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Series and Integral Representations of Particular Solutions of the Inhomogeneous Airy's Equations

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Abstract
In this work an overview of the inhomogeneous, classic Airy's equation and the inhomogeneous generalized Airy's equation is provided. When the forcing terms are constant, these equations are directly applicable to modelling variations in permeability and the study of flow through porous media with a transition layer. Particular solutions to these equations are expressed in terms of the Nield-Kuznetsov functions of the first find. When the forcing terms are non-constant continuous functions, particular solutions are expressed in terms of the Nield-Kuznetsov functions of the second kind. An investigation of these functions and some of their properties, representations, and higher derivatives, is carried out in this work. Resulting series and integrals when the forcing functions are the Einstein energy functions are presented to illustrate and identify their computational needs.

Keyword: Inhomogeneous Airy's Equations, Variable Forcing Terms, Generalized Integral Functions

Notation

The following is a list of symbols used in this work.

α_1	Airy particle, $1/[3^{\frac{2}{3}}\Gamma(\frac{2}{3})]$
α_2	Airy particle, $1/[3^{\frac{1}{3}}\Gamma(\frac{1}{3})]$
$\Gamma(.)$	Gamma function
γ_n	Parameter
ζ	Transition variable, $2p(x)^{\frac{1}{2p}}$
λ	Parameter
ρ_n	Parameter, $(p)^{1-p}/\ \Gamma(1-p)$
φ_n	Parameter, $(p)^p/\Gamma(p)$

v	Function of x
χ	Function of x
a_1	Arbitrary constant
b_1	Arbitrary constant
$(b)_k$	Pochhammer symbol, $\Gamma(b+k)/\Gamma(b)$
c	Parameter
j	Imaginary unit $\sqrt{-1}$
k	Derivative order
n	Index
p	Parameter, $1/(n+2)$
$A_i(x)$	Airy's function of the first kind
$B_i(x)$	Airy's function of the second kind
$A_n(x)$	Generalized Airy's function of the first kind
$B_n(x)$	Generalized Airy's function of the second kind
$E_1(x), E_2(x), E_3(x), E_4(x), E_i(x)$	Einstein functions; $i = 1,2,3,4$
f, F	Functions of x
$g(x), h(x), k(x), l(x)$	Power series
$g_{ni}(x)$	Generalized Airy's functions' series representations; $i = 1,2$
$G_i(x), H_i(x)$	Scorer functions
$G_n(x)$	Generalized Scorer function
$L_{i2}(x)$	Di log function
$I_p(\zeta), K_p(\zeta)$	Modified Bessel functions
$K_i(x)$	Standard Nield-Kuznetsov function of the second kind
$K_n(x)$	Generalized Nield-Kuznetsov function of the second kind
$N_i(x)$	Standard Nield-Kuznetsov function of the first kind
$N_n(x)$	Generalized Nield-Kuznetsov function of the first kind
$P_k(x), Q_k(x)$	Polynomials (coefficient functions)
$R_k(x)$	Coefficient function
W, W_1, W_2, W_3, W_4	Wronskians
y, y_n	Functions/Dependent variables
y_c	Complementary solution

1. Introduction

Recent advances in mathematical analysis and computational intelligence sparked renewed, and continued interest in special integral functions arising from classical second order equations whose many applications and roles in the advancement of our scientific knowledge base have stood the test of time, (*cf.* and the references therein) [1-7]. This statement could not be truer when one considers the interest generated for close to two centuries in Airy's and Airy-type equations (equations that describe oscillatory system behaviour), and whose solutions are Airy's functions (functions with both oscillatory and asymptotic/exponential behaviour) [1,5,8,9]. These equations have a wide spectrum of applications in quantum theory, electromagnetism and optics, wave theory, and fluid mechanics [6,11-13]. A large number of equations in mathematical physics can be reduced to Airy-type equations by suitable approximations and changes of variables [12,14-16].

The following static snapshot provides a broad classification of Airy's and Airy-type equations:

1. Airy's equations: This class refers to the classic Airy's equation, and equations that take the following form, [11,12,14-17]

$$y'' \mp \lambda^2 xy = 0 \qquad (1)$$

where λ^2 is a scalar (typically, $\lambda^2 = 1$).

2. Generalized Airy's equations: This class refers to any equation that has as its solution an Airy function or a related function. Examples include:

a) The generalized Airy's equation of index $n \geq -2$, reported in the work of Swanson and Headly [18], and takes the form:The generalization of the Airy differential equation, reported in [12] and [17]:

$$y_n'' - x^n y_n = 0 \qquad (2)$$

b) The Airy-type differential equations that include the following examples, discussed in [1,2,3]:

$$y''' - 4xy' - 2y = 0 \qquad (3)$$

c) The Airy-type differential equations that include the following examples, discussed in [1,2,3]:

Higher-order Airy-type equation:

$$y^{(n-1)} + cxy = 0 \qquad (4)$$

$$\chi^{(n)}\ (x) + \gamma_n\, x\chi\ (x) = 0 \qquad (5)$$

$$x^{(4)} + xy = 0 \qquad (6)$$

Higher-order Scorer-type equation:

$$y^{(n-1)} - cxy = 1 \qquad (7)$$

Equation governing the hyper-Airy functions:

$$y^{(2n)} + (-1)^n\, xy = 0 \qquad (8)$$

Details of the analysis of equations (4)-(8), and conditions on the variables and parameters, can be found in the elegant works reported in [1-3].

Our interest in the current work is the methods of solution of the inhomogeneous forms of Airy's and generalized Airy's equations (1) and (2), written as:

$$y'' - xy = f(x) \qquad (9)$$

$$y_n'' - x_n y_n = f(x) \qquad (10)$$

wherein the forcing function $f(x)$ is a continuous function of its argument.

Abramovich, and Skidmore and Leighton studied the behaviour of solution of equation [8,9]. This equation, however, was first considered by Miller and Mursi who showed that solution to (9) is possible when $f(x)$ is be written in the form [15].

$$f(x) = g(x).y_c + h(x).y_c' \qquad (11)$$

where $g(x)$ and $h(x)$ are expressed as power series. The complementary solution y_c is a linear combination of Airy's functions (see Table 1, below) and the solution to (9) may then be expressed in the same form given by (11), or as a series of derivatives of y_c. When $f(x)$ is a power series it self, solution to (9) can be expressed in the form:

$$y = k(x) + l(x)v(x) \qquad (12)$$

where $v(x) = y_c + \pi G_i(x)$ and $k(x)$ and $l(x)$ are expressed as power series and $G_i(x)$ is the Scorer function (see Table 1, below).

Aside from the Miller and Mursi approach, the literature reports on the following general solutions, Table 1, for equations (9) and (10) [15]. Each solution is presented with its associated references.

Entry #	Equation	General Solution	Ref.
1	$y'' - xy = 0$	$y_c = a_1 A_i(x) + b_1 B_i(x)$	[14,19]
2	$y'' - xy = -1/\pi$	$y = a_1 A_i(x) + b_1 B_i(x) + G_i(x)$	[20]
3	$y'' - xy = 1/\pi$	$y = a_1 A_i(x) + b_1 B_i(x) + H_i(x)$	[20]
4	$y'' - xy = \kappa$	$y = a_1 A_i(x) + [b_1 + \frac{\kappa\pi}{3}]B_i(x) - \kappa\pi G_i(x)$	[21]
5	$y'' - xy = \kappa$	$y = a_1 A_i(x) + [b_1 - \frac{2}{3}\kappa\pi]B_i(x) + \kappa\pi H_i(x)$	[21]
6	$y'' - xy = \kappa$	$y = a_1 A_i(x) + b_1 B_i(x) - \kappa\pi N_i(x)$	[6]
7	$y'' - xy = f(x)$	$y = a_1 A_i(x) + b_1 B_i(x) - \pi K_i(x)$	[22]
8	$y_n'' - x^n y_n = 0$	$y_n = a_n A_n(x) + b_n B_n(x)$	[18]
9	$y_n'' - x^n y_n = \kappa$	$y_n = a_n A_n(x) + b_n B_n(x) - \frac{\kappa\pi}{2\sqrt{p}\sin(p\pi)} N_n(x)$	[23,24]
10	$y_n'' - x^n y_n = -\frac{1}{\pi}$	$y_n = a_n A_n(x) + c_n B_n(x) + \frac{1}{2\sqrt{p}\sin(p\pi)}\left[G_n(x) - \frac{\sin p\pi}{\pi}.\frac{\Gamma(p).\Gamma(2p)}{p^{(3p-2)}} B_n(x)\right]$	[21]

11	$y_n'' - x^n y_n = \frac{1}{\pi}$	$y_n = a_n A_n(x) + c_n B_n(x) - \frac{1}{2\sqrt{p}\sin(p\pi)}\left[G_n(x) - \frac{\sin p\pi}{\pi}.\frac{\Gamma(p).\Gamma(2p)}{p^{(3p-2)}}B_n(x)\right]$	[21]
12	$y_n'' - x^n y_n = f(x)$	$y_n = a_n A_n(x) + b_n B_n(x) - \frac{\pi}{2\sqrt{p}\sin(p\pi)}K_n(x)$	[15,25]

Table 1: General Solutions to Airy's Equations

The integral functions appearing in the solutions, Table 1 above, are summarized in the following Table 2, together with their associated references.

Entry #	**Function**	**Definitions**	**Ref.**
1	Airy functions	$A_i(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(xt + \frac{t^3}{3}\right)dt$ $B_i(x) = \frac{1}{\pi}\int_0^{\infty} [\sin\left(xt + \frac{t^3}{3}\right) + \exp\left(xt - \frac{t^3}{3}\right)]dt$ Wronskian: $W\left(A_i(x), B_i(x)\right) = \frac{1}{\pi}$	[14,19]
2	Scorer functions	$G_i(x) = A_i(x)\int_0^x B_i\,(t)dt + B_i(x)\int_x^{\infty} A_i\,(t)dt$ $H_i(x) = B_i(x)\int_{-\infty}^x A_i\,(t)dt - A_i(x)\int_{-\infty}^x B_i\,(t)dt$	[20]
3	Wronskians	$W_1 = W\left(A_i(x), G_i(x)\right) = A_i(x)G'_i(x) - G_i(x)A'_i(x)$ $W_2 = W\left(A_i(x), H_i(x)\right) = A_i(x)H'_i(x) - H_i(x)A'_i(x)$ $W_3 = W\left(B_i(x), G_i(x)\right) = B_i(x)G'_i(x) - G_i(x)B'_i(x)$ $W_4 = W\left(B_i(x), H_i(x)\right) = B(x)H'_i(x) - H_i(x)B'_i(x)$	[11,12]
4	Standard Nield-Kuznetsov functions	$N_i(x) = A_i(x)\int_0^x B_i\,(t)dt - B_i(x)\int_0^x A_i\,(t)dt$ $K_i(x) = A_i(x)\int_0^x f(t)B_i(t)dt - B_i(x)\int_0^x f(t)A_i(t)dt$	[6,22]

5	Modified Bessel functions	$I_p(\zeta) = (j)^{-p} K_p(j\zeta) = \sum_{m=1}^{\infty} \frac{1}{m!\,\Gamma(m+p+1)} (\frac{\zeta}{2})^{2m+p}$ $K_p(\zeta) = \frac{\pi}{2} \frac{\left(I_{-p}(\zeta) - I_p(\zeta)\right)}{\sin(p\pi)}$ $j = \sqrt{-1};\ p = \frac{1}{n+2}, \zeta = 2p(x)^{\frac{1}{2p}}$, and $\Gamma(.)$ is the gamma function	[11,18]
6	Generalized Airy functions	$A_n(x) = \frac{2p}{\pi} sin(p\pi)(x)^{\frac{1}{2}} K_p(\zeta)$ $B_n(x) = (px)^{\frac{1}{2}} \left(I_{-p}(\zeta) + I_p(\zeta)\right)$ Wronskian: $W\left(A_n(x), B_n(x)\right) = \frac{2}{\pi} p^{\frac{1}{2}} \sin(p\pi)$	[11,18]
7	Airy functions in terms of modified Bessel functions	When $n = 1, p = \frac{1}{n+2} = \frac{1}{3}, \zeta = \frac{2}{3}(x)^{\frac{3}{2}}$, $A_1(x) \equiv A_i(x)\ and\ \ B_1(x) \equiv B_i(x)$ and $A_i(x) = \frac{\sqrt{x}}{3}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right]$ $B_i(x) = \sqrt{\frac{x}{3}}\left[I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right]$	[11,18]
8	Generalized Nield-Kuznetsov functions	$N_n(x) = A_n(x)\int_0^x B_n\,(t)dt - B_n(x)\int_0^x A_n\,(t)dt$ $K_n(x) = A_n(x)\int_0^x f(t)B_n(t)dt$ $- B_n(x)\int_0^x f(t)A_n(t)dt$	[15,23, 24,25]
9	Generalized Scorer function	$G_n(x) = A_n(x)\int_0^x B_n\,(t)dt + B_n(x)\int_x^{\infty} A_n\,(t)dt$ $G_n(x) = N_n(x) + \frac{\sin p\pi}{\pi}.\frac{\Gamma(p).\Gamma(2p)}{p^{(3p-2)}} B_n(x)$	[21]

Table 2: Definitions of Functions Related to Airy's Equations

The general solutions are presented in Table 1 for equations (9) and (10) and expressed in terms of Airy's functions and the Nield-Kuznetsov functions of the second kind, $K_i(x)$ and $K_n(x)$. The forcing terms could be special functions, and their special integrals might be of importance in electromagnetism, particle and radiation physics, or in energy transfer [26]. Some of these forcing functions, $f(x)$, in equations (9) and (10), that will be considered in this work are the Einstein energy functions, and future work will focus on the Whittaker inhomogeneous equation [27]. Some elegant work has been carried out recently on Einstein functions and their relations to other functions and integrals [28-30].

The scope of the current work is to document further properties and representations of K_i (x) and K_n (x) and to investigate the special integrals arising from forcing terms $f(x)$ that are functions such as Einstein functions. A computational procedure for K_n (x) is introduced, and higher derivatives of K_n (x) and the arising generalized Airy's polynomials are discussed. To accomplish this, the manuscript is organized as follows.

In section 2, three different integral representations of each of K_i (x) and K_n (x) are provided, in addition to the modified Bessel function expressions for the main definition of these functions. Moreover, expressions for K_i (x) in terms of primitives of the Scorer functions are given.

In section 3, higher derivatives of K_i (x) and K_n (x) are discussed. Iterative definition of the derivatives of K_n (x) is obtained together with the arising polynomials. Forms and degrees of these polynomials are provided, and dependence of the polynomials on index n is discussed.

In section 4, asymptotic and ascending series representations of K_i (x) are discussed and used in representing the particular solution to equation (9) when the forcing term is any of the four Einstein energy functions. Furthermore, series representations of Airy's generalized functions are used to obtain series representation for $K_n(x)$. Particular solutions to equation (10) are then obtained when the forcing terms are the four Einstein energy functions. Integration of the Einstein functions has been obtained in this work using Wolfram-Alpha [17].

Section 5 is the conclusion, where we discuss what has been accomplished and future aspects of this work. In particular, computation of the integrals and series obtained in section 4 is an ambitious task that is time consuming and requires design and testing of algorithms. This aspect is not provided in this work.

2. Representation of K_i (x) and K_n (x)

2.1 Expressions for K_i (x)

General solution to equation (9) is given by entry 7 in Table 1. The function K_i (x) is given by entry 4 in Table 2 as:

$$K_i(x) = A_i(x)\int_0^x f(t)B_i(t)dt - B_i(x)\int_0^x f(t)A_i(t)dt. \tag{13}$$

Airy's functions A_i (x) and B_i (x) expressed in terms of modified Bessel functions are given as entry 7 in Table 2. Their integrals take the forms:

$\int_0^x A_i(t)dt = \frac{1}{3}\int_0^x \sqrt{t}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}t^{\frac{3}{2}}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}t^{\frac{3}{2}}\right)\right]dt,$	(14)
$\int_0^x B_i(t)dt = \frac{1}{\sqrt{3}}\int_0^x \sqrt{t}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}t^{\frac{3}{2}}\right) + I_{\frac{1}{3}}\left(\frac{2}{3}t^{\frac{3}{2}}\right)\right]dt.$	(15)

Upon using (14) and (15) in (13), we obtain the following expression of K_i (x) in terms of modified Bessel functions:

$$K_i(x) = \frac{2\sqrt{x}}{3\sqrt{3}}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\int_0^x \sqrt{t}\, f(t)\, I_{\frac{1}{3}}\left(\frac{2}{3}t^{3/2}\right)dt - I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\int_0^x \sqrt{t}\, f(t)\, I_{-\frac{1}{3}}\left(\frac{2}{3}t^{3/2}\right)dt\right] \tag{16}$$

Other representations of K_i (x) include using integration by parts in (13) to obtain the following forms:

$$K_i(x) = -\left\{A_i(x)\int_0^x F(t)B_i'(t)dt - B_i(x)\int_0^x F(t)A_i'(t)dt\right\} \tag{17}$$

where $F' \equiv f$ and

$$Ki(x) = f(x)Ni(x) - [Ai(x)\int_0^x \left\{\int_0^t Bi(\tau)d\tau\right\} f'(t)dt - Bi(x)\int_0^x \left\{\int_0^t Ai(\tau)d\tau\right\} f'(t)dt]. \quad (18)$$

Equation (18) expresses the connection between $K_i(x)$ and $N_i(x)$. Relationships between $K_i(x)$ and the Scorer functions can be established by considering the Wronskians (entry 3) in Table 2, and expressing $\int_0^x A_i(t)dt$ and $\int_0^x B_i(t)dt$, as, [12]:

$$\int_0^x A_i(t)dt = \frac{1}{3} - \pi W_1 \quad (19)$$

$$\int_0^x A_i(t)dt = -\frac{2}{3} + \pi W_2 \quad (20)$$

$$\int_0^x B_i(t)dt = -\pi W_3 \quad (21)$$

$$\int_0^x B_i(t)dt = \pi W_4. \quad (22)$$

Using (19)-(22) in equation (18) yields:

$$K_i(x) = -\pi\left\{A_i(x)\int_0^x W_3\, f'(t)dt - B_i(t)\int_0^x W_1\, f'(t)dt\right\} - \frac{1}{3}f(x)B_i(x) \quad (23)$$

$$K_i(x) = \pi\left\{A_i(x)\int_0^x W_4\, f'(t)dt - B_i(t)\int_0^x W_2\, f'(t)dt\right\} + \frac{2}{3}f(x)B_i(x) \quad (24)$$

2.2. Expressions for $K_n(x)$

General solution of equation (10) is given by entry 12 of Table 1, with $K_n(x)$ defined by entry 8 in Table 2 as:

$$K_n(x) = A_n(x)\int_0^x f(t)B_n(t)dt - B_n(x)\int_0^x f(t)A_n(t)dt \quad (25)$$

Integrals of the generalized Airy's functions $A_n(x)$ and $B_n(x)$, given by entry 6 in Table 2, can be written as:

$$\int_0^x A_n(t)dt = p\int_0^x t^{1/2}[I_{-p}\left(2pt^{\frac{1}{2p}}\right) - I_p\left(2pt^{\frac{1}{2p}}\right)]dt, \quad (26)$$

$$\int_0^x B_n(t)dt = \sqrt{p}\int_0^x t^{1/2}\left[I_p\left(2pt^{\frac{1}{2p}}\right) + I_{-p}\left(2pt^{\frac{1}{2p}}\right)\right]dt. \quad (27)$$

Upon using (26) and (27), together with the definitions of $A_n(x)$ and $B_n(x)$ from entry 6 of Table 2, in (25), we obtain the following expressions for $K_n(x)$:

$$K_n(x) = p\sqrt{p}\left(I_{-p}\left(2px^{\frac{1}{2p}}\right) - I_p\left(2px^{\frac{1}{2p}}\right)\right)\int_0^x t^{1/2}f(t)\left[I_p\left(2pt^{\frac{1}{2p}}\right) + I_{-p}\left(2pt^{\frac{1}{2p}}\right)\right]dt \quad (28)$$

$$-p(x)^{\frac{1}{2}}\left(I_{-p}\left(2px^{\frac{1}{2p}}\right) + I_p\left(2px^{\frac{1}{2p}}\right)\right)\int_0^x t^{1/2}f(t)\left[I_p\left(2pt^{\frac{1}{2p}}\right) + I_{-p}\left(2pt^{\frac{1}{2p}}\right)\right]dt.$$

We can express $K_n(x)$, in (25), in the following forms using integration by parts:

$$K_n(x) = -\left\{A_n(x)\int_0^x F(t)B_n'(t)dt - B_n(x)\int_0^x F(t)A_n'(t)dt\right\} \tag{29}$$

where $F' \equiv f$.

$$K_n(x) = f(x)N_n(x) - A_n(x)\int_0^x \left\{\int_0^t B_n(\tau)d\tau\right\} f'(t)dt \tag{30}$$

$$+B_n(x)\int_0^x \left\{\int_0^t A_n(\tau)d\tau\right\} f'(t)dt]$$

Equation (30) expresses the connection between $K_n(x)$ and $N_n(x)$.

3. Higher Derivatives of $K_i(x)$ and $K_n(x)$

Higher derivatives of the Airy functions, their products, and resulting polynomials have received considerable attention in the literature (cf. and the references therein) [31]. Higher derivatives of $K_i(x)$ and resulting polynomials have been discussed in where two approaches were used to obtain generalizations of the derivatives [32]. In what follows, we discuss higher derivatives of $K_n(x)$ using an iterative procedure.

3.1 Higher Derivatives of $K_n(x)$

The function $K_n(x)$ is defined by (25) and has a first derivative given by:

$$K_n'(x) = A_n'(x)\int_0^x f(t)B_n(t)dt - B_n'(x)\int_0^x f(t)A_n(t)dt. \tag{31}$$

Since $K_n(x)$ satisfies the particular solution to the inhomogeneous generalized Airy's equation (10), we have the following expression for its second derivative:

$$K_n''(x) = x^n K_n(x) - f(x)W\big(A_n(x), B_n(x)\big). \tag{32}$$

The following few higher derivatives are obtained by repeated differentiation of (32):

$$K_n'''(x) = x^n K'_n(x) + nx^{n-1}K_n(x) - f'(x)W\big(A_n(x), B_n(x)\big) \tag{33}$$

$$K_n^{iv}(x) = 2nx^{n-1}K'_n(x) + [x^{2n} + n(n-1)x^{n-2}]K_n(x)$$
$$-[f''(x) + x^n f(x)]W\big(A_n(x), B_n(x)\big) \tag{34}$$

$$K_n^{v}(x) = [x^{2n} + 3n(n-1)x^{n-2}]K_n'(x)$$
$$+ [4nx^{2n-1} + n(n-1)(n-2)x^{n-3}]K_n(x) \tag{35}$$
$$-[f'''(x) + x^n f'(x) + 3nx^{n-1}f(x)]W\big(A_n(x), B_n(x)\big)$$

$$K_n^{vi}(x) = [6nx^{2n-1} + 4n(n-1)(n-2)x^{n-3}]K_n'(x)$$
$$+[x^{3n} + 7n(2n-1)x^{2n-2} + n(n-1)(n-2)(n-3)x^{n-4}]K_n(x) \tag{36}$$
$$-[f^{iv}(x)+x^n f''(x) + 4nx^{n-1}f'(x) + \{x^{2n} + 6n(n-1)x^{n-2}\}f(x)] *$$
$$W\big(A_n(x), B_n(x)\big)$$

$$K_n^{vii}(x) = [x^{3n} + 13n(2n-1)x^{2n-2} + 5n(n-1)(n-2)(n-3)x^{n-4}]K_n'(x)$$
$$+[9nx^{3n-1} + 11n(2n-1)(2n-2)x^{2n-3} + n(n-1)(n-2)(n-3)(n-4)x^{n-5}]K_n(x) \quad (37)$$
$$-[f^v(x)+x^n f'''(x) + 5nx^{n-1}f''(x) + \{x^{2n} + 10n(n-1)x^{n-2}\}f'(x) + \{8nx^{2n-1} + 10n(n-1)(n-2)x^{n-3}\}f(x)]W\big(A_n(x), B_n(x)\big)$$

Each of the above derivatives of $K_n(x)$ is expressed in terms of $K_n(x)$ and $K_n'(x)$. The kth derivatives of $K_n(x)$ is thus expressed in the following form:

$$K_n^{(k)}(x) = P_k(x)K_n'(x) + Q_k(x)K_n(x) - R_k(x)W\big(A_n(x), B_n(x)\big), \quad (38)$$

where $P_k(x)$ and $Q_k(x)$ are the polynomial coefficients of $K_n'(x)$, $K_n(x)$, respectively and $R_k(x)$ is the coefficient function of the Wronskian $W(A_n(x), B_n(x))$, in the kth derivative of $K_n(x)$. These coefficients are shown in the Table 3, below.

k	$P_k(x)$	$Q_k(x)$	$R_k(x)$
2	0	x^n	$f(x)$
3	x^n	nx^{n-1}	$f'(x)$
4	$2nx^{n-1}$	$x^{2n} + n(n-1)x^{n-2}$	$f''(x) + x^n f(x)$
5	$x^{2n} + 3n(n-1)x^{n-2}$	$4nx^{2n-1} + n(n-1)(n-2)x^{n-3}$	$f'''(x) + x^n f'(x) + 3nx^{n-1}f(x)$
6	$6nx^{2n-1} + 4n(n-1)(n-2)x^{n-3}$	$x^{3n} + n(11n-7)x^{2n-2} + n(n-1)(n-2)(n-3)x^{n-4}$	$f^{iv}(x) + x^n f''(x) +4nx^{n-1}f'(x) + [x^{2n} +6n(n-1)x^{n-2}]f(x)$
7	$x^{3n} + n(23n-13)x^{2n-2} + 5n(n-1)(n-2)(n-3)x^{n-4}$	$9nx^{3n-1} + 2n(n-1)(13n-11)x^{2n-3} + n(n-1)(n-2)(n-3)(n-4)x^{n-5}$	$f^v(x) + x^n f'''(x) + 5nx^{n-1}f''(x) + [x^{2n} + 10n(n-1)x^{n-2}]f'(x) + [8nx^{2n-1} + 10n(n-1)(n-2)x^{n-3}]f(x)$

Table 3: $P_k(x)$, $Q_k(x)$ and $R_k(x)$

3.2. Iterative Definition of the Higher Derivatives of $K_n(x)$

Degrees of the polynomials $P_k(x)$ and $Q_k(x)$ are shown in Table 4, expressed in terms of the order of the derivative, k and the index, n, of the generalized Airy's equation. These polynomials are needed to define higher derivatives iteratively and are the same polynomial coefficients that arise in higher derivatives of the Airy generalized functions $A_n(x)$ and $B_n(x)$ [21]. The coefficient function $R_k(x)$ may or may not be a polynomial, depending on whether $f(x)$ is a polynomial.

Polynomial	Degree (when k is even)	Degree (when k is odd)
$P_k(x)$	$\left(\frac{k}{2}-1\right)n-1\ ;\ k\geq 4$	$\left(\frac{k-1}{2}\right)n\ ;\ k\geq 3$
$Q_k(x)$	$\left(\frac{k}{2}\right)n\ ;\ k\geq 2$	$\left(\frac{k-1}{2}\right)n-1\ ;\ k\geq 3$

Table 4: Degrees of the Polynomials $P_k(x)$ and $Q_k(x)$

The $(k+1)th$ derivative of $K_n(x)$ takes the following form:

$$K_n^{(k+1)}(x) = P_{k+1}(x)K'_n(x) + Q_{k+1}(x)K_n(x) - R_{k+1}(x)W\big(A_n(x), B_n(x)\big), \quad (39)$$

The polynomial coefficients $P_{k+1}(x)$ and $Q_{k+1}(x)$ and the function $R_{k+1}(x)$ in the $(k+1)th$ derivative are obtained from $P_k(x)$, $Q_k(x)$ and $R_k(x)$, in the kth derivative using the following relationships:

$$P_{k+1}(x) = P'_k(x) + Q_k(x), \quad (40)$$

$$Q_{k+1}(x) = Q'_k(x) + x^n P_k(x), \quad (41)$$

$$R_{k+1}(x) = R'_k(x) + P_k(x)f(x). \quad (42)$$

Equation (39) can thus be written in the following final forms that include the value of the Wronskian, $W(A_n(x), B_n(x))$, given in entry 6 of Table 2:

$$K_n^{(k+1)}(x) = \{P'_k(x) + Q_k(x)\}K'_n(x) + \{Q'_k(x) + x^n P_k(x)\}K_n(x) - \frac{2}{\pi}p^{\frac{1}{2}}\sin(p\pi)\{R'_k(x) + f(x)P_k(x)\}. \quad (43)$$

Equation (43) gives the value $K_n^{(k+1)}(0) = -R_{k+1}(0)W\big(A_n(x), B_n(x)\big)$.

3.3 Dependence of the Coefficients on Index *n*

For a given value of index n, if a term in a coefficient involves x to a negative power, that term is dropped out. For the sake of illustration consider Table 3 when n=3. The coefficients take the forms shown in Table 5, below.

k	$P_k(x)$	$Q_k(x)$	$R_k(x)$
2	0	x^3	$f(x)$
3	x^3	$3x^2$	$f'(x)$
4	$6x^2$	x^6+6x	$f''(x)+x^3f(x)$
5	x^6+18x	$12x^5+6$	$f'''(x)+x^3f'(x)+9x^3f(x)$
6	$18x^5+24$	x^9+78x^4	$f^{iv}(x)+x^3f''(x)+12x^2f'(x)+[x^6+36x]f(x)$

7	$x^9 + 168x^4$	$27x^8 + 336x^3$	$f^v(x) + x^3 f'''(x) + 15x^2 f''(x) + [x^6 + 60x]f'(x) + [24x^5 + 60]f(x)$

Table 5: $P_k(x)$, $Q_k(x)$ and $R_k(x)$ for n=3

4. Series Representations of $K_i(x)$ and $K_n(x)$

4.1. Asymptotic and Ascending Series of $K_i(x)$

The function $K_i(x)$ can be evaluated using the following asymptotic and ascending series expression, wherein $F'(x) = f(x)$. Using the asymptotic expressions for $A_i(x)$ and $B_i(x)$, namely [11,12].

$$A_i(x) \approx \frac{\exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right)}{2\sqrt{\pi}x^{\frac{1}{4}}} \quad (44)$$

and

$$B_i(x) \approx \frac{\exp\left(\frac{2}{3}x^{\frac{3}{2}}\right)}{\sqrt{\pi}x^{\frac{1}{4}}} \quad (45)$$

in (13), gives the following expression that is valid for large values of x:

$$K_i(x) = \frac{\exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right)}{2\pi x^{\frac{1}{4}}}\int_0^x \frac{\exp\left(\frac{2}{3}t^{\frac{3}{2}}\right)}{t^{\frac{1}{4}}}f(t)dt - \frac{\exp\left(\frac{2}{3}x^{\frac{3}{2}}\right)}{2\pi x^{\frac{1}{4}}}\int_0^x \frac{\exp\left(-\frac{2}{3}t^{\frac{3}{2}}\right)}{t^{\frac{1}{4}}}f(t)dt. \quad (46)$$

Ascending series of $K_i(x)$ takes the following form (based on equation (17)), valid for all x, [22]:

$$K_i(x) = 2\sqrt{3}\,\alpha_1\,\alpha_2\left[\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k\frac{3^k x^{3k}}{(3k)!}\right\}\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k\int_0^x F(t)\frac{3^k t^{3k}}{(3k)!}\right\} - \left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k\int_0^x F(t)\frac{3^k t^{3k-1}}{(3k-1)!}\right\}\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k\frac{3^k x^{3k+1}}{(3k+1)!}\right\}\right] \quad (47)$$

wherein the Airy particles α_1 and α_2 are given by:

$$\alpha_1 = A_i(0) = \frac{1}{3^{2/3}\Gamma(\frac{2}{3})} \approx 0.3550280538878172$$
$$\alpha_2 = -A'_i(0) = \frac{1}{3^{1/3}\Gamma(\frac{1}{3})} \approx 0.2588194037928067 \quad (48)$$

$(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = b(b+1)(b+2)\dots(b+k-1); k > 0, (b)_0 = 1,$ is the Pochhammer symbol.

4.2. Series and Integral Representations of $K_n(x)$

Following Swanson and Headley, the generalized Airy's functions are evaluated using the following expressions [18].

Let $p=\frac{1}{n+2}$, $\rho_n=\frac{(p)^{1-p}}{\Gamma(1-p)}$ and $\varphi_n=\frac{(p)^p}{\Gamma(p)}$. Then,

$$g_{n1}(x)=1+\sum_{k=1}^{\infty}p^{2k}\prod_{j=1}^{k}\frac{x^{(n+2)k}}{j(j-p)} \tag{49}$$

$$g_{n2}(x)=x\left[1+\sum_{k=1}^{\infty}p^{2k}\prod_{j=1}^{k}\frac{x^{(n+2)k}}{j(j+p)}\right] \tag{50}$$

$$A_n(x)=\rho_n g_{n1}(x)-\varphi_n g_{n2}(x) \tag{51}$$

$$B_n(x)=\frac{1}{\sqrt{p}}[\rho_n g_{n1}(x)+\varphi_n g_{n2}(x)]. \tag{52}$$

From (49) and (50) we obtain:

$$g_{n1}(t)f(t)=f(t)\left\{1+\sum_{k=1}^{\infty}p^{2k}\prod_{j=1}^{k}\frac{t^{(n+2)k}}{j(j-p)}\right\} \tag{53}$$

$$g_{n2}(t)f(t)=tf(t)\left[1+\sum_{k=1}^{\infty}p^{2k}\prod_{j=1}^{k}\frac{t^{(n+2)k}}{j(j+p)}\right]. \tag{54}$$

Using (51)-(54), equation (25) becomes:

$$K_n(x)=\frac{2}{\sqrt{p}}\rho_n\varphi_n\left[g_{n1}(x)\int_0^x g_{n2}(t)f(t)dt-g_{n2}(x)\int_0^x g_{n1}(t)f(t)dt\right], \tag{55}$$

Or

$$K_n(x)=\frac{2}{\sqrt{p}}\rho_n\varphi_n\left[g_{n1}(x)\int_0^x\left\{tf(t)\left[1+\sum_{k=1}^{\infty}p^{2k}\prod_{j=1}^{k}\frac{t^{(n+2)k}}{j(j+p)}\right]\right\}dt\right. \tag{56}$$
$$\left.-\,g_{n2}(x)\int_0^x\left[f(t)\left\{1+\sum_{k=1}^{\infty}p^{2k}\prod_{j=1}^{k}\frac{t^{(n+2)k}}{j(j-p)}\right\}\right]dt\right]$$

4.3. An Illustration When $f(x)$ is an Einstein Energy Function

The general solutions to Airy's inhomogeneous equations (9) and (10) are given by entries (7) and (12), respectively, of Table 1, with particular solitions given by $-\pi K_i(x)$ and $-\frac{\pi}{2\sqrt{p}\sin(p\pi)}K_n(x)$, respectively. The function $K_i(x)$ can be evaluated using representations (46) and (47), and $K_n(x)$ using representation (56). These are illustrated in what follows using Einstein's energy functions as forcing term, $f(x)$, in Airy's equations (9) and (10).

Einstein's four energy functions, defined on $(0,+\infty)$, are [28,29].

$$E_1(x) = \frac{x}{e^x - 1} \tag{57}$$

$$E_2(x) = \log(1 - e^{-x}) \tag{58}$$

$$E_3(x) = \frac{x}{e^x - 1} - \log(1 - e^{-x}), \tag{59}$$

$$E_4(x) = \frac{x^2 e^x}{(e^x - 1)^2}. \tag{60}$$

4.3.1. Asymptotic Series Evaluations

Substituting (57)-(60) in (46), we obtain the following asymptotic representations for $E_i(x)$, $i = 1,2,3,4$, respectively:

$$K_i(x) = \frac{\exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right)}{2\pi x^{\frac{1}{4}}} \int_0^x \left[\frac{(t)^{\frac{3}{4}} \exp\left(\frac{2}{3}t^{\frac{3}{2}}\right)}{e^t - 1}\right] dt - \frac{\exp\left(\frac{2}{3}x^{\frac{3}{2}}\right)}{2\pi x^{\frac{1}{4}}} \int_0^x \left[\frac{(t)^{\frac{3}{4}} \exp\left(-\frac{2}{3}t^{\frac{3}{2}}\right)}{e^t - 1}\right] dt \tag{61}$$

$$K_i(x) = \frac{\exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right)}{2\pi x^{\frac{1}{4}}} \int_0^x \frac{\exp\left(\frac{2}{3}t^{\frac{3}{2}}\right)}{t^{\frac{1}{4}}} \log(1 - e^{-t})\, dt - \frac{\exp\left(\frac{2}{3}x^{\frac{3}{2}}\right)}{2\pi x^{\frac{1}{4}}} \int_0^x \frac{\exp\left(-\frac{2}{3}t^{\frac{3}{2}}\right)}{t^{\frac{1}{4}}} \log(1 - e^{-t})\, dt \tag{62}$$

$$K_i(x) = \frac{\exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right)}{2\pi x^{\frac{1}{4}}} \int_0^x \exp\left(\frac{2}{3}t^{\frac{3}{2}}\right)\left[\frac{t^{\frac{3}{4}}}{e^t - 1} - \frac{\log(1 - e^{-t})}{t^{\frac{1}{4}}}\right] dt - \frac{\exp\left(\frac{2}{3}x^{\frac{3}{2}}\right)}{2\pi x^{\frac{1}{4}}} \int_0^x \exp\left(-\frac{2}{3}t^{\frac{3}{2}}\right)\left[\frac{t^{\frac{3}{4}}}{e^t - 1} - \frac{\log(1 - e^{-t})}{t^{\frac{1}{4}}}\right] dt \tag{63}$$

$$K_i(x) = \frac{\exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right)}{2\pi x^{\frac{1}{4}}} \int_0^x \left[\frac{t^{\frac{7}{4}} \exp\left(t + \frac{2}{3}t^{\frac{3}{2}}\right)}{(e^t - 1)^2}\right] dt - \frac{\exp\left(\frac{2}{3}x^{\frac{3}{2}}\right)}{2\pi x^{\frac{1}{4}}} \int_0^x \left[\frac{t^{\frac{7}{4}} \exp\left(t - \frac{2}{3}t^{\frac{3}{2}}\right)}{(e^t - 1)^2}\right] dt \tag{64}$$

4.3.2. Ascending Series Evaluations

Ascending series for $K_i(x)$ is given by equation (47). This series requires the function $F(x)$, which is the integral of the forcing term $f(x)$. For the Einstein functions $E_i(x)$, $i = 1,2,3,4$, the improper integrals are evaluated as follows.

$$\int_0^t E_i(\tau)d\tau = \lim_{r\to 0^+}\int_r^t E_i(\tau)d\tau \tag{65}$$

Using Wolfram-Alpha, (17) we obtain the following integrals in terms of the dilog function:

$$\int_0^x E_1(t)dt = \lim_{r\to 0^+}\int_r^x \frac{t}{e^t-1}dt = xlog(1-e^{-x}) - L_{i2}(e^{-x}) - \frac{\pi^2}{6} \tag{66}$$

$$\int_0^x E_2(t)dt = \lim_{r\to 0^+}\int_r^x \log(1-e^{-t})\,dt = L_{i2}(e^{-x}) - \frac{\pi^2}{6} \tag{67}$$

$$\int_0^x E_3(t)dt = \lim_{r\to 0^+}\int_r^x [\frac{t}{e^t-1} - \log(1-e^{-t})]\,dt \tag{68}$$

$$= xlog(1-e^{-x}) - 2L_{i2}(e^{-x}) + \frac{\pi^3}{3}$$

$$\int_0^x E_4(t)dt = \lim_{r\to 0^+}\int_r^x \frac{t^2e^t}{(e^t-1)^2}dt = 2L_{i2}(e^x) + 2\,x\log(1-e^x) + \frac{x^2e^x}{1-e^x} - \frac{\pi^2}{3} \tag{69}$$

Each of integrals (66)-(69) represents $F(t)$ in equation (47). Upon their substitution, in turn, in (47), the following expressions for $K_i(x)$ are obtained, respectively:

$$K_i(x) =$$

$$2\sqrt{3}\alpha_1\alpha_2\left[\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k\frac{3^kx^{3k}}{(3k)!}\right\}\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k\int_0^x\left[xlog(1-e^{-x}) - L_{i2}(e^{-x})\right.\right.\right.$$

$$\left.\left.\left.-\frac{\pi^2}{6}\right]\frac{3^kt^{3k}}{(3k)!}\right\}\right] \tag{70}$$

$$-2\sqrt{3}\alpha_1\alpha_2\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k\int_0^x\left[xlog(1-e^{-x}) - L_{i2}(e^{-x})\right.\right.$$

$$\left.\left.-\frac{\pi^2}{6}\right]\frac{3^kt^{3k-1}}{(3k-1)!}\right\}\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k\frac{3^kx^{3k+1}}{(3k+1)!}\right\}$$

$$K_i(x) = 2\sqrt{3}\alpha_1\alpha_2\left[\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k\frac{3^kx^{3k}}{(3k)!}\right\}\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k\int_0^x\left[L_{i2}(e^{-x}) - \frac{\pi^2}{6}\right]\frac{3^kt^{3k}}{(3k)!}\right\}\right.$$
$$\left.-\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k\int_0^x\left[L_{i2}(e^{-x}) - \frac{\pi^2}{6}\right]\frac{3^kt^{3k-1}}{(3k-1)!}\right\}\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k\frac{3^kx^{3k+1}}{(3k+1)!}\right\}\right] \tag{71}$$

$$K_i(x) = 2\sqrt{3}\alpha_1\alpha_2\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!}\right\}\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \int_0^x \left[x log(1-e^{-x}) - 2L_{i2}(e^{-x}) + \frac{\pi^3}{3}\right]\frac{3^k t^{3k}}{(3k)!}\right\} \tag{72}$$

$$-2\sqrt{3}\alpha_1\alpha_2\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \int_0^x \left[x log(1-e^{-x}) - 2L_{i2}(e^{-x}) + \frac{\pi^3}{3}\right]\frac{3^k t^{3k-1}}{(3k-1)!}\right\}\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!}\right\}$$

$$K_i(x) = 2\sqrt{3}\alpha_1\alpha_2\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!}\right\}\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \int_0^x \left[2L_{i2}(e^x) + 2\,x\log(1-e^x) + \frac{x^2 e^x}{1-e^x} - \frac{\pi^2}{3}\right]\frac{3^k t^{3k}}{(3k)!}\right\} \tag{73}$$

$$-2\sqrt{3}\alpha_1\alpha_2{}^* \left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \int_0^x \left[2L_{i2}(e^x) + 2\,x\log(1-e^x) + \frac{x^2 e^x}{1-e^x} - \frac{\pi^2}{3}\right]\frac{3^k t^{3k-1}}{(3k-1)!}\right\}\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!}\right\}$$

4.3.3. Generalized Series Evaluations

To find expressions for $K_n(x)$ when the forcing terms are Einstein functions, we substitute (57)-(60) in (56) to obtain, respectively, the following integrals and series:

$$K_n(x) = \frac{2}{\sqrt{p}}\rho_n\varphi_n\left[g_{n1}(x)\int_0^x \left\{\frac{t^2}{e^t-1}\left[1 + \sum_{k=1}^{\infty} p^{2k}\prod_{j=1}^{k}\frac{t^{(n+2)k}}{j(j+p)}\right]\right\}dt - g_{n2}(x)\int_0^x \left[\frac{t}{e^t-1}\left\{1 + \sum_{k=1}^{\infty} p^{2k}\prod_{j=1}^{k}\frac{t^{(n+2)k}}{j(j-p)}\right\}\right]dt\right] \tag{74}$$

$$K_n(x) = \frac{2}{\sqrt{p}}\rho_n\varphi_n\left[g_{n1}(x)\int_0^x \left\{t\log(1-e^{-t})\left[1 + \sum_{k=1}^{\infty} p^{2k}\prod_{j=1}^{k}\frac{t^{(n+2)k}}{j(j+p)}\right]\right\}dt - g_{n2}(x)\int_0^x \left[\log(1-e^{-t})\left\{1 + \sum_{k=1}^{\infty} p^{2k}\prod_{j=1}^{k}\frac{t^{(n+2)k}}{j(j-p)}\right\}\right]dt\right] \tag{75}$$

$$K_n(x) = \frac{2}{\sqrt{p}}\rho_n\varphi_n\left[g_{n1}(x)\int_0^x\left\{t\left[\frac{t}{e^t-1} - \log(1-e^{-t})\right]\left[1 + \sum_{k=1}^{\infty}p^{2k}\prod_{j=1}^{k}\frac{t^{(n+2)k}}{j(j+p)}\right]\right\}dt - g_{n2}(x)\int_0^x\left[\left[\frac{t}{e^t-1} - \log(1-e^{-t})\right]\left\{1 + \sum_{k=1}^{\infty}p^{2k}\prod_{j=1}^{k}\frac{t^{(n+2)k}}{j(j-p)}\right\}\right]dt\right] \quad (76)$$

$$K_n(x) = \frac{2}{\sqrt{p}}\rho_n\varphi_n\left[g_{n1}(x)\int_0^x\left\{\frac{t^3e^t}{(e^t-1)^2}\left[1 + \sum_{k=1}^{\infty}p^{2k}\prod_{j=1}^{k}\frac{t^{(n+2)k}}{j(j+p)}\right]\right\}dt - g_{n2}(x)\int_0^x\left[\frac{t^2e^t}{(e^t-1)^2}\left\{1 + \sum_{k=1}^{\infty}p^{2k}\prod_{j=1}^{k}\frac{t^{(n+2)k}}{j(j-p)}\right\}\right]dt\right] \quad (77)$$

5. Conclusion

In this work we provided a review of what has been accomplished to date in solving the inhomogeneous Airy's and generalized Airy's equations when the inhomogeneity is due to a variable function. We investigated and documented further properties and representations of the main functions that appear in the particular solutions, namely $K_i(x)$ and $K_n(x)$, in terms of modified Bessel functions and using asymptotic and ascending series. We also obtained higher derivatives of $K_n(x)$ and the associated generalized Airy's polynomials. In obtaining solutions to the inhomogeneous Airy's equations (9) and (10) with Einstein's energy functions as the forcing terms, special integrals arise together with infinite series the evaluation of which is ambitious. However, this will be left for future work where design and testing of algorithms will be needed.

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