. However, both ones are special-relativistic eflects, so that special relativity has implicitly been sleptin. For the sake of Hertz's program (see Subsection 4.3), *i* The function $g(v)$ exhibits the following obvious properties.
1. The mass of a body is always positive. Hence, $g(v)$ is a *positive* function.
2. As noted above, in the state at rest, there is no influence of the velocity on the mass, because there is no velocity; therefore, $g(0) = 1$.
3. $g(v)$ is dimensionless. For this, there is a reference speed V such that actually $g = g(v/V)$. Without loss of generality, V is independent of *v.* Moreover, V is independent of $m(v)$ as well as space and time, since the mass does so, see above. Hence, V is a *unirersal* constant. In what follows, the dependency on V will be omitted if not needed. t is preferable to stay entirely within mechanics.

2.4. Newton's Original Law (Axiom) 1

Further, the original formulation of Newton's first axiom is this [69]:

Law (Axiom) 1 *Erery body perseveres in its state of being at rest or of moving uniformly foriard, except insofar as it is compelled to change its state by forces impressed.*

In other words, w.r.t. an (at this time formal) Galilean coordinate system with the spatial and temporal variables, free bodies move along straight lines.

Here, "state" means 'stationary state of constant velocity $\vec{v}$'. However, we agree with Sommerfeld [78] in that Newton's three axioms become more coherent if the state is represented by the momentum $\vec{p}$. Then, Law (Axiom) 1 would describe the law of state conservation, while Law 2 (see Subsection 2.6) — the law of state change [22][23][29][34][81]. Accordingly, Newton's equation of conservation of state reads

$$\vec{p} = \vec{p}_0 = m_0\vec{v} = \mathrm{const.} \tag{2}$$

2.5. New Axiom 1. Extended Principle of Relativity

We propose the following generalization.

New Axiom 1. All coordinate systems, which do not rotate relatively to each other, are physically equivalent.

In contrast to the common treatments (notably, [69] Corollary 5 to the axioms, [19] p. 639), this *principle of relativity* is *not* limited to inertial systems.

We will show that it is not needed for the transition from non-relativistic to special-relativistic mechanics through not using it in what follows.
Corollary 2.5.1 By the principle of sufficient reason, w.r.t. an (at this time for-mal) Galilean coordinate system with the spatial and temporal variables, free bodies move along straight lines.

2.6. Newton's Original Law (Axiom) 2

Further, the original formulation of Newton's second axiom is this [69]:

Law (Axiom) 2 *A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.*

As a formula,

$$\Delta \vec{p} := \Delta (m\vec{v}) \propto \vec{F}_{\text{impress}} \,. \tag{3}$$

Ignoring Newton's preceding Denition 2 (see Subsection 2.2), the corresponding differential 'equation of state change' reads (cf. [9])

$$\mathrm{d}\vec{p} = \mathrm{d}(m\vec{v}) = \vec{v}\,\mathrm{d}m + m\,\mathrm{d}\vec{v} = \vec{F}_{\text{impress}}\,\mathrm{d}t \,. \tag{4}$$

It allows for a velocity-dependent mass, see Subsection 2.3 above. However, as described in Subsection 2.2, Newton treats the mass as being an invariant property of a body. Therefore, Newton's dilerential equation of state change reads [29]

$$\mathrm{d}\vec{p} = \mathrm{d}\vec{p}_0 := m_0 \mathrm{d}\vec{v} = \vec{F}_{\text{ext},0}\,\mathrm{d}t \,. \tag{5}$$

Here, we have generalized Newton's "impressed force"$\vec{F}_{\text{impress}}$(3) to the more general external force $\vec{F}$ext,0.

2.7. New Axiom 2: Equilibrium of Forces

Definition 2.7.1 "A body so small that, *for the purpose of our investigation,* the distances between its different parts may be neglected, is called a material particle." (Maxwell [60]).

This article deals solely with the motion of particles, i.e., translations, no rotations.

The fundamental principle of statics consists in the equilibrium of forces (and torques which are not considered here). Following basically d'Alembert [1][46], we are considering dynamics as the equilibrium of the force of inertia $\vec{F}_{\text{inert}}$, external moving forces $\vec{F}_{\text{ext}}$, forces by constraints $\vec{F}_{\text{constr}}$, and fictitious forces $\vec{F}_{\text{fict}}$ in non-inertial frames.

Definition 2.7.2 An external moving force $\vec{F}_{\text{ext}}$ is a quantity which changes the momentum $\vec{p}$ of a particle w.r.t. an inertial system, cf. formula (5).

Definition 2.7.3 The force of inertia of a particle is

$$\vec{F}_{\text{inert}} = -\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \,; \quad \vec{p} := m\vec{v} \,. \tag{6}$$

Recall that the mass m may depend on the speed v of the particle.

New Axiom 2 The sum of all force vectors acting upon a particle always equals the zero vector.

$$\vec{F}_{\text{inert}} + \vec{F}_{\text{ext}} + \vec{F}_{\text{constr}} + \vec{F}_{\text{fict}} = \vec{0} \,. \tag{7}$$

Equation (7) is the fundamental dynamical law of the 'dynamics of discrete mass points' [48].
For a speed-independent mass, $m(v) = m(0) =$ m0, the force of inertia (6) simplies to d'Alembert's expression [1]

$$\vec{F}_{\text{inert}} = \vec{F}_{\text{inert},0} := -m_0\,\vec{a} = -m_0\,\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -m_0\,\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} \,. \tag{8}$$

Inserting that into the right eq. (7) yields Newton's equation of motion,

$$m_0\,\vec{a} = \vec{F}_{\text{ext},0}, \tag{9}$$

first published in Euler's 1736 'Mechanica' [37] in the form $\mathrm{d}\vec{v} = \vec{F}\,\mathrm{d}t/m_0$ (cf. № 154).

2.8. Newton's Original Law (Axiom) 3

Finally, the original formulation of Newton's third law (axiom) reads:

Law (Axiom) 3 *To any action there is a always an opposite and equal reaction; in other words, the actions of two bodies upon each other are a/iays equal and opposite in direction.*

Whatever presses or draws something else is pressed or drawn just as much by it. . . .

It is usually understand as expressing the conservation of the total momentum of two particles during a collision [51][76], or a more general interaction,

$$\vec{p}_1 + \vec{p}_2 = \mathrm{const.} \tag{10}$$

Strictly speaking, that equation is not really an axiom as axioms 1 and 2 are, for it follows from the homogeneity of space by Noether's theorem. However, Newton's explanation also includes the requirement that, during contact interaction, the total *momentum flux density* (pressure) vanishes.

2.9. New Axiom 3: Conservation of Total Momentum and Momentum Flux Density

For this and the sake of completeness, the following generalization is proposed:

New Axiom 3 The total momentum of a system is constant except insofar asa net external force acts upon it. During contact interaction, the total momentum ux density vanishes.

3. Generalization of Newton's Equation of State Change

3.1. Symmetry, or Galilean and Non-Galilean Invariance

All physically correct equations must be invariant (not necessarily covariant!)w.r.t. the symmetry operations of the physical spaces involved, notably, position space and time.

With the function $g(v)$ dened in Subsection 2.3, the momentum changes from $\vec{p}_0 = m_0\vec{v}$ to

$$\vec{p} = m_0\, g(v)\, \vec{v}\,, \tag{11}$$

so that

$$\mathrm{d}\vec{p} = \mathrm{d}[m_0\, g(v)\, \vec{v}] = m_0\, \mathrm{d}[g(v))\, \vec{v}] = \vec{F}_{\mathrm{ext}}\mathrm{d}t\,. \tag{12}$$

Accordingly, the equation of motion changes from

$$\frac{\mathrm{d}\vec{p}_0}{\mathrm{d}t} = m_0\,\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{F}_{\mathrm{ext},0} \quad \text{to} \quad \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = m_0\,\frac{\mathrm{d}\,[g(\vec{v})\,\vec{v}]}{\mathrm{d}t} = \vec{F}_{\mathrm{ext}}\,. \tag{13}$$

Here, while d'Alembert's force of inertia (8) is Galilean invariant, the nond'Alembertian force of inertia

$$\vec{F}_{\mathrm{inert}} = -\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = -m_0\,\frac{\mathrm{d}\,[g(\vec{v})\,\vec{v}]}{\mathrm{d}t} \tag{14}$$

is *not* Galilean invariant. A velocity-dependent mass is connected with a *non-* Galilean transformation of the coordinates. This transformation has to be found. We will calculate it in what follows.

3.2. Calculation of the Function $g(v)$

A most simple way to find that *non*-Galilean transformation consists in constructing a covariant, i.e. *manifest* invariant theory in which the momentum and force of inertia transform in the same manner as the coordinates do,

$$\vec{p} = m_0 \, \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}\,, \quad \vec{F}_{\mathrm{inert}} = -m_0 \, \frac{\mathrm{d}^2\vec{x}}{\mathrm{d}\tau^2}\,. \tag{15}$$

Here, τ is a new time parameter which is invariant against the coordinate transformation sought for. It is assumed to be a scalar because there is not yet any indication to work with more than one time parameter. It will be given in formula (23) below.
Both formulas (11) and (15) represent the *kinetic* momentum $\vec{p}_{\mathrm{kin}}$ of a body. Equating them yields

$$\vec{p}_{\mathrm{kin}} = m_0 \, g(v) \, \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \overset{!}{=} m_0 \, \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}\,; \quad \text{hence,} \quad \mathrm{d}\tau = \frac{\mathrm{d}t}{g(v)}\,. \tag{16}$$

This way, the time *t* changes its character from a universal background parameter to a coordinate ([54] Sect. 11.1).

Now, within a manifest invariant theory, the Lagrangian *L* is derived from an invariant action *S*, i.e., here, from an action which is invariant against the coordinate transformations which leaves dτ (16) invariant. For a force-free particle, we choose (*V* being the universal reference speed in Subsection 2.3). The invariant factor $-m_0 V^2$ guarantees dimensional correctness and compatibility with Lagrange's classical expression $\frac{1}{2} m_0 v^2$, see below.

$$S := -m_0 V^2 \int_{\tau_1}^{\tau_2} \mathrm{d}\tau = -m_0 V^2 \int_{t(\tau_1)}^{t(\tau_2)} \frac{\mathrm{d}t}{g(v)} = \int_{t_1}^{t_2} L \, \mathrm{d}t\,;$$
$$L = -\frac{m_0 V^2}{g(v)} \tag{17}$$

Accordingly, the *canonical* momentum equals

$$\vec{p}_{\mathrm{can}} = \frac{\partial L}{\partial \vec{v}}\,, \quad \text{where} \quad \vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \quad \text{as before.} \tag{18}$$

For a force-free particle, the *kinetic* momentum (16) equals the *canonical* momentum (18),

$$\vec{p}_{\mathrm{kin}} = m_0 \, g(v) \, \vec{v} \overset{!}{=} \vec{p}_{\mathrm{can}} = -m_0 V^2 \, \frac{\mathrm{d}}{\mathrm{d}\vec{v}} \, \frac{1}{g(v)} = \frac{m_0 V^2}{g^2(v)} \, \frac{\mathrm{d}g}{\mathrm{d}v}(v) \, \frac{\vec{v}}{v}\,. \tag{19}$$

That equation is easily simplied to

$$\frac{\mathrm{d}g}{g^3}(v) = \frac{v \, \mathrm{d}v}{V^2}\,. \tag{20}$$

Since $g(0) = 1$ (see Subsection 2.3), the solution is

$$g(v) = \frac{1}{\sqrt{1 - \frac{v^2}{V^2}}}\,. \tag{21}$$

Therefore,

$$L = -m_0 V^2 \sqrt{1 - \frac{v^2}{V^2}}\,; \quad \vec{p}_{\mathrm{can}} = \frac{\partial L}{\partial \vec{v}} = \frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{V^2}}} = \vec{p}_{\mathrm{kin}}\,, \tag{22}$$

see formula (16), as it should be.

4. Special-relativistic Mechanics

4.1. The Poincare Group

Knowing $g(v)$ (21), we are able to determine the coordinate transformations which leave

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{g(v)} = \sqrt{1 - \frac{v^2}{V^2}}\,\mathrm{d}t \tag{23}$$

invariant. Taking $V = c$, the fundamental constant of special relativity (STR),from experiment [87] or the Lorentz force [72], we rewrite its square as

$$(c\,\mathrm{d}\tau)^2 = (c\,\mathrm{d}t)^2 - (\mathrm{d}\vec{x})^2\,. \tag{24}$$

The set of all ane linear differentiable coordinate transformations, which leave the r.h.s. unchanged, is well known to be the Poincare group, the group of Minkowski spacetime isometries [65][66].

Independent of that, the linearity of the Poincare transformation can be reasoned as follows [2][33]. They interrelate coordinate systems to each other. Consider the case of two free bodies w.r.t. two Galilean frames. According to Corollary 2.5.1, both ones move along straight world lines. The physically reasonable coordinate transformations between straight lines are linear, see, e.g. [45] p. 167, [71] p. 554; some subtleties are dealt with in [33] App. A.
Confirming our statement above that $\boldsymbol{V}$ is a universal constant, that linearity implies V to be a *universal* constant in all coordinate systems, in which the action $\boldsymbol{S}$ (17) holds true. This represents another proof of the constancy of the speed of light w.r.t. all inertial systems.

In agreement with the Erlangen program [55], the invariant innitesimal length $cd\tau$ (24) defines one of the two non-Euclidean Minkowski geometries.

Hence, the contra- and covariant 4-coordinates are dierent. We choose (μ =0, 1, 2, 3, 4)

$$x^\mu = (ct, x, y, z), \quad x_\mu = (ct, -x, -y, -z), \quad \text{i.e.,} \quad \eta_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1). \tag{25}$$

This signature $(+---)$ exhibits the advantage of that the proper length s is directly related to the proper time τ as $s = c\tau$. All equations above refer to the contra-variant vector components.
With $V = c$, our function $g(v)$ (21) becomes

$$g(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} =: \gamma\,, \tag{26}$$

the omnipresent Lorentz factor.

4.2. Special-relativistic Equation of Motion and Force of Inertia

Inserting $g(v)$ (26) into eq. (13), one obtains the special-relativistic version of Newton's equation of motion as

$$m_0\,\frac{\mathrm{d}}{\mathrm{d}t}\,(\gamma\,\vec{v}) = \vec{F}_{\mathrm{ext}}\,. \tag{27}$$

Using $\mathrm{d}t = \gamma \mathrm{d}\tau$ (see formulas (23) and (26)), it can be brought into covariantform as

$$m_0\,\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = F^\mu_{\mathrm{ext,\,STR}}\,; \quad \vec{F}_{\mathrm{ext,\,STR}} := \gamma\,\vec{F}_{\mathrm{ext}}\,. \tag{28}$$

Consequently, the four-force of inertia equals

$$F^\mu_{\mathrm{inert}} = -m_0\,\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}\,. \tag{29}$$

Its zeroth component equals

$$F^0_{\rm inert} = -F^0_{\rm ext,STR} = -m_0 c\, \frac{{\rm d}^2 t}{{\rm d}\tau^2} = -\frac{m_0}{c}\, \frac{\vec{v}\cdot\vec{a}}{\left(1 - v^2/c^2\right)^2}\,. \tag{30}$$

Therefore, any special-relativistic expression of the external force $\vec{F}_{\rm ext,}$ STR (28) transforms Poincare-covariant with the position vector $\vec{x}$. Examples are the Lorentz force [72] and its Heaviside's gravito-electromagnetic analogue in their special-relativistic form.

A special-relativistic Lagrangian equation of motion reads

$$\frac{\rm d}{{\rm d}\tau}\frac{\partial L}{\partial \frac{{\rm d}x^\mu}{{\rm d}\tau}} = \frac{\partial L}{\partial x^\mu}. \tag{31}$$

It is not covariant but invariant.

4.3. $E = mc^2$

Einstein [19] has obtained his famous formula $E = mc^2$ ($= m_0\gamma c^2$) from considering the irradiation of light waves by a body. Born [7] and Cester et al. [9][10] have additionally used the relation $p = E/c$ between momentum p and energy $E = h\nu$ of a photon. Goenner [42] evokes the 4-wave vector of a free quantum particle within wave mechanics. In contrast, this contribution adheres to Hertz's [50] program to represent classical mechanics such that all other branches of physics can be derived from it.

For that, we will show that Einstein's formula can be derived *without* using any *non*-mechanical quantities.

Multiplying eq. (27) with dt and $\vec{v}$ yields[14]

$$m_0\, \vec{v}\cdot {\rm d}\left(\gamma\, \vec{v}\right) = \vec{v}\cdot\vec{F}_{\rm ext}\, {\rm d}t = \vec{F}_{\rm ext}\cdot {\rm d}\vec{x} = {\rm d}E_{\rm kin}\,. \tag{32}$$

Inserting formula (26), the l.h.s. becomes

$$\frac{m_0\, c^2}{2}\, \frac{{\rm d}\beta^2}{\left(1-\beta^2\right)^{3/2}} = m_0\, c^2 {\rm d}\frac{1}{\left(1-\beta^2\right)^{1/2}}\,; \quad \beta := \frac{v}{c}\,. \tag{33}$$

Using that and $E_{\rm kin}(\beta = v = 0) = 0$, the integration of eq. (32) is straight forward to yield

$$E_{\rm kin} = m_0\, c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right). \tag{34}$$

It is tempting to assign the two terms on the r.h.s. to two dierent energies. This suggest to interpret the negative of the second, velocity-*in*dependent term m_0c^2 =: Erest to be the energy of a particle at rest. Then,

$$E_{\rm kin} + E_{\rm rest} = \frac{m_0\, c^2}{\sqrt{1-\beta^2}} = m\, c^2 =: E \tag{35}$$

is the total energy of a free particle.

The fact that the conventional formula (18) yields the correct canonical momentum for a free particle suggests the denition of the Hamiltonian $H := \ \vec{v}\cdot\vec{p}_{\rm can} - L$ to yield its energy (cf. [67] (47)),

$$E = \gamma m_0 v^2 + \frac{m_0 c^2}{\gamma} = \gamma m_0 c^2. \tag{36}$$

This corroborates formula (35).

A manifest invariant treatment is this one. Multiplying both sides of theequation of motion (28) by $dx_\mu/d\tau$ leads to

$$m_0 \frac{\mathrm{d}x_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = \frac{m_0}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(\frac{\mathrm{d}ct}{\mathrm{d}\tau}\right)^2 - \left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}\right)^2\right]$$
$$= \frac{\mathrm{d}ct}{\mathrm{d}\tau}F^0_{\mathrm{ext,\,STR}} - \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}\cdot\vec{F}_{\mathrm{ext,\,STR}}\,. \quad (37)$$

The temporal components recover formula (30), the spatial components the spatial components of eq. (28).

4.4 More on the Velocity-Dependent Mass $m(v)$

In his famous article ‘Does the inertia of a body depend on its energy content?’ [19], Einstein concludes (in nowadays notation),
The mass of a body is a measure of its energy content; if the energy
changes by E, the mass changes in the same sense by E/c^2.

That means that the inert and heavy masses equal

$$m_{\mathrm{inert}}(v) = m_{\mathrm{heavy}}(v) = m(v) = \frac{m_0}{\sqrt{1-v^2/c^2}}\,. \quad (38)$$

Later, Einstein has corroborated that view [20].

On the other hand, all Lorentz covariant formulations of a theory contain 4-scalars, 4-vectors, and 4-tensors only. While the rest mass m^0 is a 4-scalar, $m(v)$ (38) is not. And it is not always possible, in Newtonian equations, simply to replace the rest-mass m^0 with the dynamic *aka* observed mass $m(v)$ [57].

The solution to this seeming contradiction consists in the formulation of 4-vectors. The basis is the contra-variant 4-position x^μ (25). From it, the 4-velocity follows as

$$u^\mu := \left(\frac{\mathrm{d}ct}{\mathrm{d}\tau}, \frac{\mathrm{d}x}{\mathrm{d}\tau}, \frac{\mathrm{d}y}{\mathrm{d}\tau}, \frac{\mathrm{d}z}{\mathrm{d}\tau}\right) = \gamma\left(\frac{\mathrm{d}ct}{\mathrm{d}t}, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right) = \gamma(c,\vec{v})\,. \quad (39)$$

Multiplication with m_0 yields the 4-momentum

$$p^\mu := m_0\gamma\,(c,\vec{v}) = (m_0\gamma\,c, m_0\gamma\,\vec{v}) = (m(v)\,c, \gamma\,\vec{p}_0)\,. \quad (40)$$

Thus, $m(v)$ is the temporal component $\frac{1}{c}\,p^0$ of the 4-vector $\frac{1}{c}\,p^\mu$.

Notice that, in contrast to his pioneering article ([16] § 10, p. 919), Einstein [19] does not refer to the longitudinal, $m_{\mathrm{long}} = m_0\gamma^3$, and transverse inert masses, $m_{\mathrm{trans}} = m_0\gamma^2$. There, he remarks that those formulas depend on the denitions of force and acceleration.

The fact that $m(v)$ is not invariant has a fatal consequence for Heaviside’s gravito-electromagnetic equations as we will see next.

4.5 On Heaviside’s gravito-electromagnetic equations

Einstein’s result quoted at the beginning of the foregoing Subsection 4.4 exhibits crucial implications for Heaviside’s gravito-electromagnetic equations [47][58].

Notice that, surprisingly enough, the structure of them can be derived in a rather straightforward manner [36]. An elementary Lorentz covariant theory of real-valued vector elds contains at least a real-valued 4-vector eld amplitude $A^\nu(x^\mu)$ and a Lorentz covariant equation of motion. The simplest non-trivial covariant dierential equation reads

$$\partial_\nu A^\nu(x^\mu) = 0\,. \tag{41}$$

That, however, is not an equation of motion but a constraint. It corresponds to the Lorenz gauge of the electromagnetic potentials.

Another covariant first-order differential equation is

$$\partial^\sigma A^\nu(x^\mu) - \partial^\nu A^\sigma(x^\mu) =: F^{\sigma\nu}(x^\mu)\,. \tag{42}$$

Again, that is not an equation of motion but, this time, the denition of the Faraday tensor and its gravito-electromagnetic analogue, respectively.
Furthermore, there is the Lorentz covariant vector wave equation as an equation of motion,

$$\Box A^\nu(x^\mu) = f^\nu(x^\mu)\,, \tag{43}$$

where $\Box := \partial^\sigma\partial_\sigma$ denotes the d'Alembertian. The r.h.s. is the 4-source, i.e., $f_v = \mu_0 j^v$ for the electromagnetic eld and $f\,v = -(4\pi G/c^2)\,j^v_{\ g}$ for the gravitoelectromagnetic eld, respectively. Here, μ_0 is the magnetic permeability *aka* magnetic constant, G the gravitational constant, and $j_{\mathrm{g}}^\nu = (c\rho_{\mathrm{g}}, \vec{j}_{\mathrm{g}})$ the gravitocurrent 4-density. From that, Heaviside [47] has concluded that gravitational waves propagate with the speed c of light in a vacuum.

By virtue of the Lorenz gauge 41 and the wave equation (43), the gravitocurrent 4-density should obey the equation of continuity,

$$\partial_\nu j_{\mathrm{g}}^\nu = \frac{\partial\rho_{\mathrm{g}}}{\partial t} + \nabla\cdot\vec{j}_{\mathrm{g}} = 0\,. \tag{44}$$

However, that equation requires the source quantity to be a *Lorentz scalar.* While the electrical charge is such one, the heavy mass $m_g(v) = \iiint \rho_g\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ (38) is not. Hence, $j^v_{\ g}$ is only approximately a 4-vector for $v \ll c$ and eq. (44) does not exactly hold. This makes the in-homogeneous gravito-electromagnetic equations (43) to be *not* Lorentz invariant. That and other diculties of the gravito-electromagnetic theory are beyond the scope of this contribution, however (for a short review, see [58]).

Finally, there is the Bianchi identity

$$\partial_\gamma F_{\nu\sigma} + \partial_\nu F_{\sigma\gamma} + \partial_\sigma F_{\gamma\nu} = 0\,. \tag{45}$$

It represents the homogeneous (gravito-) electromagnetic equations. Obviously, they both are Lorentz covariant.

5. Summary and Conclusion

"Die Spezielle Relativitätstheorie hat mit Licht nichts zu tun. Es ist genau umgekehrt. Licht hat mit Relativitätstheorie zu tun! Denn auch das Verhalten von Licht unterliegt den Rahmenbedingungen der Kinematik." (Romano Rupp 2022 [75] p. 97)

En: The special relativity theory has nothing to do with light. It is exactly the other way round. Light has to do with relativity! Because also the behavior of light is subject to the basic conditions of kinematics

We have presented a dynamical derivation of the Poincare transformation starting from exploring how Newton [69] has dealt with the mass of a body. While Newton's Law (Axiom) 2 allows the mass of a body to depend on its velocity, his Definition 2 does not. For this, we have proposed a corresponding generalization of Newton's [69] Definition 2 and Laws (Axioms) of motion. That *ineritably* leads to special-relativistic mechanics but the value c of a universal speed. Hence, special-relativistic mechanics does *not* arise from some require-ment(s) *added* to classical mechanics — on the contrary, classical mechanics contains assumptions which suppress the speed of light *c*.

". . . special relativity is fully determined by the development of Galileo's thoughts. Why and how did it take so long for this to have been realized? . . . the epistemology of the theory is totally decoupled from any knowledge of the behavior of light. . . " (Feigenbaum 2008 [39] p. 29).

“Why hadn’t Galileo determined the full range of systems that embody his thoughts? More seriously, why hadn’t the mathematically superior Newton? Why hadn’t the poser of the brachistochrone, Leibniz, nor various of the Bernoulli? More significantly, why hadn’t the extraordinarily powerful Lagrange? And then why not Lorentz? Unbelievably, why not Poincare? And why did Einstein need to, and always continued to, base it inextricably linked to light?” (Feigenbaum 2008 [39] p. 30).

It [time as a parameter] could as well have been introduced in the nineteenth century, before relativity, motivated by the elegance it brings to the treatment of canonical transformations and other advanced topics. (Oliver David Johns 2005 [54] Introduction to Ch. 11, p. 267). Obviously, there was not any experimental hint to deviate from Newton’s equation of motion and the corresponding Galilean transformation. This resembles the history of the energy law which — mathematically — already was in the hands of Huygens.

This contribution simplifies, improves, and completes our previous contributions on the dynamical derivation of the Lorentz transformation [30][32][81][83]. The dependence of the mass of a body on its velocity is derived in a rigorous manner *without* using any non-mechanical quantities as done, e.g., by Born [7] and Cester *et al.* [9][10]. Planck [72] has exploited the Lorentz transformation properties of the Lorentz force for deriving the Lorentz covariant equation of motion for an electrically charged particle. Here, his pioneering result is comprised in a more general manner.

As within any theory of this kind [80], the numerical values of universal constants in it like the reference speed V cannot be determined by the theory itself. It can be stated, however, that V is a space-time constant and Poincar”e scalar, and so c is. Therefore, as first shown by Ignatowsky [52] and Frank & Rothe [40], it is not necessary to require the speed of light to have one and the same value w.r.t. all inertial systems (Einstein’s second postulate [16] in its second formulation in § 3, pp. 900f.). Feigenbaum even writes, “. . . that the constant speed of light is unnecessary for the construction of the theories of relativity, but overwhelmingly more, there is no room for it in the theory” [39] p. 3. Our treatment confirms the first statement. Rupp stresses that the asymptotic limit c for a body with a rest mass m0 > 0 is a pure/y kinematic rather than dynamical eflect ([75] p. 100; see also [33]).

The consequences of the Poincare transformation when compared with the Galilean transformation, in particular, the synchronization of clocks in the ab-sence of absolute time, is beyond the scope of this article (see, e.g. [39] pp. 15f. for a straight mechanical method). Knowing the Poincare transformation, however, that should be much easier than without knowing it.

Homogeneity and isotropy of space and time when space is empty are not postulated but based on the principle of sufficient reason. Exploiting that and the equality of kinetic and canonical momenta for a force-free body, the com-mon difficulties of the derivation of the Lorentz and Poincare transformations, respectively, in 3 + 1 dimensions are avoided. The possible velocity-dependence of the mass implies the introduction of the new time parameter τ which — con-trary to the ordinary time within non-relativistic mechanics — is not invariant against the Galilean transformation. The set of affine linear diflerentiable coor-dinate transformations, w.r.t. which $\mathrm{d}\tau = \sqrt{1-v^2/c^2}\,\mathrm{d}t$ is invariant, is the set of Poincar”e transformations.

Gomori & Szabo” have claimed that “Lorentz covariance and the principle of relativity are not completely equivalent” ([43] Sect. 5, [84] Abstract). Our view on that issue is rather pragmatical. Any equation is invalid, if both of its sides do not transform in one and the same manner. This makes the purely point-mechanical arguing for the Lorentzian case in [84] (20) fl. doubtful (in contrast to the Galilean description of two point masses connected with a spring in [84] pp. 5fl.). Any expression for the external 3-force $\vec{F}_{\mathrm{ext,STR}}$(28) is correct if and only if it is the spatial part of a 4-vector.

All kinematic derivations of the Lorentz and Poincare transformations, respectively, we are aware of presuppose the *principle of relativity.* In contrast, we do *not* need it because we are not beginning with the transformation between reference frames. This frees us from the subtleties of the relationship between Lorentz covariance and the principle of relativity just mentioned. Instead, we have explored the consequences of a velocity-dependence of the mass of a body for the symmetry of the laws of mechanics. If the mass is allowed to depend on the velocity, the symmetry of the covariant form of the inertial force,

$$F^{\mu}_{\mathrm{inert}} = -m_0\,\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\tau^2}, \quad \mathrm{d}\tau = \sqrt{1-\frac{v^2}{c^2}}\,\mathrm{d}t, \qquad (46)$$

changes from Galilean ($c \to \infty$) to Poincare covariance (0 < c <). However, any coordinate transformation within one reference frame can be interpreted as a coordinate transformation between two reference frames. Many forces used within non-relativistic mechanics, notably, potential forces, are not compatible with special relativity. For this, most probably, a rigorous special-relativistic description of interactions needs /e/d interaction. An alter-native approach may be Lagrangian mechanics with time as a coordinate [54].

The symmetry of a single solution of an equation can be smaller than that of the equation itself. A well-known example is given by the two solutions $x_{1,2} = 1$ to the equation $x^2 = 1$. The set of all solutions (sometimes even some subsets of them) exhibits the symmetry of the equation [31]. Are there solutions to Poincare-invariant equations which are not Poincare-invariant?

When 'Lorentz covariant' is understood as 'manifest invariant' as formulated in terms of four-scalars, -vectors, -tensors, and spinors, Lorentz covariance is sufficient but not necessary; really necessary is Lorentz invariance. Consider, for instance, the Lorentz covariant equation $A^v = B^v$ for two 4-vectors A^v and B^v. Now, if the function $f(x^\mu)$ is not a Lorentz scalar, the equation $f(x^\mu)A^v = f(x^\mu)$Bv is not Lorentz covariant but still Lorentz invariant.

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87. For instance, one can give a particle in inertial system S' the constant velocity u′ and measure its velocity u in inertial system S; V 2 = vu′u/(v + u′ − u).