Sequential Tests and Reliability Integral Theory

Fausto Galetto*

Department of Manufacturing Systems & Economics, Politecnico of Turin, Turin, Italy

*Corresponding Author
Fausto Galetto, Department of Manufacturing Systems & Economics, Politecnico of Turin, Turin, Italy.

Citation: Galetto, F. (2025). Sequential Tests and Reliability Integral Theory. *Eng OA, 3*(7), 01-37.

Abstract

We use the data of two papers "Statistical Inference on the Shape Parameter of Inverse Generalized Weibull Distribution" (Zhuang et al.), and "On Designing of Bayesian Shewhart-Type Control Charts for Maxwell Distributed Processes with Application of Boring Machine" (Alshahrani et al.) to compare the above authors findings with ours. From the analysis we get different results: the cause is that they use the Probability Limits of the PI (Probability Interval) as they were the Confidence Limits (Control Limits of the Control Charts, CCs). The Control Limits in the Shewhart CCs are based on the Normal Distribution (Central Limit Theorem, CLT) and are not valid for non-normal distributed data: consequently, the decisions about the "In Control" (IC) and "Out Of Control" (OOC) states of the process are wrong. The Control Limits of the CCs are wrongly computed, due to unsound knowledge of the fundamental concept of Confidence Interval. Minitab and other software (e.g. JMP, SAS) use the "T Charts", claimed to be a good method for dealing with "rare events", but their computed Control Limits of the CCs are wrong. The same happens for the Confidence Limits of the parameters of the distribution involved in the papers (Weibull, Inverse Weibull, Gamma, Binomial, Maxwell). We will show that the Reliability Integral Theory (RIT) is able to solve these problems and the Sequential way of dealing with data.

Keywords: Control Charts, Exponential Distribution, TBE, T Charts, Minitab, JMP, Reliability Integral Theory

MSC:

1. Introduction

Since 1989, the author (FG) tried to inform the Scientific Community about the flaws in the use of *("wrong") quality methods for making Quality* and in 1999 about the GIQA (Golden Integral Quality Approach) showing how to manage Quality during all the activities of the Product and Process Development in a Company, including the Process Management and Control Charts (CC) for Process Control [1,2]. Control Charts (CC) use *sequentially* the collected data to assess if a Production or Service process output is to be considered In Control (IC) or Out of Control (OOC); the decision is very important for taking Corrective Actions (CA), if needed.

To show our Theory we will use some of the data found in the papers [3-5].
But before that we mention the very interesting the statements in the Excerpt 1:

In the recent paper "*Misguided Statistical Process Monitoring Approaches*" by W. Woodall, N. Saleh, M. Mahmoud, V. Tercero-Gómez, and S. Knoth, published in *Advanced Statistical Methods in Process Monitoring, Finance, and Environmental Science*, 2023, We read in the **Abstract**: *Hundreds of papers on flawed statistical process monitoring (SPM) methods have appeared in the literature over the past decade or so. The presence of so many ill-advised methods, and so much incorrect theory, adversely affects the SPM research field. Critiques of some of the various misguided, and/or misrepresented, approaches have been published in the past 2 years in an effort to stem this tide. These critiques are briefly reviewed here.* References…

Excerpt 1. From the paper "Misguided Statistical Process Monitoring Approaches"

We agree with the authors in the excerpt 1, but, nevertheless, they did not realise the problem that we are showing here: *wrong Control Limits in CCs for Rare Events, with data exponentially or Weibull or Maxwell distributed.* Several papers compute "a-scientific" control limits… See References…

We will show that the Test of Hypotheses and the Confidence Intervals (CI) are intimately related and so equivalent for decision making. Using the data in with good statistical methods we give our "reflections on Sequential Methods and Control Charts (CCs)" [3-33].

We will try to state that *several papers* (that are not cited here, but you can find in the "*Garden of flowers*" [24] and some in the Appendix A) compute in an *a-scientific way* (see the formulae in the Appendix C) the Control Limits of CCs for "Individual Measures or Exponential, Weibull, Maxwell and Gamma distributed data", indicated as I-CC (Individual Control Charts); we dare to show, to the Scientific Community, how to compute the *True Control Limits (True Confidence Limits)*. If the author is right, then all the <u>decisions, taken up today, have been very costly to the Companies using those Control Limits</u>; therefore, "*Corrective Actions*" are needed, according to the <u>Quality Principles</u>, because NO "*Preventive Actions*" were taken [1,2, 27-36]: this is shown through the suggested published papers. Humbly, given our strong commitment to Quality [34-58], we would dare to provide the "truth": *Truth makes you free* [hen ("hic et nunc"=here and now)].

On 22nd of February 2024, we found the paper "<u>*Publishing an applied statistics paper: Guidance and advice from editors*</u>" published in *Quality and Reliability Engineering International* (QREI-2024, 1-17) [by C. M. Anderson-Cook, Lu, R. B. Gramacy, L. A. Jones-Farmer, D. C. Montgomery, W. H. Woodall; the authors have important qualifications and Awards]; since I-CC is a part of "*applied statistics*" we think that their hints will help: the authors' sentence "*Like all decisions made in the face of uncertainty, Type I (good papers rejected) and Type II (flawed papers accepted) errors happen since the peer review process is not infallible.*" is very important for this paper: the interested readers can see [34-58].

To let the reader follow our way of approaching the problem of estimation we will use various figures and some data: this is caused by the fact that there are wrong ideas in the literature.

By reading [24] and other papers, the readers are confronted with this type of practical problem: we have a warehouse with two departments
1. in the 1st of them, we have a sample (the "*The Garden of flowers*… in [24]") of "products (*papers*)" produced by various production lines (*authors*)
2. while, in the other, we have some few products produced by the same production line (*same author*)
3. several inspectors (*Peer Reviewers, PRs*) analyse the "quality of the products" in the two departments; the PRs can be the same (but we do not know) for both the departments
4. The *final result*, according to the judgment of the inspectors (PRs), is the following: the products stored in the 1st dept. is good, while the products in the 2nd dept. are defective. It is a very clear situation, as one can guess by the following statement of a PR: "*Our limits* [in the 1st dept.] *are calculated using* <u>standard mathematical statistical *results/methods*</u> as is typical in the vast literature of similar papers [24]." See the <u>standard mathematical statistical *results/methods*</u> in the Appendix A and meditate (see the formulae there)!

Hence, the problem becomes "…the *standard ... methods* as is typical …": are those standards typical methods (in the "*The Garden … in [24]*" scientific?

To understand we need to give now "*Some ideas on Hypothesis Test and The Statistical Hypotheses with the related risks*"; in alternative, you can read [6-21, 25-36].

We define as *statistical hypothesis* a statement about a *population parameter* θ (e.g. the *"true"* mean, the *"true"* shape, the *"true"* variance, the *"true"* reliability, the *"true"* failure rate, …n that we *assume* to exists and has *a value* even though it is *unknown* to us), related to the *statistical model* $F(x|\theta)$ associated with a rand*om variable* (RV) X. The set of all the possible values of the parameter is called the *parameter space* Θ. The goal of a *hypothesis test* is to decide, based on a sample drawn from the population, which value *hypothesized* for the population parameter of the parameter space Θ can be accepted as *true*. Remember: nobody knows the truth…

Generally, two competitive hypotheses are defined, the *null hypothesis* H_0 and the *alternative hypothesis* H_1.

Table 1: Statistical Hypotheses and Risks

UNKNOWN REALITY	Decision taken due the Reliability test data (sample)	
	Accept H_0	Reject H_0
H_0 *True*	correct decision probability 1-α	*Type I error* *risk α*
H_0 *False, i.e.* H_1 *True*	*Type II error* *risk β*	correct decision probability 1-β

A hypothesis testing procedure (or simply a *hypothesis test*) is a *rule (decision criterion)* that specifies

- for which sample values the decision is made to «accept» H_0 as true,
- for which sample values H_0 is rejected and then H_1 is accepted as true.

based on managerial/Statistics which defines

- the *test statistic* (a formula to analyse the data)
- the *critical region* C (rejection region)

to be used for decisions, with the stated risks: *decision criterion*.

The subset of the sample space for which H_0 will be rejected is called *rejection region* (or *critical region*). The complement of the rejection region is called the *acceptance region*.

If θ denotes the population parameter, the general form of the null hypothesis is H_0: {θ∈Θ₀} versus the alternative hypothesis H_1: {θ∈Θ₁}, where Θ_0 is a subset of the *parameter space* Θ and Θ_1 a subset disjoint from Θ_0.; Θ₀∪Θ₁= Θ and Θ₀∩Θ₁=∅; *before* collecting any data, with H_0 we accept a probability *α* of wrong

decision, while with H_1 we accept a probability *β* of wrong decision. A hypothesis test of H_0: {θ∈Θ₀} versus the alternative hypothesis H₁: {θ∈Θ₁} might make one of two types of errors, traditionally named Type I Error and Type II Error; their probabilities are indicated as α and *β*.

If «actually (but we do not know)» H_0: {θ∈Θ₀} is true and the hypothesis test (the *rule, the computed quantity S*, in the figure 1), due to the collected *data*, incorrectly decides to reject H_0 then the test (and the Experimenter, the Manager, the Researcher, the Scholar who follow the rule) makes a Type I Error, whose probability is α. If, on the other hand, «*actually* (but we do not know)» θ∈Θ₁ but the test (the *rule*), due to the collected *data*, incorrectly decides to accept H0 then the test (and the Experimenter, the Manager, the Researcher, the Scholar who follow the rule) makes a Type II Error, whose probability is β.

These two different situations are depicted in the table 1 (for simple parametric hypotheses).

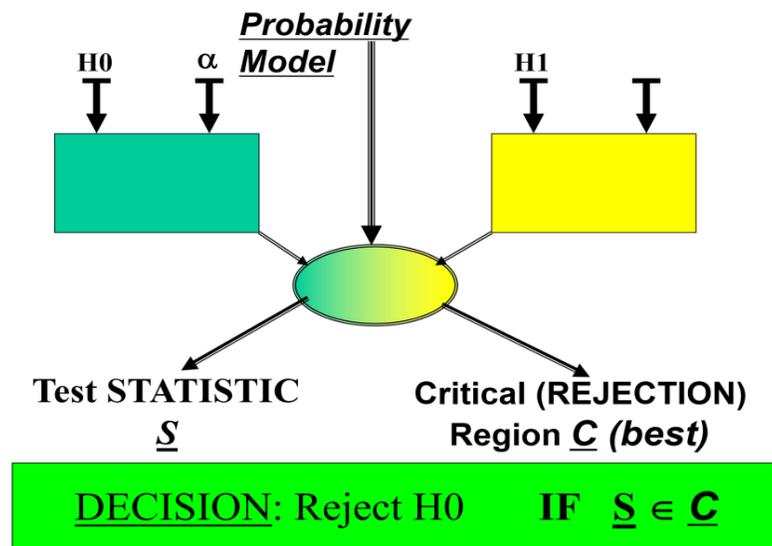The framework of a test of hypothesis is depicted in the figure 1.

Figure 1: The Pictorial Framework of a Statistical Hypothesis (based on a Probability Model)

Notice that when we decide to "*accept the null hypothesis*" in reality we *use a short-hand statement saying that* "*we do not have enough evidence to state the contrary*". It is evident that

$$\alpha = P[reject\ H_0|H_0\ true]\ and\ \beta = P[accept\ H_0|H_0\ false] \qquad (1)$$

A *likelihood ratio test* is any test that has a rejection region of the following form {s(D): q(D)≥c}, where c is any number satisfying 0≤c≤1 and s(D) is the "statistic" by which we elaborate the data of the empirical sample D. This test is a measure of how much the evidence, provided by the data D, supports H_0.

This has great importance for Control Charts, as you can see in the figure 3.

Suppose C is the "critical" (or rejection) region for a test, based on a «*statistic s(D)*» (the formula to elaborate the sampled data D, providing the value *s(D)*.

Then for testing H_0: {θ∈Θ₀}, the test makes a mistake if «*s(D) ϵC*», so that the probability of a Type I Error is α=P(S(D)ϵC) [S(D) is the random variable giving the result s(D)]. It is important the *power of the test 1-β*, which is the probability of rejecting H_0 when *in reality* H_0 is false

$$1 - \beta = P[reject\ H_0|H_0\ false] \qquad (2)$$

Therefore, the *power function* of a hypothesis test with rejection region C is the function of θ defined by β(θ)=P(S(D)ϵC). The function 1-β(θ), *power function*, evaluated at the value θ, is often

named the *Operating Characteristic curve* [*OC curve*].

To find the RV S(D) and the region C, we use the likelihood function L(θ|D={x$_1$, x$_2$, …, x$_n$])

$$L(\theta|D) = \prod_{1}^{n} f(x_i) \qquad (3)$$

Let L$_0$ be the Likelihood function L(θ$_0$|D) and L$_1$ be the Likelihood function L(θ$_1$|D): the most powerful test is the one that has the most powerful critical region C={s(D): q(n)=L$_1$/L$_0 \geq$k$_\alpha$}, where q(n) is the Likelihood Ratio L$_1$/L$_0$ and the quantity k$_\alpha$ is chosen in such a way that the Type I Error has a risk (probability) α as in the formula (4), with fixed *n* (the *sample size*),

$$\int\int \dots\dots \int_{q(n)\geq k_\alpha} L_0 \, dx_1 dx_2 \dots\dots dx_n = \alpha \qquad (4)$$

The most powerful critical region C has the highest power 1-β(θ).

Let CR$_n$ be the "Critical Region" found by (4) and β$_n$ be the probability (5), function of n,

$$\beta_n = \int\int \dots\dots \int_{q(n)\leq CR_n} L_1 \, dx_1 dx_2 \dots\dots dx_n \qquad (5)$$

By (4) and (5), increasing n, we arrive to select a final sample size n, such that β$_n$ = β, the desired risk.

Usually when an efficient estimator exists, this provides then a powerful statistic, giving the most powerful test. [6-33]

We will use *α = β* in the following discussion. *After the data analysis,* we can decide if the data suggest us to "accept (= not reject)" H$_0$: {θεΘ$_0$} or "accept" H$_1$: {θεΘ$_1$},and after that we can compute the Confidence Interval, CI=θ$_L$-------θ$_U$, of the parameter θ, with Confidence Level *CL* = 1 - *α* = 1 - (*α* / 2 + *β* / 2) = 1 - (*α* / 2 + *α* / 2).

When we consider the *Control Charts* we want to test the two Hypotheses H$_0$: {the process is "IC (In Control)"} against H$_1$: { the process is "OOC (Out of Control)"}, and after the data analysis we can compute the *Control Interval* (which is *actually* a Confidence Interval), LCL-------UCL.

If we use the Table 3 data (remission time of 128 bladder cancer patients) it is easy to see that (as said with the above warehouse example) the practical problem becomes a *Theoretical one* [1-58] (all references and figure 21). Since those data are well "exponentially distributed" we anticipate here, immediately, *the wrong formulae* (either using the parameter θ = θ$_0$ or its estimate $\bar{t}_0$, with α= 0.0027) in the formula (6) (as you can find in [24])

$$LCL = \theta_0 ln(1 - \alpha/2) = 0.00135 \, \bar{t}_0$$
$$UCL = \theta_0 ln(\alpha/2) = 6.6077 \, \bar{t}_0 \qquad (6)$$

The readers should understand clearly the *Theoretical and Practical* Difference between L------U (the Probability Interval) and LCL------UCL (the Confidence Interval), pictorially shown in the figure 2: the two lines L and U depends on the parameter θ (to be estimated) and on the two probabilities α and β, while the two *points* L and U depends on the *assumed value* θ$_0$ of the parameter and on the two *chosen* probabilities α and β; *after the data analysis*, we compute the estimate $\bar{t}_0$ of the parameter θ and from that the Confidence Interval LCL------UCL, with Confidence Level *CL* = 1 – *α*. It is clear now the wrong ideas in the formulae (6).
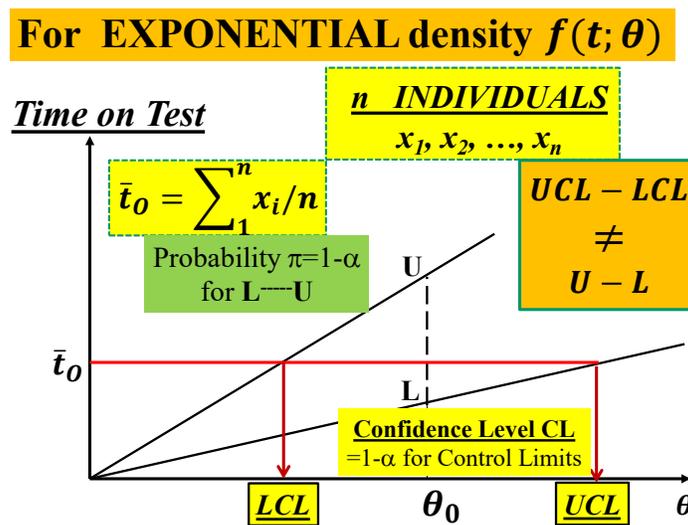
e 2. Theoretical and Practical Difference between L-----U and LCL-----UCL.

Figure 2: Theoretical and Practical Difference between L------U and LCL------UCL

In the formulae (6), for the interval LCL‑‑‑‑‑‑UCL (*named* Control Interval, by the authors [24]), the LCL *ac‑tually* must be L and the UCL *actually* must be U, vertical interval L‑‑‑‑‑‑U (figure 2); the actual interval LCL‑‑‑‑‑‑UCL is the horizontal one in the figure 2, which is not that of the formulae (6). Since the errors have been continuing for at least 25 years, we dare to say that this paper is an *Education Advance* for all the Scholars, for the software sellers and the users: they should study the books and papers in [1-58].

The readers could think that the I-CCs are well known and well dealt in the scientific literature about Quality. We have some doubt about that: we will show that, at least in one field, the I-CC_TBE (with TBE, Time Between Event data) usage, it is not so: there are several published papers, in "scientific magazines and Journals (well appreciated by the Scholars)" with wrong Control Limits; a sample of the involved papers (from 1994 to January 2024) can be found in [23-24]". Therefore, *those authors do not extract the maximum information from the data in the Process Control. "The Garden…"* [24] and the excerpts 1, with the Deming's statements, constitute the Literature Review.

> *"Management need to grow-up their knowledge because experience alone, without theory, teaches nothing what to do to make Quality" "Experience alone, without theory, teaches management nothing about what to do to improve quality and competitive position, nor how to do it.), ... understanding of quality requires education. There is no substitute for knowledge. It is a hazard to copy. It is necessary to understand the theory of what one wishes to do or to make..... hundreds of people are learning what is wrong. .... I make this statement on the basis of experience, seeing every day the devastating effects of incompetent teaching and faulty applications. Again, teaching of beginners should be done by a master, not by a hack".*

Excerpt 2: Some Statements of Deming about Knowledge and Theory (Deming 1986, 1997)

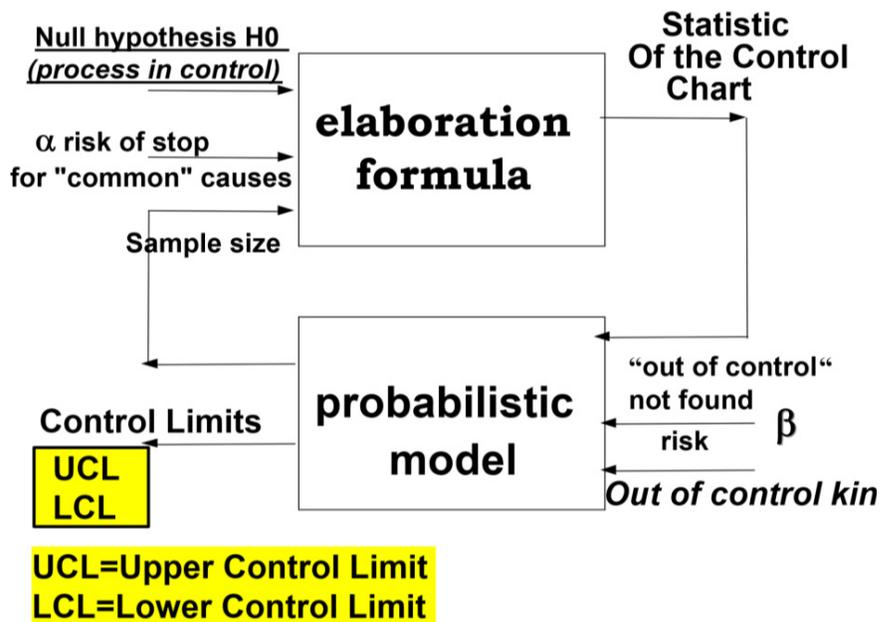We hope that the Deming statements about knowledge will interest the Readers (Excerpt 2).

Figure 3: LCL and UCL of Control Charts with their Risks

The good Managers, Researchers, Scholars do not forget that the two risks always are present and therefore they must take care of the power of the test 1-β, they use for the decision (*as per the principles F1 and F2*) [24-30].

Such Managers, Researchers, Scholars use the Scientific Method. It is important to state immediately and in an explicit way that
➢ *the risks must be stated,*
➢ *together with the goals (the hypotheses),*
➢ *BEFORE any statistical (reliability) test is carried out and data are analysed.*

For demonstration of reliability characteristics, with reliability tests, Managers, Students, Researchers and Scholars must take into account, according the F1 principle, the very great importance of W. E. Deming statements (Excerpt 2): from these, unfortunately for Quality, for the Customers, for the Users and for the Society, this devastating result

➢ *The result is that* <u>*hundreds of people are learning what is wrong*</u>*. I make this statement on the basis of experience, seeing every day the devastating effects of incompetent teaching and faulty applications.*

In many occasions and several *Conferences on Total Quality Management for Higher Education Institutions*, [Toulon (1998), Verona (1999), Derby (2000), Mons (2001), Lisbon (2002), Oviedo (2003), Palermo (2005), Paisley (2006), Florence (2008), Verona (2009)] the author (FG) showed <u>many real cases, found in books and magazines specialized on Quality</u> related to concepts, methods and applications wrong, linked to Quality [21-58]. All the very many documents published (more than 250) by F. Galetto show the profound truth that

facts and figures are useless, if not dangerous, without a sound theory (F. Galetto, 2000), *Brain is the most important asset: let's not forget it*. (F. Galetto, 2003),

All that is particularly important for the analysis of any type of data (quality or reliability).

Sequential sampling

Sequential sampling refers to a routine in which each unit is "measured" about a kind of quantity of interest (length, weight, defectiveness, duration, reliability, failure rate, …) and the "cumulated" quantity is employed in decision taking about the acceptance of the null hypothesis H_0, with $1-\alpha$ as the probability of Accepting H_0, when it is true. At any "measurement" 1, 2, …, k, decision rules are required to provide a decision between three alternatives, a) Acceptance of H_0, b) Rejection of H_0, or c) continuation of sampling (by taking a new unit); this process continues until a decision a) or b) is taken; the number of items then drawn defines the *sample size*: sequential sampling, in general, leads to an expected sample size smaller than other sampling methods.

As seen before, the *likelihood* ratio test is used; the <u>*likelihood ratio test statistic*</u> for testing H_0 versus H_1 is the ratio q(k)=L1(k)/L$_0$(k), where k is the present (variable) sample size, with the rules (after Wald, 1945), a) if $q(k) \leq B \cong \beta / (1 - \alpha)$ then retain H_0, b) if $q(k) \geq A \cong (1 - \beta) / \alpha$ then choose H_1 and reject H_0, if $B \cong \beta /(1 - \alpha) \leq q(k) \leq A \cong (1 - \beta) / \alpha$ then continue sampling. The two quantities A and B are not computed easily: likely, Wald provided the approximations $A \cong (1 - \beta) / \alpha$ and $B \cong \beta / (1 - \alpha)$. These rules, under a suitable transformation of scale, lead to two Decision parallel lines, the *Acceptance line* and the *Rejection line*: the successive points of $q*(k)$ the "transformed value of q(k)" generates a *random walk path*; when the path reaches a decision line the inspection ceases, while

when the path is contained within the two lines, the sampling is continued and new items are tested and measured.

2. Materials and Methods
2.1 A reduced Background of Statistical Concepts
After the ideas given in the Introduction, we provide the following ones essential to understand the "problems related to I-CC and sequential estimation" as we found in the literature. We suggest it for the formulae given and for the difference between the concepts of PI (*Probability Interval*) and CI (<u>*Confidence Interval*</u>): this is overlooked in "The Garden … [24]"

Engineering Analysis is related to the investigation of phenomena underlying products and processes; the analyst can communicate with the phenomena only through the *observed data*, collected with sound experiments (designed for the purpose): any phenomenon, in an experiment, can be considered as a *measurement-generating process* [MGP, a black box that we do not know] that provides us with information about its behaviour through a *measurement process* [MP, known and managed by the experimenter], giving us the observed data (the "message").

It is a law of nature that the data are variable, even in conditions considered fixed, due to many unknown causes.

MGP and MP form the Communication Channel from the phenomenon to the experimenter.

The information, necessarily incomplete, contained in the data, has to be extracted using sound statistical methods (the best possible, if we can). To do that, we consider a *statistical model* $F(x|\theta)$ associated with a *random variable* (RV) X giving rise to the measurements, the "*determinations*" D=$\{x_1, x_2, …, x_n\}$ of the RV, constituting the "*observed sample*" D; n is the sample size. Notice the function $F(x|\theta)$ [a function of real numbers, whose form we assume we know] with the symbol θ accounting for an unknown quantity (or some unknown quantities) that we want to estimate (assess) by suitably analysing the sample D.

We indicate by $f(x|\theta) = dF(x|\theta) / dx$ the pdf (probability density function) and by $F(x|\theta)$ the Cumulative Function, where θ is the set of the parameters of the functions.

We state in the Table 2 a sample of models where θ is a set of parameters:

Name	$F(x\|\theta)$	parameters				Symbol
Exponential	$1 - exp(-x/\theta)$		θ			$E(x\|\theta)$
Weibull	$1 - exp\left[-(x/\eta)^\beta\right]$	β	η			$W(x\|\beta,\eta)$
Inverted Weibull	$1 - e^{-(\eta/x)^\beta}$	β	η			$IW(x\|\beta,\eta)$
General Inverted Weibull	$\left[1 - e^{-(\eta/x)^\beta}\right]^\omega$	β	η	ω		$GIW(x\|\beta,\eta,\omega)$
Modified Power Generalised Weibull	$1 - e^{1-\left(1+(x/\eta)^\beta\right)^\omega} e^{-(x/\theta)}$	β	η	ω	θ	$MPGW(x\|\beta,\eta,\omega,\theta)$
Maxwell	$\dfrac{\sqrt{2/\pi}}{\sigma^3} \displaystyle\int_0^x t^2\, e^{-t^2/(2\sigma^2)} dt$		σ^2			$MW(x\|\sigma)$
Normal	$\dfrac{1}{\sqrt{2\pi}\sigma} \displaystyle\int_0^x e^{-(t-\mu)^2/(2\sigma^2)}\, dt$	μ	σ^2			$N(x\|\mu, \sigma^2)$

Table 2: Some Probability Models for Data Analysis

Two important models are the Normal and the Exponential, but we consider also the others for comparison. When $\theta = \{\mu,\sigma^2\}$ we have the *Normal model*, written as $N(x\|\mu,\sigma^2)$, with (parameters) mean $E[X]=\mu$ and variance $Var[X]=\sigma^2$ with pdf

$$f(x\|\mu,\sigma^2)= n(x\|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \qquad (7)$$

When $\theta = \{\theta\}$ we have *Exponential model*, $E(x\|\theta)$, with (the single parameter) mean $E[X]=\theta = 1/\lambda$ (variance $Var[X]=\theta^2 = 1/\lambda^2$), whose pdf is written in two equivalent ways $f(x\|\theta) = e^{-x/\theta}/\theta = \lambda e^{-\lambda x} = f(x\|\lambda)$.

When we have the *observed sample* $D=\{x_1, x_2, \ldots, x_n\}$, our general problem is to estimate the value of the parameters of the model (representing the parent population) from the <u>information given by the sample</u>. We define some criteria which we require a "good" estimate to satisfy and see whether there exist any "best" estimates. We assume that the parent population is distributed in a form, the model, which is completely determinate but for the value θ_0 of some parameter, e.g. unidimensional, θ, or bidimensional $\theta=\{\mu, \sigma^2\}$, or $\theta=\{\beta,\eta,\omega\}$) as in the $GIW(x\|\beta,\eta,\omega)$, or $\theta=\{\beta,\eta,\omega,\theta\}$) as in the $MPGW(x\|\beta,\eta,\omega,\theta)$.

We seek some function of θ, say $\tau(\theta)$, named *inference function*, and we see if we can find a RV T which can have the following properties: unbiasedness, sufficiency, efficiency. Statistical Theory allows us the analysis of these properties of the estimators (RVs).

We use the symbols $\bar{X}$ and S^2 for the unbiased estimators T_1 and T_2 of the mean and the variance.

Luckily, we have that T_1, in the Exponential model $f(x\|\theta)$, is efficient [6-21, 25-33], and it extracts the total available information from any random sample, while the couple T_1 and T_2, in the <u>Normal model</u>, are jointly sufficient statistics for the inference function $\tau(\theta)=(\mu, \sigma^2)$, so extracting the maximum possible of the total available information from any random sample. The estimators

(which are RVs) have their own "distribution" depending on the parent model $F(x\|\theta)$ and on the sample D: we use the symbol $\varphi(t,\theta,n)$ for that "distribution". It is used to assess their properties. For a given (collected) sample D the estimator provides a value t (real number) named the *estimate* of $\tau(\theta)$, unidimensional.

As said before, a way of finding the estimate is to compute the *Likelihood Function* $L(\theta|D)$ [LF] and to maximise it: the solution of the equation $\partial L(\theta|D)/\partial\theta = 0$ is termed *Maximum Likelihood Estimate* [MLE]. Both are used for sequential tests.

The LF is important because it allows us finding the MVB (*Minimum Variance Bound*, Cramer-Rao theorem) [1-2, 6-16, 26-36] of an unbiased RV T [related to the inference function $\tau(\theta)$], such that

$$Var(T) \geq \frac{[\tau(\theta)]^2}{E\left\{\left[\frac{\partial ln L(\theta|D)}{\partial\theta}\right]^2\right\}} = MVB(T) \qquad (8)$$

The inverse of the MVB(T) provides a *measure of the total available amount of information* in D, relevant to the inference function $\tau(\theta)$ and to the *statistical model* $F(x\|\theta)$.

Naming IT(T) the information extracted by the RV T we have that [6-21, 26-36]
$I_T(T)=1/MVB(T) \Leftrightarrow$ T is an Efficient Estimator.
If T is an *Efficient Estimator* there is no better estimator able to extract more information from D.

The estimates considered before were "*point estimates*" with their properties, looking for the "best" single value of the inference function $\tau(\theta)$.

We recap the very important concept of *Confidence Interval* (CI) and Confidence Level (CL) [6-21, 26-36].

The "*interval estimates*" comprise all the values between τ_L (Lower confidence limit) and τ_U (Upper confidence limit); the CI is defined by the *numerical* interval CI=$\{\tau_L \text{-----} \tau_U\}$, where τ_L and τ_U are two quantities computed from the *observed sample* D: when we make the statement that $\tau(\theta)\epsilon$CI, we accept, before any computation, that, doing that, we can be right, in a long run of applications, $(1-\alpha)\%$=CL of the applications, BUT we cannot know IF we are right in the single application (CL=Confidence Level).

We know, before any computation, that we can be wrong $\alpha\%$ of the times but we do not know when it happens.

The reader must be very careful to distinguish *between* the *Probability Interval* PI=$\{L\text{-----}U\}$, where the endpoints L and U depends on the distribution $\varphi(t,\theta,n)$ of the estimator T (that we decide to use, which *does not depend on the "observed sample" D*) and, on the probability $\pi=1-\alpha$ (that we fix before any computation), as follows by the probabilistic statement (9) [se the figure 2 for the

exponential density, when n=1]

$$P[L \le T \le U] = \int_L^U \varphi(t,\theta,n)dt = 1 - \alpha \tag{9}$$

and the Confidence Interval CI=$\{\tau_L\text{-----}\tau_U\}$ which depends on the "observed sample" D.

Notice that the Probability Interval PI=$\{L\text{-----}U\}$, given in the formula (9), *does not depend* on the data D, as you can pictorially see in fig. 2: L and U are the Probability Limits. Notice that, on the contrary, the Confidence Interval CI=$\{\tau_L\text{-----}\tau_U\}$ *does depend* on the data D, pictorially seen in fig. 2. This point is essential for all the papers in the References.

Shewhart identified this approach, L and U, on page 275 of [19] where he states:

"For the most part, however, we never know $f_\Theta(\Theta,n)$ [this is the symbols of Shewhart for our $\varphi(t,\theta,n)$] in sufficient detail to set up such limit… We usually chose a *symmetrical* range characterised by limits $\overline{\Theta} \pm t\sigma_\Theta$ *symmetrically spaced* in reference to Θ. Tchebycheff's Theorem tells us that the probability P that an observed value of Θ will lie within these *symmetric* limits so long as the quality standard is maintained satisfies the inequality P>1-1/t². We are still faced with the choice of t. Experience indicated that t=3 seems to be an acceptable economic value". See the excerpts 3,…

The Tchebycheff Inequality: IF the RV X is arbitrary with density f(x) and finite variance σ^2 THEN we have the probability $P[|X-\mu| \ge k\sigma] \le 1/k^2$, where $\mu = E[X]$. This is a "Probabilistic Theorem".

It can be transferred into *Statistics*. Let's suppose that we want to determine experimentally the unknown mean μ within a "stated error ε". From the above (Probabilistic) Inequality we have $P[\mu - \varepsilon < X < \mu + \varepsilon] \ge 1 - \sigma^2/\varepsilon^2$; IF $\sigma \ll \varepsilon$ THEN the event $\{|X - \mu| < \varepsilon\}$ is "very probable" in an experiment: this means that the observed value x of the RV X can be written as $\mu - \varepsilon < x < \mu + \varepsilon$ and hence $x - \varepsilon < \mu < x + \varepsilon$. In other words, using x as an estimate of μ we commit an error that "most likely" does not exceed ε. IF, on the contrary, $\sigma \not\ll \varepsilon$, we need n data in order to write $P[\mu - \varepsilon < \overline{X} < \mu + \varepsilon] \ge 1 - \sigma^2/(n\varepsilon^2)$, where $\overline{X}$ is the RV "mean"; hence we can derive $\overline{x} - \varepsilon < \mu < \overline{x} + \varepsilon$., where $\overline{X}$ is the "empirical mean" computed from the data. In other words, using $\overline{x}$ as an estimate of μ we commit an error that "most likely" does not exceed ε. See the excerpts 3, 3a, 3b.

Notice that, when we write $\overline{x} - \varepsilon < \mu < \overline{x} + \varepsilon$, we consider the Confidence Interval CI [6-21, 25-33], and no longer the Probability Interval PI [6-21, 25-33].

These statistical concepts are very important for our purpose when we consider the Sequential tests and the Control Charts, especially with Individual data.

Notice that the error made by several authors [4, 5, 24] is generated

by *lack of knowledge* of the difference between PI and CI [6-21, 25-33]: they think *wrongly* that CI=PI, a diffused disease [4, 5, 24]! They should study some of the books/papers [6-21, 25-33] and remember the Deming statements (excerpt 2).

The Deming statements are important for Quality. Managers, scholars; the professors must learn Logic, Design of Experiments and Statistical Thinking to draw good decisions. The authors must, as well. Quality must be their number one objective: they must learn *Quality methods as well*, using Intellectual Honesty [1-2, 6-21, 25-33]. Using (9), *those authors do not extract the maximum information from the data in the Process Control*. To *extract the maximum information from the data* one needs statistical valid Methods [1-2, 6-21, 25-33].

As you can find in any good book or paper [6-21, 25-33] there is a strict relationship between CI and Test Of Hypothesis, known also as Null Hypothesis Significance Testing Procedure (NHSTP). In Hypothesis Testing (see the Introduction), the experimenter wants to assess if a "thought" value of a parameter of a distribution is confirmed (or rejected) by the collected data: for example, for the mean μ (parameter) of the Normal $n(x|\mu,\sigma^2)$ density, he sets the "null hypothesis" $H_0=\{\mu=\mu_0\}$ and the probability P=α of being wrong if he decides that the "null hypothesis" H_0 is true, when actually it is opposite: H_0 is wrong. When we analyse, at once, the *observed sample* D=$\{x_1, x_2, …, x_n\}$ and we compute the empirical (observed) *mean* $\overline{x}$ and the *empirical* (observed) *standard deviation*

s , we define the *Acceptance interval*, which *is the CI*

$$LCL = \bar{x} - t_{1-\alpha/2}s/\sqrt{n} < \mu < \bar{x} + t_{1-\alpha/2}s/\sqrt{n} = UCL \qquad (10)$$

Notice that the interval (for the Normal model, μ'' assumed)

$$\mu'' - t_{1-\alpha/2}\sigma/\sqrt{n} \overline{} \mu'' - t_{1-\alpha/2}\sigma/\sqrt{n} \qquad (11)$$

is the Probability Interval such that
$P[\mu'' - t_{1-\alpha/2}\sigma/\sqrt{n} < \bar{X} < \mu'' - t_{1-\alpha/2}\sigma/\sqrt{n}] = 1 - \alpha.$

A fundamental reflection is in order: the formulae (10) and (11) tempt the unwise guy to think that he can get the *Acceptance interval*, which is *the CI* [1-23], by substituting the assumed values μ_0, σ_0 of the parameters with the *empirical* (observed) *mean* $\bar{x}$ and *standard deviation s*. *This trick is valid **only** for the Normal distribution*.

The formulae (10) can be used sequentially to test $H_0=\{\mu=\mu_0\}$ versus $H1=\{\mu=\mu_1<\mu_0\}$; for any value $2<k\leq n$; we obtain n-2 CIs,

decreasing in length; we can continue until either μ_1<LCL or UCL<μ_0, or both (verify) μ_1<LCL and UCL<μ_0.

More ideas about these points can be found in [34-58].

In the field of Control Charts, with Shewhart, instead of the formula (10), we use (12)

$$\bar{x} - \frac{z_{1-\alpha/2}s}{c_4\sqrt{n}} < \mu < \bar{x} + \frac{z_{1-\alpha/2}s}{c_4\sqrt{n}} \qquad (12)$$

where the t distribution value $t_{1-\alpha/2}$ is replaced by the value $z_{1-\alpha/2}$ of the Normal distribution, actually $z_{1-\alpha/2} = 3$, and a coefficient c_4 is used to make "unbiased" the estimate of the standard deviation, computed from the information given by the sample.

Actually, Shewhart does not use the coefficient c_4 is as you can see from page 294 of Shewhart book (1931), where $\bar{X}$ is the "Grand Mean", computed from D [named here *empirical* (observed) mean $\bar{x}$], σ is "estimated standard of each sample" (named here s, with sample size n=20, in excerpt 3)

Excerpt 3. From Shewhart book (1931), on page 294

2.2 Control Charts, as Sequential Testing, for Process Management

Statistical Process Management (SPM) entails Statistical Theory and tools used for monitoring any type of processes, industrial or not. The Control Charts (CCs) are the tool used for monitoring a process, to assess its two states: the first, when the process, named IC (In Control), operates under the common causes of variation (variation is always naturally present in any phenomenon) and the second, named OOC (Out Of Control), when the process operates under some *assignable causes of variation*. The CCs, using the observed data, allow us to decide if the process is IC or OOC. CCs are a statistical test of hypothesis for the process null hypothesis

$H_0=\{IC\}$ versus the alternative hypothesis $H_1=\{OOC\}$. Control Charts were very considered by Deming [9-10] and Juran [12] after Shewhart invention [19-20].

We start with Shewhart ideas (see the excerpts 3, 3a and 3b). In the excerpts, $\bar{X}$ is the (experimental) "Grand Mean", computed from D (we, on the contrary, use the symbol $\bar{x}$), σ is the (experimental) "estimated standard of each sample" (we, on the contrary, use the symbol s, with sample size n=20, in excerpts 3a, 3b), $\bar{\sigma}$ is the "estimated mean standard deviation of all the samples" (we, on the contrary, use the symbol $\bar{s}$).

Excerpt 3a. From Shewhart book (1931), on page 89

On page 95, he also states that

"*even when nothing is known about the condition under which the distribution was observed, we find that the average and the standard deviation enable us to estimate … the number of observations lying within any symmetrical range $\bar{X} \pm z\sigma$, where z>1. In fact, the proportion of the total number of observed values within any such limits is always greater than 1-1/z^2. This follows from … Tchebycheff's Theorem.*" He then adds "*we see that no matter what set of n observed values we may have, the number of these values N_z lying within the closed range $\bar{X} \pm z\sigma$ is greater than n(1-1/z^2)*".

$$\bar{X} \pm 3\frac{\sigma}{\sqrt{n}} = 13,540 \pm 3\frac{440}{\sqrt{20}} \qquad \bar{\sigma} \pm 3\frac{\sigma}{\sqrt{2n}} = 423 \pm 3\frac{440}{\sqrt{40}}$$

Excerpt 3b. From Shewhart book (1931), on page 294

So, we clearly see that Shewhart, the inventor of the CCs, *used the data to compute the Control Limits,* LCL (Lower Control Limit, which is the Lower *Confidence* Limit) and UCL (Upper Control Limit, the Upper *Confidence* Limit) both for the mean μ_X (1st parameter of the Normal pdf) and for σ_X (2nd parameter of the Normal pdf). They are considered the limits comprising 0.9973n of the observed data. Similar ideas can be found in [5-21, 25-42] (with Rozanov, 1975, we see the idea that CCs can be viewed as a Stochastic Process).

We invite the readers to consider that *if* one assumes that the process is In Control (IC) and *if* he knows the parameters of the distribution he can test if the assumed known values of the parameters are confirmed or disproved by the data, *then* he does not need the Shewhart Control Charts; it is sufficient to use NHSTP or the Sequential Test Theory!

Remember the ideas in the previous section and compare Excerpts 3, 3a, 3b (where *LCL, UCL depend on the data*) with the following Excerpt 4 (where *LCL, UCL depend on the Random Variables*) and appreciate the profound "logic" difference: this is the cause of the many errors in the CCs for TBE [Time Between Events (see [4, 5, 24]).

Let X_{ij}, i=1, 2, …, j =1, 2, …, n be independent and identically distributed (IID) normal $N(\mu_0, \sigma^2_0)$ r.v.'s, where μ_0 and σ^2_0 are the _specified IC_ mean and variance, respectively. The X_{ij} represents the jth observation from the ith rational subgroup (sample) obtained at the i-th sampling stage, and Y_i denotes the i-th charting statistic based on the i-th sample. When n≥2, at sampling stage i, $Y_i = \bar{X}_i$ the subgroup mean, is typically used for monitoring the process mean, while in case of individual observations (i.e., n=1) $Y_i=X_{i1}\equiv X_i$. The control limits of the standard Shewhart chart (X chart or the $\bar{X}$ chart) are given by **UCL₁=μ_Y+kσ_Y and LCL₁=μ_Y-kσ_Y** (1) where μ_Y and σ_Y are the _specified IC_ mean and standard deviation of the charting statistic Y_i, and k denotes the distance of the control limits from the center line (CL) in the units of the standard deviation of the charting statistic. An OOC signal is triggered when for the first time $Y_i \notin [LCL_1, UCL_1]$. It should be mentioned that in the above setup, the control charts are used to monitor the process in real time, by comparing the value of the i-th charting statistic ($\bar{X}_i$ or X_i) to the control limits. Therefore, as long as an OOC signal is not triggered, samples are continued to be drawn from the process.

The same type of arguments are used in another paper [4] _JQT_, 2017 where the data are Erlang distributed with λ_0 is the scale parameter and the Control Limits LCL and UCL are defined [copying Xie et al.] **erroneously** as

$$P[T_r > UCL] = \alpha_0/2 \text{ and } P[T_r < LCL] = \alpha_0/2.$$

Excerpt 4. From a paper in the "Garden… [24]". Notice that one of the authors wrote several papers…

The formulae, in the excerpt 4, LCL_1 and UCL_1 are actually the Probability Limits (L and U) of the _Probability Interval_ PI in the formula (9), when $\varphi(t,\theta,n)$ is the pdf of the Estimator T, related to the Normal model $F(x; \mu, \sigma^2)$. Using (9), _those authors do not extract the maximum information from the data in the Process Control_. From the Theory [6-36] we derive that the interval $L=\mu_Y - 3\sigma_Y \text{-----} \mu_Y+3\sigma_Y=U$ is the PI such that the RV $Y=\bar{X}$

$$P[\mu_Y -3\sigma_Y \leq Y=\bar{X} \leq \mu_Y +3\sigma_Y]= 0.9973 \quad (12a)$$

and it is not the CI of the mean $\mu=\mu_Y$ [as wrongly said in the Excerpt 4, where actually $(LCL_1\text{-----}UCL_1)=PI$].

The same error is in other books and papers (not shown here but the reader can see in [21-24]).

The data plotted in the CCs [6-21, 25-36] (see the fig. 3) are the means $\bar{x}(t_i)$, determinations of the RVs $\bar{X}(t_i)$, i=1, 2, ..., n (n=number of the samples) computed from the _sequentially_ collected data of the i-th sample $D_i=\{x_{ij}, j=1, 2, ..., k\}$ (k=sample size)}, determinations of the RVs $X(t_{ij})$ at _very close instants_ t_{ij}, j=1, 2, ..., k. In other applications I-CC (see the fig. 3), the data plotted are the Individual Data $x(t_i)$, determinations of the Individual Random Variables $X(t_i)$, i=1, 2, ..., n (n=number of the collected data), modelling the measurement process (MP) of the "Quality Characteristic" of the product: this model is very general because it is able to consider every distribution of the Random Process $X(t)$, as we can see in the next section. From the excerpts 3, 3a, 3b and formula (10) it is clear that Shewhart was using the Normal distribution, as a consequence of the _Central Limit Theorem_ (_CLT_) [6-20, 26-36]. In fact, he wrote on page 289 of his book (1931) "... _we saw that, no matter what the nature of the distribution function of the quality is, the distribution of the arithmetic mean approaches normality rapidly with increase in n_ (his n is our k), _and in all cases the expected value of means of samples of n_ (our k) _is the same as the expected value of the universe_" (CLT in Excerpt 3, 3a, 3b).
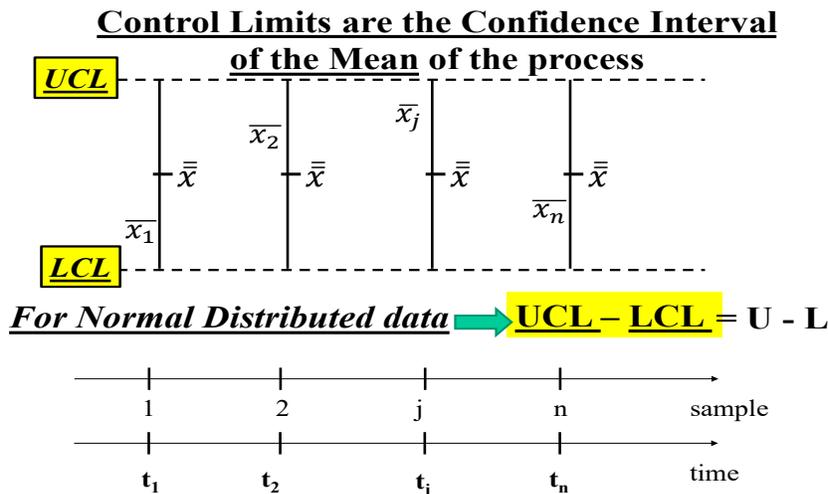

Figure 3: Control Limits $LCL_X\text{-----}UCLX=L\text{-----}U$ (Probability interval), for Normal data (Individuals xij, sample size k) "sample means" $\bar{x}_i$ and "grand mean" $\bar{\bar{x}}$
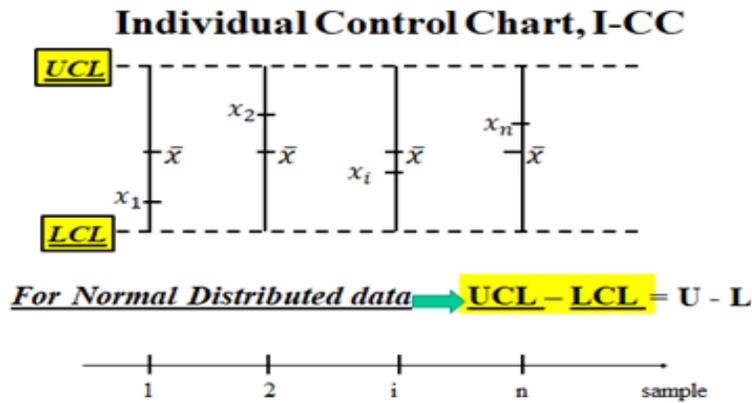
Figure 4: Individual Control Chart (sample size k=1). Control Limits LCL⁻⁻⁻⁻UCL=L⁻⁻⁻⁻U (Probability interval), for Normal data (Individuals xi) and "grand mean" $\bar{x}$

Let k be the sample size; the RVs $\bar{X}(t_i)$ are assumed to follow a normal distribution and uncorrelated; $\bar{X}(t_i)$ [i^{th} rational subgroup] is the mean of RVs IID $X(t_{ij})$ j=1, 2, ..., k, (k data sampled, at very near times t_{ij}).

To show our way of dealing with CCs we consider the process as a "*stand-by system* whose transition times from a state to the subsequent one" are the collected data. The lifetime of "stand-by system" is the sum of the lifetimes of each unit. The process (modelled by a "stand-by …") behaves as a Stochastic Process $X(t)$ [25-33], that we can manage by the Reliability Integral Theory (RIT): see the next section; this method is very general because it is able to consider every distribution of $X(t)$.

If we assume that $X(t)$ is distributed as f(x) [probability density function (pdf) of "transitions from a state to the subsequent state" of a stand-by subsystem] the pdf of the (RV) mean $\bar{X}(t_i)$ is, due the CLT (page 289 of 1931 Shewhart book), $\bar{X}(t_i) \sim N(\mu_{\bar{X}(t_i)}, \sigma^2_{\bar{X}(t_i)})$ [experimental mean $\bar{x}(t_i)$] with mean $\mu_{\bar{X}(t_i)}$ and variance $\sigma^2_{\bar{X}(t_i)}$. $\bar{\bar{X}}$ is the "grand" mean and $\sigma^2_{\bar{\bar{X}}}$ is the "grand" variance: the pdf of the (RV) grand mean $\bar{\bar{X}} \sim N(\mu_{\bar{\bar{X}}}, \sigma^2_{\bar{\bar{X}}})$ [experimental "grand" mean $\bar{\bar{X}}$]. In fig. 2 we show the determinations of the RVs $\bar{X}(t_i)$ and of $\bar{\bar{X}}$.

When the process is Out of Control (OOC, *assignable causes of variation*, some of the means $\mu_{\bar{x}(t_i)}$, estimated by the experimental means $\bar{x}_i = \bar{x}(t_i)$, are "statistically different)" from the others [6-21, 25-36]. We can assess the OOC state of the process via the Confidence Intervals (provided by the Control Limits) with CL=0.9973. **Remember** the *trick valid __only__ for the Normal Distribution* ….; consider the **L**=μ_Y-3σ_Y-----μ_Y+3σ_Y=**U**; putting $\bar{\bar{x}}$ in place of μ_Y and $\bar{s}/\sqrt{k}$ in place of σ_Y we get the CI of $\mu_{\bar{\bar{x}}}$ when the sample size k is considered for each $\bar{X}(t_i)$, with CL=0.9973. The quantity $\bar{s}$ is the mean of the standard deviations of each sample. This allows us to compare each (subsystem) mean $\mu_{\bar{x}(t_q)}$, q=1,2, …, n, to any other (subsystem) mean $\mu_{\bar{x}(t_r)}$ r=1,2, …, n, and to the (Stand-by system) grand mean $\mu_{\bar{\bar{x}}} = \mu$. If two of them are different, the process is classified as OOC. The quantities $LCL_X = \bar{\bar{x}} - 3\bar{s}/\sqrt{k}$ and $UCL_X = \bar{\bar{x}} + 3\bar{s}/\sqrt{k}$ are

the Control Limits of the CC, which are the *Confidence Limits*. When the Ranges R_i=max(x_{ij})-min(x_{ij}) are considered for each sample we have $LCL_X = \bar{\bar{x}} - A_2\bar{R}$, $UCL_X = \bar{\bar{x}} + A_2\bar{R}$ and $LCL_R = D_3\bar{R}$, $UCL_R = D_4\bar{R}$, where $\bar{R}$ is the "mean range" and the coefficients A_2, D_3, D_4 are tabulated and depend on the sample size k [6-21, 25-36].

We stress that the interval LCL_X-------UCL_X is the "*Confidence Interval*" with "Confidence Level" CL=1-α=0.9973 for the unknown mean μ_X(t) of the Stochastic Process X(t) [25-36]. The interval LCL_R----------UCL_R is the "Confidence Interval" with "Confidence Level" CL=1-α=0.9973 for the unknown Range of the Stochastic Process X(t) [25-36].

Notice that, **ONLY** for underlined normally distributed data, the length of the Control Interval (UCL_X-LCL_X, which is the Confidence Interval) equals the length of the Probability Interval, PI (U-L): **UCL_X-LCL_X=U-L**.

The error highlighted, i.e. the *confusion between* the Probability Interval and the Control Limits (Confidence Interval!) has *no consequences* for decisions *when* the data are *Normally distributed*, as considered by Shewhart. On the contrary, it has BIG consequences for decisions **WHEN the data are Non-Normally distributed** [4, 5,24].

We think that the paper "Quality of Methods for Quality is important", [1] appreciated and mentioned by J. Juran at the plenary session of the EOQC (European Organization for Quality Control) Conference (1989), should be considered and meditated.

2.3 Statistics and Reliability Integral Theory (RIT)
We are going to present the fundamental concepts about RIT (Reliability Integral Theory) that we use for computing the Control Limits (*Confidence* Limits) of CCs. RIT is the natural way for Sequential Tests, because the transitions happen *sequentially*; to provide the ideas, we use a "4 units Stand-by system", depicted by 5 states (figure 5): 0 is the state with all units not-failed; 1 is the state with the first unit failed; 2 is the state with the second unit

failed; and so on, until the system enters the state 5 where all the 4 units are failed (down state, in yellow): any transition provides a datum to be used for the computations. RIT can be found in the author's books…
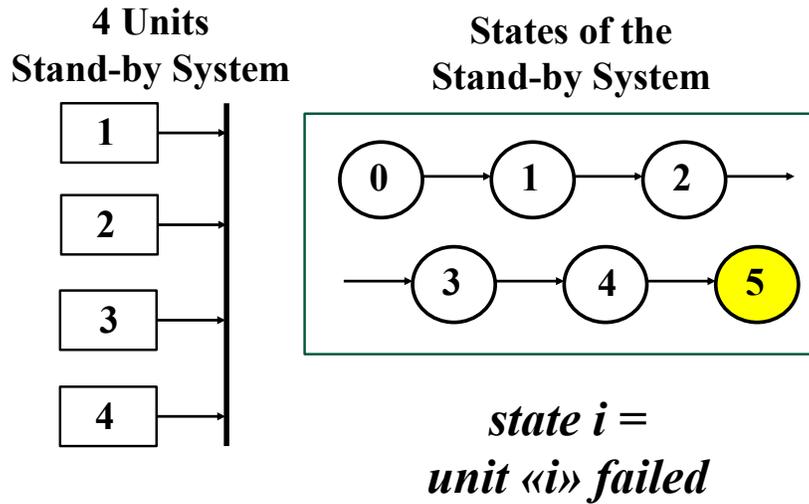

Figure 5: A "4 Units Stand-by System" and its States

RIT can be used for parameters estimation and Confidence Intervals (CI), (Galetto 1981, 1982, 1995, 2010, 2015, 2016), in particular for Control Charts (Deming, 1986, 1997, Shewhart 1931, 1936, Galetto 2004, 2006, 2015). In fact, any Statistical or Reliability Test can be depicted by an "Associated Stand-by System" [25-36] whose transitions are ruled by the kernels $b_{k,j}(s)$; we write the *fundamental system of integral equations for the reliability tests*, whose duration t is related to interval $0^{-----}t$; the collected data tj can be viewed as the times of the various failures (of the units comprising the System) [$t_0=0$ is the start of the test, t is the end of the test and g is the number of the data (4 in the figure 5)]

Firstly, we assume that the kernel $b_{j,j+1}(s - t_j)$ is the pdf of the exponential distribution $f(s - t_j \,|\, \mu, \sigma^2) = \lambda e^{-\lambda(s-t_j)}$ where λ is the failure rate of each unit and $\lambda = 1 / \theta$: θ is the MTTF of each unit. We state that $R_j(t - t_j)$ is the probability that the stand-by system does not enter the state g (5 in fig. 5), at time t, when it starts in the state j (0, 1, …, 4) at time t_j, $\overline{W}_j(t - t_j)$ is the probability that the system does not leave the state j, $b_{j,j+1}(s - t_j)ds$ is the probability that the system makes the transition j→j+1, in the interval $s^{-----}s+ds$.

The system reliability $R_0(t)$ is the solution of the mathematical system of the Integral Equations (13)

$$R_j(t - t_j) = \overline{W}_j(t - t_j) + \int_{t_j}^{t} b_{j,j+1}(t - t_j)R_{j+1}(t - s)ds \tag{13}$$

$$\text{for } j = 0, 1, \dots,, \qquad g - 1, \ R_g(t|t_g) = \overline{W}_g(t|t_g)$$

With $\lambda e^{-\lambda(s-t_j)}$ we obtain the solution (see fig. 5, putting the Mean Time To Failure MTTF of each unit = θ, $\lambda = 1/\theta$) (see the figure 6)

$$R_0(t) = e^{-\lambda t}\left[1 + \lambda t + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^4}{4!}\right] \tag{13a}$$

The reliability system (13) can be written in matrix form,

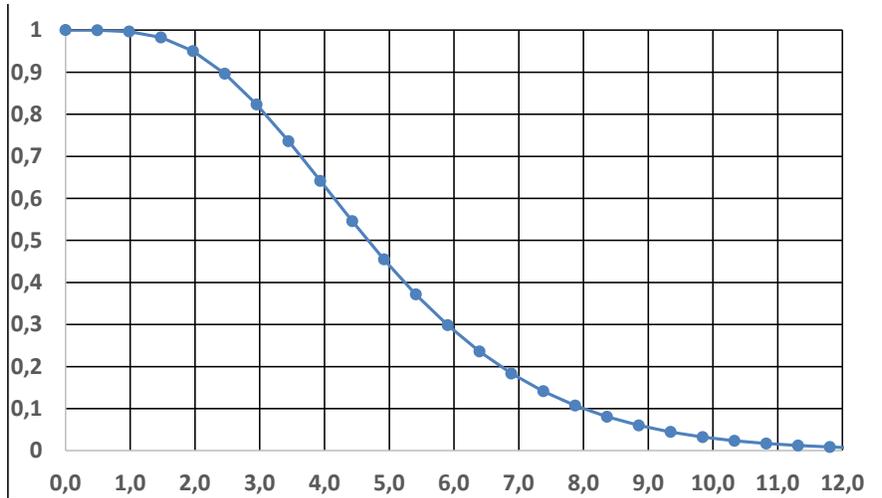$$R(t - r) = \overline{W}(t - r) + \int_{r}^{t} B(s - r)R(s)ds \tag{14}$$

Figure 6: Example of Reliability R_0 (λt_0) of a "4 units Stand-by system" with MTTF=θ=123 days; t_0 is the total time on test of the 4 units

To compute the CI (with CL=0.8), find the abscissas of the intersections at R_0 $(\lambda_L\, t_0)$=0.9 and R_0 $(\lambda_U\, t_0) = 0.1$ …

At the end of the reliability test, at time t, we know the data (the times of the transitions t_j) and the "*observed*" *empirical sample* D={x_1, x_2, …, x_g}, where $x_j = t_j - t_{j-1}$ is the length between the transitions; the transition instants are $t_j = t_{j-1} + x_j$ giving the "*observed*" *transition sample* D*={t_1, t_2, …, t_{g-1}, t_g, t=end of the test} (times of the transitions t_j).

We consider now that we want to estimate the unknown MTTF=θ=1/λ of each item comprising the "associated" stand-by system [24-30]: each datum is a measurement from the exponential pdf; we compute the determinant det $B(s\,|\,r;\,\theta,D^*) = (1\,/\,\theta)^g$ exp [-$T(t)$] of the integral system (14), where $T(t)$ is the "Total Time on Test" $T(t) = \sum_1^g x_i$ [t_0 in the figure 5]: the "Associated Stand-by System" [25-33] in the Statistics books provides the pdf of the sum of the RV X_i of the "*observed*" *empirical sample* D={x_1, x_2, …, x_g}. At the end time t of the test, the integral equations, constrained by the <u>constraint</u> D*, provide the equation

$$\partial ln det B(s|r;\,\theta,D^*)/\partial\theta = \theta/g - T(t) = 0 \qquad (15)$$

It is important to notice that, in the case of exponential distribution [11-16, 25-36], it is <u>exactly the same result as the one provided by the <u>*MLM Maximum Likelihood Method*</u></u>.

If the kernel $b_{j,j+1}(s - t_j)$ is the pdf $f(s - t_j|\mu,\sigma^2)= (1/\sqrt{2\pi}\sigma)e^{-(s-t_j-\mu)^2/(2\sigma^2)}$ the data are normally distributed, $X \sim N(\mu_X,\sigma_X^2) = (1/\sqrt{2\pi}\sigma_X)e^{-(x-\mu_X)^2/(2\,\sigma_X^2)}$, with sample size n, then we get the usual estimator $\bar{X} = \sum X_i\,/n$ such that $E(\bar{X}) = \mu_X$

The same happens with any other distribution (e.g. see the Table 2) provided that we write the kernel $b_{i,i+1}$ (s).

The reliability function R_0 $(t|\theta)$, [formula (13)], with the parameter θ, of the "Associated Stand-by System" provides the *Operating Characteristic Curve* (OC Curve, reliability of the system) [6-36] and allows to find the Confidence Limits (θ_L Lower and θ_U Upper) of the "unknown" mean θ, to be estimated, for any type of distribution (Exponential, Weibull, Rayleigh, Normal, Gamma, Inverted Weibull, General Inverted Weibull, …); by solving, with (a general) unknown (indicated as) θ, the two equations $R_0(t_0|\theta) = 1 - \alpha\,/\,2$ and R_0 $(t_0\,|\,\theta) = \alpha\,/\,2$; we get the two values (θ_L, θ_U) such that

$$R_0(t_o|\theta_L) = \alpha/2 \;\; and \;\; R_0(t_o|\theta_U) = 1 - \alpha/2 \qquad (16)$$

where t_o is the (computed) "total of the length of the transitions x_i=t_j - t_{j-1} data of the *empirical sample* D" and CL=1 - α is the Confidence Level. CI = θ_L -------- θ_U is the Confidence Interval: $\theta_L = 1/\lambda_U$ and $\theta_U = 1\,/\,\lambda_L$.

For example, with figure 6, we can derive θ_L = 62.5 days=$1/\lambda_U$ and θ_U=200 *days* = 1 / λ_L, with CL=0.8. It is quite interesting that the book [14] Meeker et al., "*Statistical Intervals: A Guide for Practitioners and Researchers*", John Wiley & Sons (2017) use the same ideas of FG (shown in the formula 16) for computing the CI; the only difference is that the author FG defined the procedure in 1982 [26], 35 years before Meeker et al.

As said before, we can use RIT for the Sequential Tests; we have only to consider the various transitions and the Total Time on Test to the last transition we want to consider.

2.4 Control Charts for TBE Data. Some Ideas for Phase I Analysis
Let's consider now TBE (Time Between Event, time between transitions) data, *exponentially or Weibull distributed*. Quite a lot of authors (in the "*Garden … [24]*") *compute wrongly the Control Limits* (which are the Confidence Limits) *of these CCs*.

The formulae, shown in the section "*Control Charts for Process Management*", are based on the Normal distribution (thanks to the CLT; see the excerpts 3, 3a and 3b); unfortunately, they are used also for NON_normal data (e.g. see formulae (6)): for that, sometimes, the NON_normal data are transformed "with suitable transformations" in order to "produce Normal data" and to apply those formulae (above) [e.g. Montgomery in his book].

Sometimes we have few data and then we use the so called "*Individual Control Charts*" I-CC. The I-CCs are very much *used for exponentially (or Weibull) distributed data*: they are also named "rare events Control Charts for TBE (Time Between Events) data", I-CC_TBE.

In the previous section, we computed the CI = θ_L ------- θ_U of the parameter θ, using the (subsample) "transition times durations": t_o = "total of the *transition times durations* (length of the transitions $x_i = t_j - t_{j-1}$ data) in the *empirical sample* (subsample with n=4 only, as an example)" and Confidence Level CL=1 - α.

When we deal with a I-CC_TBE we compute the LCL and UCL of the mean θ through the *empirical mean* $\bar{t}_o = t_o / n$ of each transition, for the… ; we solve the two following equations (17) for the two

unknown values LCL and UCL, for $R(\bar{t}_o | \theta)$ of each item in the sample, similar to (16)

$$R(\bar{t}_o|LCL) = \alpha/2, \qquad R(\bar{t}_o|UCL) = 1 - \alpha/2 \qquad (17)$$

where now $\bar{t}_o = t_o/n$ is the "mean, to be attributed, to the single lengths of the single transitions $x_i = t_j - t_{j-1}$ data in the *empirical sample* D with the Confidence Level CL=1 - α: $LCL = 1 / \lambda_U$ and $UCL = 1 / \lambda_L$.

In the next sections we can see the Scientific Results found by a Scientific Theory (we anticipate them: the Control Limits are LCL=18.0 days and UCL=88039.3 days).

3. Results
In this section we provide the scientific analysis of the "remission time" data [3] and compare our result with those of the authors: the findings are completely different and the decisions, consequently, should be different, with different costs of wrong decisions.

3.1 Control Charts for TBE data. Phase I Analysis
The "remission time of 128 bladder cancer patients" data are in the table 3.

7.09	7.59	1.4	4.23	8.66	25.74	6.97	3.02	2.83	19.13
3.48	1.76	32.15	25.82	2.62	2.64	5.71	5.41	10.66	4.51
2.02	17.36	14.77	0.5	7.32	3.25	7.66	7.63	5.32	7.28
1.26	23.63	5.32	5.09	21.73	79.05	2.54	8.37	26.31	5.17
3.57	3.52	7.62	2.87	2.69	2.09	10.34	15.96	18.1	6.94
4.4	4.26	17.12	12.02	36.66	0.4	2.26	2.69	0.9	8.65
7.39	43.01	0.2	12.03	11.98	7.93	8.53	6.93	2.46	14.24
9.22	5.49	17.14	2.23	14.76	46.12	3.36	13.8	5.06	6.76
4.5	0.51	11.79	4.33	5.34	1.19	22.69	3.31	10.75	13.11
3.88	2.02	4.18	4.34	34.26	5.62	1.46	0.08	0.81	10.06
16.62	14.83	6.54	3.64	12.63	2.75				

Table 3: Data "of remission time of 128 bladder cancer patients" from "Statistical Inference on the Shape Parameter of Inverse Generalized Weibull Distribution", Mathematics (2024) [3] and Modified generalized Weibull distribution: theory and applications, Scientific Reports (2023):12828

Using all the 128 Cancer data the authors [3] write:

> *It is reasonable to assume that the remission times of bladder cancer patients follow an GIW distribution. Based on the given data for the 128 patients, as well as the information from …, we further assume the distribution is GIW($\hat{\omega}$=61.38, $\hat{\beta}$=0.51, $\hat{\eta}$=8.19).*

Excerpt 5. Zhuang et al., Statistical Inference on … Generalized Weibull Distribution. 2024

They add also:

> *Figure 3 (our Excerpt 7) displays the Q-Q plot for the observed data versus the theoretical IGW distribution, given that $\hat{\omega}$=61.38, $\hat{\beta}$=0.51, $\hat{\eta}$=8.19, which confirms that this data set would be suitable for the proposed methodologies. We have also included the histogram of the observed data, and the density curve, to provide further insight.*

Excerpt 6. Zhuang et al., Statistical Inference on … Generalized Weibull Distribution. 2024

So, the authors decided to "assume" (use) the GIW$(x|\beta, \eta, \omega)$: $[1 - e^{-(\eta/x)^\beta}]^\omega$ to analyse all the 128 data in table 3; their estimates are in Excerpts 5 ,6. Looking at Q-Q Plot and Histogram (in Excerpt 7) the readers can have some doubts about the use. of GIW.
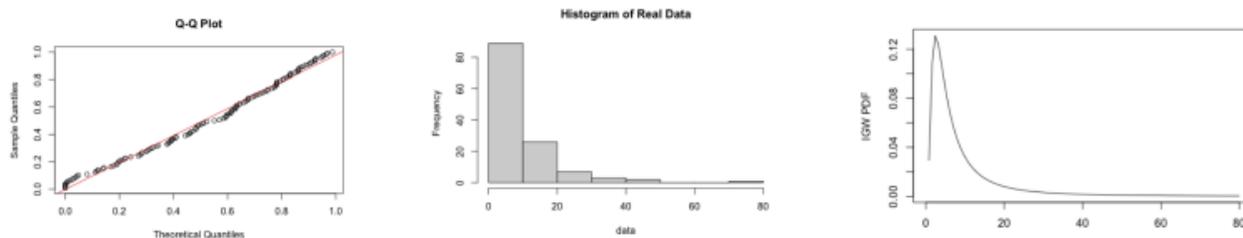
Excerpt 7. QQ plot of remission time of 128 bladder cancer patients with IGW histogram of the real data and probability density GIW (estimates of the parameters $\hat{\omega}$=61.38, $\hat{\beta}$=0.51, $\hat{\eta}$=8.19), from [3]

As a matter of fact, we can draw the figure 7, TTOT (of the data x_i) versus i/n (n=128); from the graph it is evident that the exponential distribution is suitable for the data analysis. Therefore, we will compare the models Exponential, Inverted W and GIW.
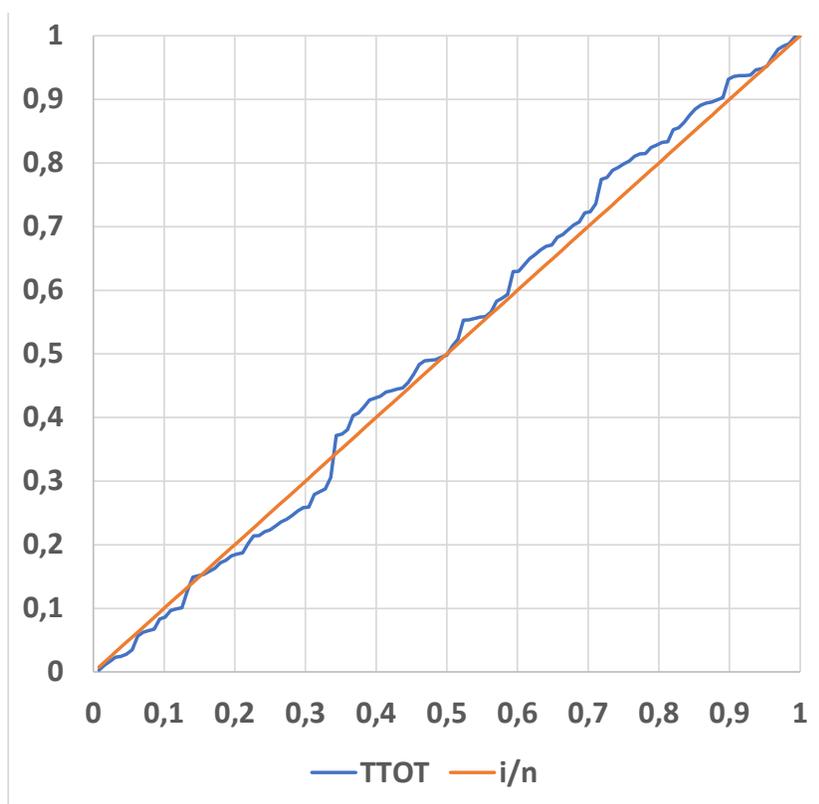
Figure 7: TTOT (Total Time on Test transform) of the Cancer Data. Exponential Distribution is Suitable

We divide the data in two sets: the first based on the first 32 data and the second considering the others.

Fitting the Weibull distribution, one finds β=1.17 and η=8.88, with -2lnL=198.58; since the 1εCI of β, with CL=80% we are allowed to use the exponential distribution (as given in figure 7). Seeing the figure 8 we find that the data show an OOC.
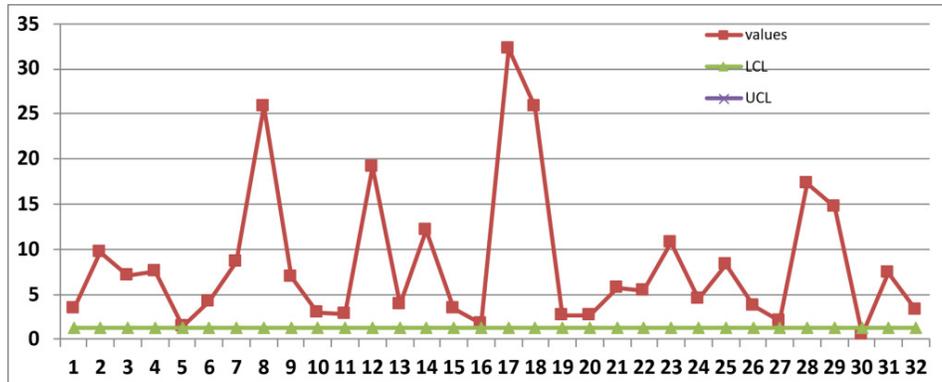
Figure 8: CC of the first 32 Cancer data (Exponential distribution): process OOC (1 point)

The figure 8 shows that the first 32 data do not allow to assess if the "null hypothesis" $H_0=\{\theta=10\}$, with $\alpha=0.025$ is to be accepted or rejected in favour of the $H_1=\{\theta=5.75\}$ with $\beta=0.025$. The Wald Sequential Test is inefficient for the 32 data. Compare with figure 9.
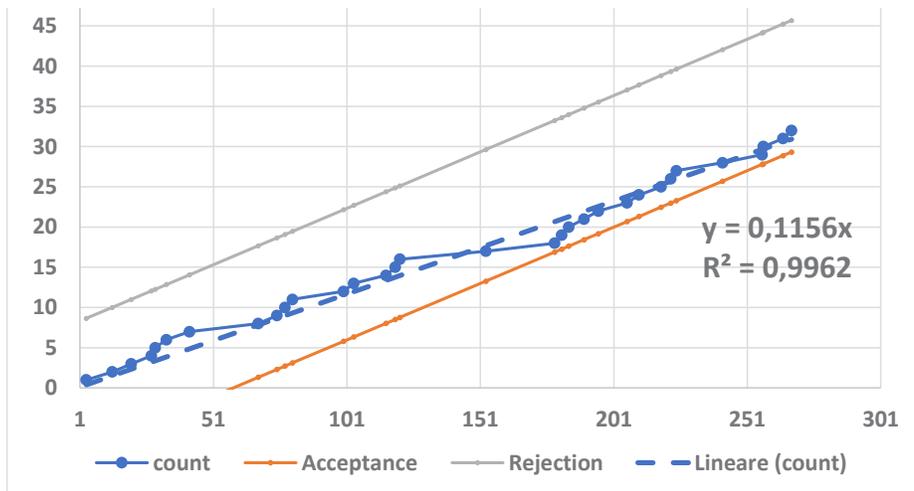
Figure 9: Sequential Test (Wald) of the 32 First Data
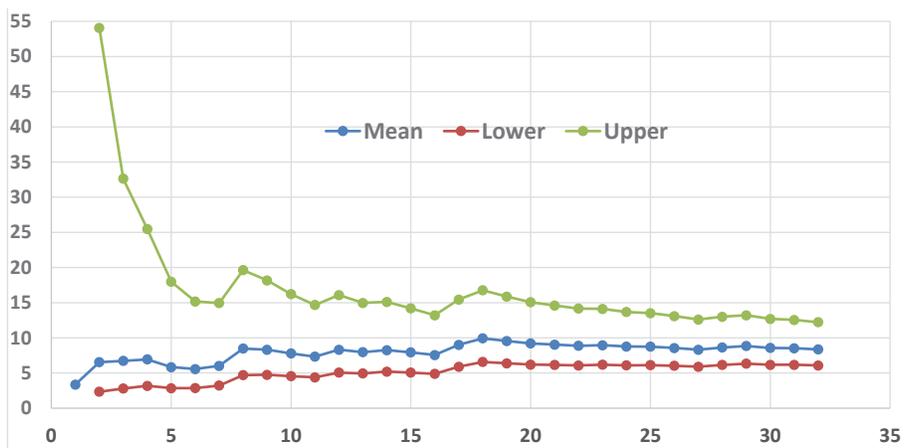
Figure 10. Sequential Confidence Intervals ($\alpha=\beta=0.025$) for the Exponential Distribution

The last CI=$\{6.08, 12.23\}$, figure 10, shows that the first 32 data allows to assess that the "null hypothesis" $H_0=\{\theta=10\}$, with $\alpha=0.025$ is to be accepted but $H_1=\{\theta=5.75\}$ rejected with $\beta>0.025$: $\{5.75<6.08<10<12.23\}$. The Sequential CIs are not more efficient than Wald Test.
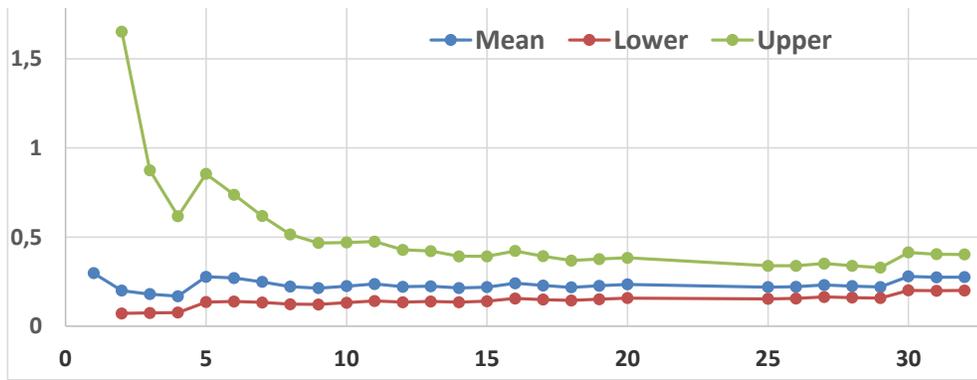
Figure 11. Sequential Confidence Intervals (α=β=0.025) for the Inverse Weibull Distribution

Fitting the Inverse Weibull distribution, on the first 32 data (1/x_i), one finds β=1.0422 and η=0.281, with -2lnL=18.59; since the 1ϵCI of β, with CL=80% we are allowed to use the exponential distribution, as we could do for the data x_i. The conclusion, the first 32 data (1/x_i), about the sequential CIs would be the same as for the data xi.

In the next session we consider all the 128 data and compare our results with the authors of [3].

The authors of the paper "Modified generalized Weibull distribution: theory and applications, Scientific Reports (2023):12828" use the Modified Power Generalised Weibull model MPGW(x|β,η,ω,θ)=MPGW(x|3.5251, 2.905, 0.1153, 19.231), but they do not compute the Confidence Intervals; so, we cannot see if ω=0 is an acceptable estimation.

For exponentially distributed data (17) becomes (18) [6-33], k=1, with CL=1 - α

$$e^{-[\bar{t}_o/LCL]} = 1 - α/2 \quad \text{and} \quad e^{-[\bar{t}_o/UCL]} = α/2 \qquad (18)$$

The endpoints of the CI=LCL┄┄┄┄UCL are the Control Limits of the I-CC_TBE.
This is the right method to extract the "true" complete information contained in the sample (see the figs. 8, 9, 10). The figures are justified by the Theory [6-33] and are related to the formulae [(12), (13) for k=1], for the I-CC_TBE charts.

Remember the book Meeker et al., "*Statistical Intervals: A Guide for Practitioners and Researchers*", John Wiley & Sons (2017): the authors use the same ideas of FG; the only difference is that FG invented 30 years before, at least.

Compare the formulae [(18), for k=1], theoretically derived with a sound Theory [6-33], with the ones in the Excerpt [in the Appendix (a small sample from the "*Garden* … [24]")] and notice that the two Minitab authors (Santiago&Smith) use the "empirical mean $\bar{t}_o$" in place of the $θ_0$ in the figure 1: it is the **_same_** trick of replacing $\bar{x}$ to the mean $μ$ which is valid for the Normal distributed data only; e.g., see the formulae (1)!

3.2 Control Charts for TBE data. Phase II Analysis
We saw in the previous section what usually it is done during the Phase I of the application of CCs: estimation of the mean and standard deviation; later, their values are assumed as "true known" parameters of the data distribution, in view of the Phase II.

We considered the first 32 (out of 128 remission times of bladder) Cancer data; using all the 128 data the authors found (Excerpts 5, 6) the distribution GIW(x|β, η, ω): $1 - [1 - e^{-(η/x)^β}]^ω$ with estimated parameters $\hat{β}$=0.51, $\hat{η}$=8.19, $\hat{ω}$=61.38; on the contrary we found that the exponential distribution (after fitting the Weibull and the Inverse Weibull) was suitable: that allowed us to make many considerations about the use of sequential sampling.

Now we consider all the 128 data and see new considerations.

In particular, for TBE individual data the exponential distribution is assumed with a known parameter $λ_0$ or $θ_0$.

We consider now what it is done during the Phase II of the application of CCs for TBE data individual exponentially distributed.

As previously we find that the Exponential distribution is well fitting the data, figure 12, opposite to the distribution GIW(x|β, η, ω): $1 - [1 - e^{-(η/x)^β}]^ω$.
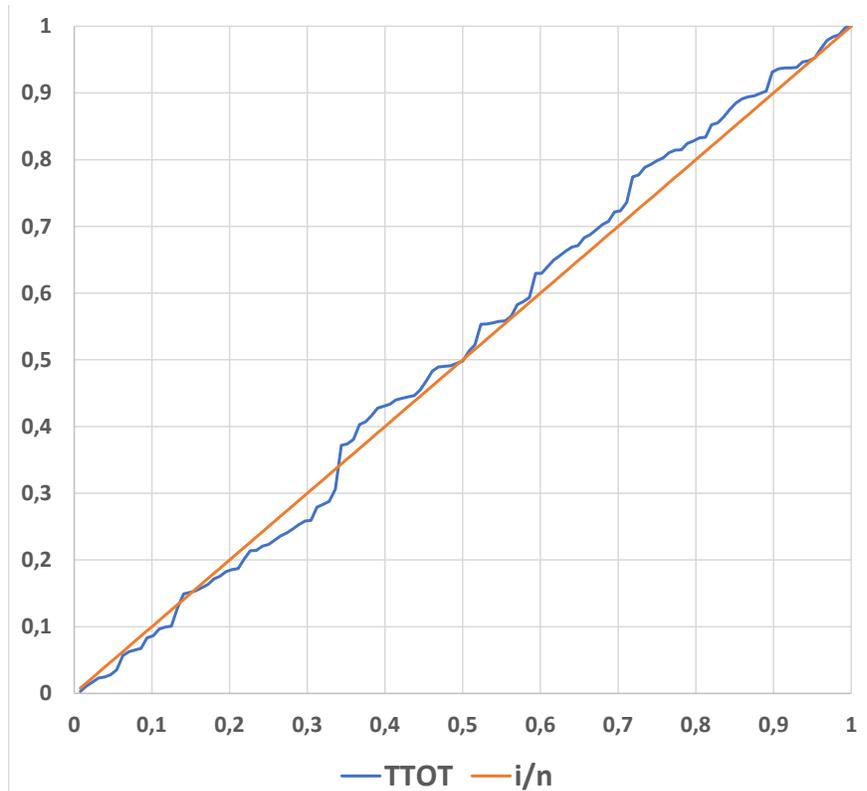
Figure 12. TTOT (Total Time On Test transform) of the 128 Cancer data. Exponential distribution is suitable
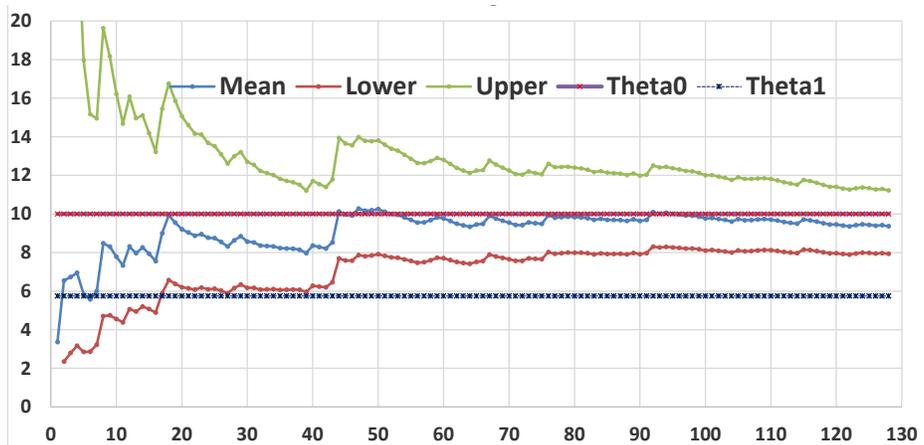

Figure 13. Sequential Confidence Intervals ($\alpha=\beta=0.025$) of the 128 Cancer data with hypotheses $H_0=\{\theta=10\}$ versus $H_1=\{\theta=5.75\}$. Exponential distribution is suitable

The last CI=$\{9.36, 11.22\}$, figure 13, shows that the 128 data allows to assess that the "null hypothesis" $H_0=\{\theta=10\}$, with $\alpha=0.025$ is to be accepted but $H_1=\{\theta=5.75\}$ rejected with $\beta>0.025$: $\{5.75<9.36<10<11.22\}$. The Sequential CIs are less efficient than Wald Test.
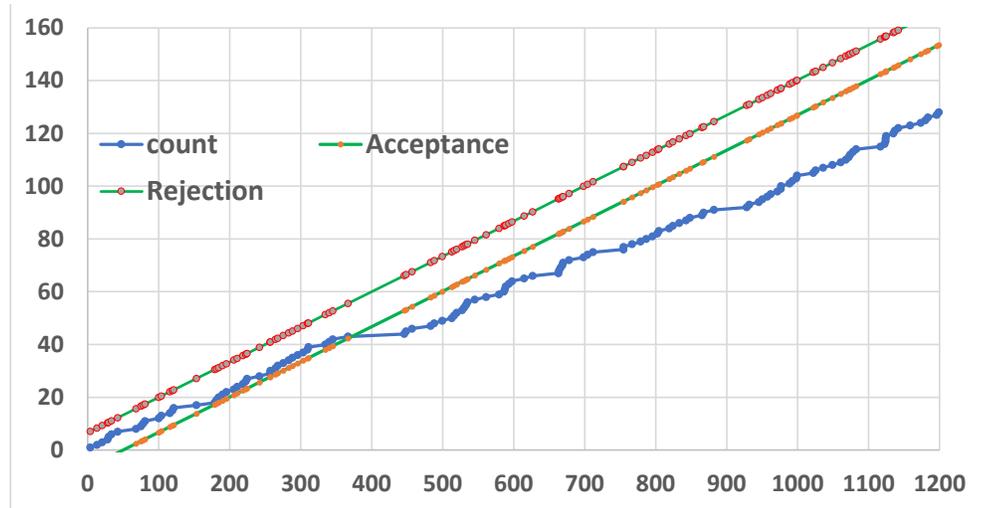
Figure 14. Sequential Test (Wald) of the 128 Cancer data; the decision to Accept $H_0=\{\theta=10\}$ happens at the 42nd point
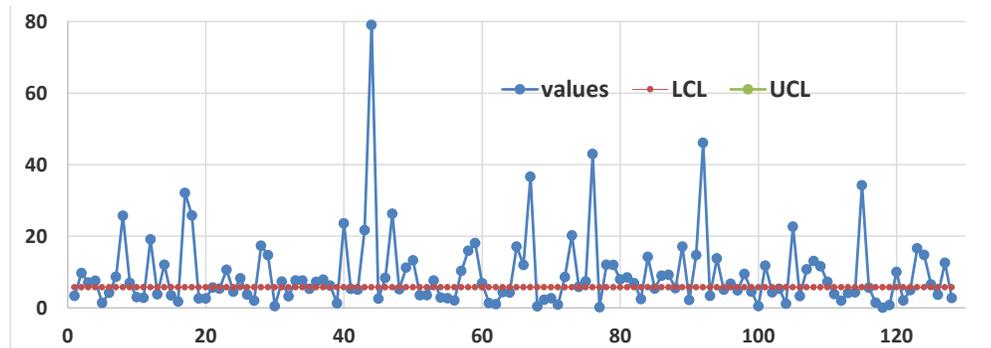
Figure 14. Control Chart of the 128 Cancer Data: The Process is OOC

As it happened previously, we find that the CC provides much more information to the Manager to allow him to take sound decisions.

Since the CCs are "sequential tests", as are the reliability flow graphs, we think that it is wise to use them.

3.3 Sequential Test by the Authors of [3]

Now we see what the authors of [3] did about their distribution GIW(x|β, η, ω): $1 - [1 - e^{-(\eta/x)^\beta}]^\omega$ (in the paper they use α in place of our ω; we introduced ω because α is the type I probability risk, associated to H_0).

They found the MLE (Maximum Likelihood Estimate $\hat{\omega}_n$ and Estimator Ω_n) of the parameter ω, with n the number of the data considered, H and B estimators of η and β

$$\hat{\omega}_n = -n/\left[\sum_1^n ln\left(1 - e^{-(\eta/x_i)^\beta}\right)\right] \quad \text{and}$$

$$\Omega_n = -n/\left[\sum_1^n ln\left(1 - e^{-(H/x_i)^B}\right)\right] = n/Y$$

and computed $\hat{\omega}_{128} = 61.38$; from that they computed the 95% Confidence Interval as (59.82, 63.07), which is defined as $(\hat{\omega}_n/d, \, d\hat{\omega}_n)$, where d is the "accuracy" of the CI. They proved a very interesting result: the distribution GIW(x|β, η, ω) depends only on n, the number of data considered, and not on the parameters of GIW(x|β, η, ω) of the Random Variable $T = \omega Y / n \sim Gamma(n, 1/n)$.

Notice that the Confidence Interval (59.82, 63.07) is actually a Probability Interval, showing the same error mentioned in [24]. The authors state (in Excerpt 8)

Excerpt 8. Zhuang et al., Statistical Inference on … Generalized Weibull Distribution. 2024

We tried to draw a "TTOT (Total Time On Test transform) of the 128 Cancer data" (similar to Figure 12) with GIW(x|0.51, 8.19, 61.38): $1 - [1 - e^{-(8.19/x)^{0.51}}]^{61.38}$; it is impossible to draw such a graph with data in table 3. To understand the reader can see the figure 16:
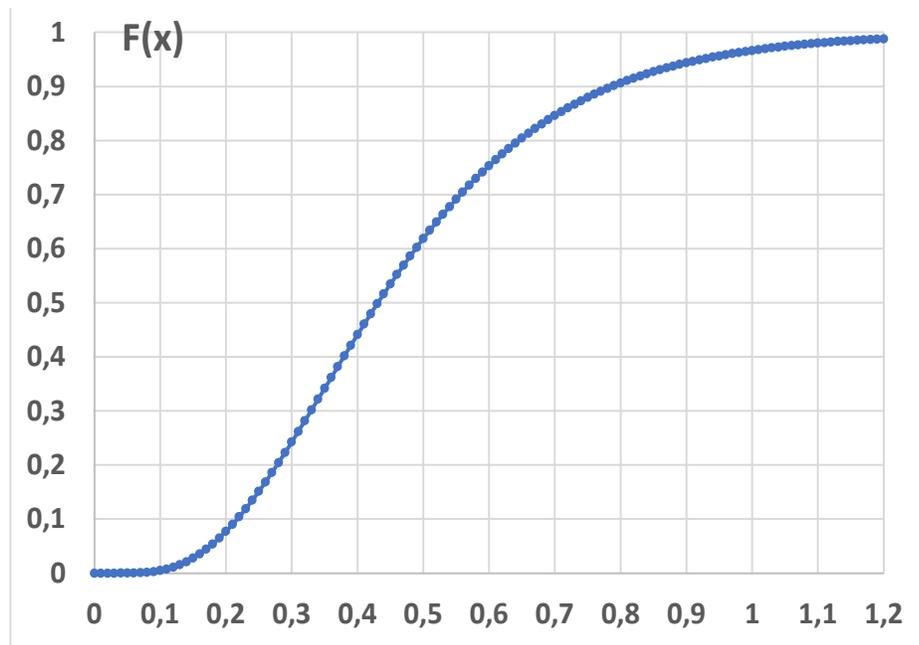

Figure 16. Distribution GIW(x|0.51, 8.19, 61.38) for values from 0 to 1.2

It is evident from table 3 that only the smallest nine data 0.08, 0.20, 0.40, 0.50, 0.51 0.81, 0.90, 1.05, 1,19 could be shown in the figure 16; the other 119 data are all near the ordinate 1 (in the figure 16).

How could GIW(x|0.51, 8.19, 61.38) fit suitably the 128 Cancer data?
So, the Excerpt 8 is quite doubtful.

To understand we created the figure 17 where we have shown, versus the number of data, the *Inverse data* (of those in Table 3), the sum of the inverse of the collected data named "*Tot_inverse*" and the "*Tot_inverse_B*" $\sum_1^n -ln\left(1 - e^{-(\eta/x_i)^\beta}\right)$ from the Distribution GIW(x|0.51, 8.19, 61.38), with their interpolating formulae (where x "actually" is the number of counts, 1, 2, 3 ,…, n-2, n-1, n).
We see that the "*Tot_inverse_B*" $\sum_1^n -ln\left(1 - e^{-(\eta/x_i)^\beta}\right)$, from the Distribution GIW(x|0.51, 8.19, 61.38), does not fit well the successive sums of the Inverse data (of those in Table 3).
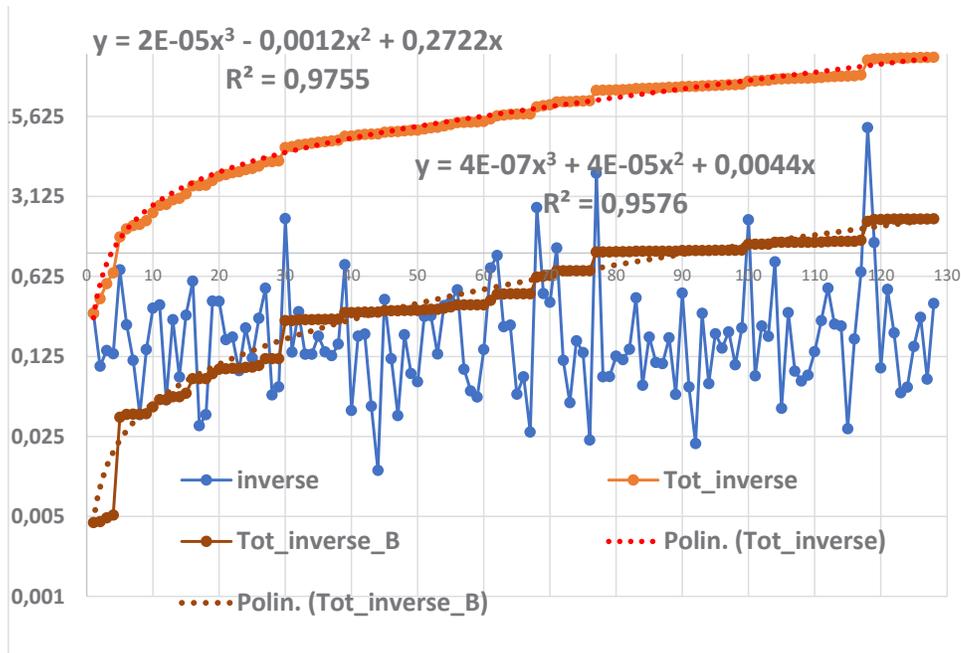
What is the consequence? We leave it to the readers…

Figure 17. Inverse data (of those in Table 3), sum of the inverse of collected data named "Tot_inverse" and "Tot_inverse_B" $\sum_1^n -ln\left(1 - e^{-(\eta/x_i)^\beta}\right)$ from the Distribution GIW(x|0.51, 8.19, 61.38). x in the interpolating formulae is the number of counts, not the collected data.

Computing the quantity "*Tot_inverse_B*"=$\sum_1^n -ln\left(1 - e^{-(8.19/x_i)^{0.51}}\right)$ =1.99 26, we find the estimate $\hat{\omega}_{128\_FG} = 64.2391$, that is different from the estimate of the authors $\hat{\omega}_{128} = 61.38$; so getting the Distribution GIW(x|0.51, 8.19, 61.38). Their 95% Confidence Interval, $CI = (\hat{\omega}_n/d, d\hat{\omega}_n)$, where d is the "accuracy" of the CI, was (59.82, 63.07): notice that the "named" CI $_{Zhuang}$=(59.82, 63.07) is actually a Probability Interval, showing the same error mentioned in [24].

It is important to notice that $\hat{\omega}_{128\_FG} > 63.07$ (the upper limit of the "named" CI $_{Zhuang}$ =(59.82, 63.07). We leave to the readers to say what that means!

Let's indicate as T the Random Variable $\Omega \sum_1^g \left[-ln\left(1 - e^{-(\eta/x_i)^\beta}\right)\right]/g$; we have that $T \sim Gamma(g, 1/g)$, with density $f(t; \beta, \theta) = t^{\beta-1}e^{-t/\theta}/[\theta^\beta \Gamma(\beta)]$ where θ is the scale parameter and β is the shape parameter ($\beta = g = 1/\theta$).

We can write the Probability statement, for any value chosen g,

$$P\left\{G_L = L < T = \Omega \sum_1^g \left[-ln\left(1 - e^{-(\eta/x_i)^\beta}\right)\right]/g < U = G_U\right\} = 1 - \alpha \quad (15)$$

where $L^{------}U$ is the interval that comprises the RV T with probability 1-α and G is the Cumulative Gamma Distribution.

From (15) can derive the "equivalent" Probability statement, for any value chosen g,

$$P\left\{G_L/\Omega = L/\Omega < \sum_1^g \left[-ln\left(1 - e^{-(\eta/x_i)^\beta}\right)\right]\right.$$

$$\left./g < U/\Omega = G_U/\Omega\right\} = 1 - \alpha \quad (15)$$

where L/Ω $^{------}U/\Omega$ is the random interval that comprises the parameter ω with probability 1-α.

After the estimation of $\hat{\omega}_n$ we have the Confidence Interval, $CI = (G_L/\hat{\omega}_n, G_U/\hat{\omega}_n)$ surely different from the "named" CI=(59.82, 63.07).

By taking advantage of the fact [3] that $Y = -\sum_1^n \left[-ln(1 - e^{-(H/x_i)^B})\right]$, follows a Gamma distribution with parameters n and 1/ω, we can compute directly the CI by computing the OC Curve (Operating Characteristic Curve) OC(%)=1 - Gamma(1.956|128, 1/ω); we show it in figure 18:
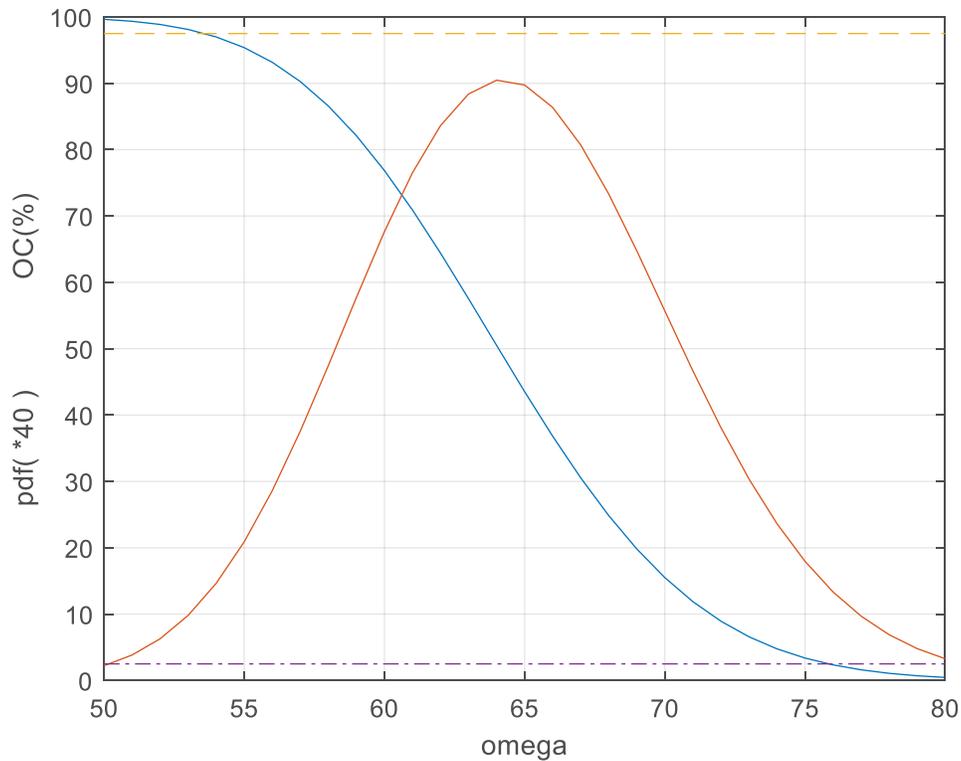
Figure 18. OC Curve and pdf (multiplied by 40) versus ω (omega), given "Tot_inverse_B" $\sum_1^n -ln\left(1 - e^{-(\eta/x_i)^\beta}\right)$ from the Distribution GIW(x|0.51, 8.19, 61.38). The intersections of the OC with the two horizontal lines y=0.025 and y=0.975, provide the limits of the CI.

It is clear that the intersections of the OC Curve (fig. 18) with the two horizontal lines y=0.025 and y=0.975, provide the limits of the CI, which are different from the "named" $CI_{\_Zhuang}$=(59.82, 63.07).

The Confidence limits are the values of the "unknown" variable ω satisfying next two equations, with D= Tot_inverse_B, computed with all the 128 data

$$\int_0^D y^{127}\omega^{128} e^{-\omega y}dy/\Gamma(128) = 1 - \alpha/2\,,$$

$$\int_0^D y^{127}\omega^{128} e^{-\omega y}dy/\Gamma(128) = \alpha/2 \qquad (16)$$

Putting $D_g=\sum_1^g -ln\left(1 - e^{-(\hat{\eta}/x_i)^{\hat{\beta}}}\right)$ computed with g data, we can

get the successive Confidence Intervals CI_g; two of them can be seen in the figure 19 with their OC Curves, for g=128 and g=73.

Our CI_{73} (figure 19) is $CI_{73}\approx$(83, 131); notice the big difference with the one given in the Excerpt 9. Notice that the value 61.38, estimated by all the 128 data, is named as "true value" (which is unknown).

Now, using the data that we setup from Table5, we just take the first 73 observations and construct the confidence interval for ω. These observations give us a 95% confidence interval as (59.84, 63.07). This interval covers the <u>true value</u> of ω, which is 61.38, by only utilizing 73 observations.

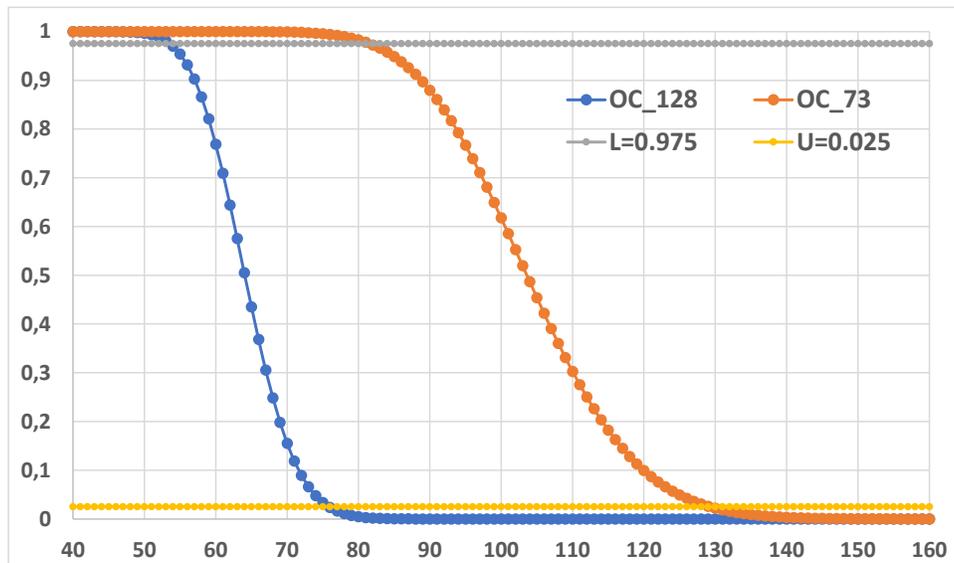**Excerpt 9.** Zhuang et al., Statistical Inference on … Generalized Weibull Distribution. 2024

Figure 19. Two OC Curves versus ω (omega). The intersections of each OC with the two horizontal lines y=0.025 and y=0.975, provide the limits of the CIs. The curve "on the right" is for g=73 (the sample size "optimum" for decision, according to Excerpt 9); the other is for g=128.

Notice that ω_{128}=64.24 while ω_{73}=104.07, quite a big difference with the "true value" 61.38, as in the Excerpt 9.

Obviously the CIs are different from the ones in [3].

The authors say:

3.4 Other cases

Another case we want to consider is in the paper [5] by Alshahrani et al., "On Designing of Bayesian Shewhart-Type Control Charts for Maxwell Distributed Processes with Application of Boring Machine. Mathematics 2023, 11, 1126"

> *The assumptions of normality for the quality characteristic may not be achieved in practice… Therefore, using the normal distribution for non-normal data may result in a false alarm in process monitoring or may allow for the later detection of changes. In the literature, several researchers suggested control charts for quality characteristic which follows a skewed distribution. … The Maxwell (or Maxwell–Boltzmann) distribution is a familiar positively skewed distribution. The Maxwell distribution has a smoothly increasing hazard risk; therefore, it is commonly used in life-testing experiments and reliability analysis where the assumption of constant hazard risk, such as in exponential distribution, is not practical. The Maxwell distribution is widely used in statistical machines, physics, chemistry, and life testing experiment, but it is recently has been applied in the SPC techniques.*

Excerpt 10. Alshahrani et al., "On Designing … of Boring Machine. Mathematics 2023"

If X is a RV having the Maxwell distribution with scale parameter σ^2, then its pdf is as follows: $f(x|\sigma^2) = \sqrt{2/\pi}\,\sigma^{-3}x^2 e^{-x^2/(2\sigma^2)}$ with $x, \sigma^2 > 0$.

The ML Estimator (which is a RV) of the parameter σ^2 is σ^2 is $\Sigma^2 = \sum_1^n X_j^2/(3n)$; the transformation $T = X^2/(2\sigma^2)$ shows that $T \sim (3/2,1)$, the Gamma distribution, and $U = 3n\Sigma^2/(2\sigma^2) \sim G(3n/2,1)$. It is interesting to note that if x is interpreted as the

velocity of a particle of unit mass (m=1), the quantity $w = mx^2/2$ is the energy of the particle and the two components X_1 and X_2 can be considered as independent RVs normally distributed with mean E[X]=0 and variance Var[X]=σ^2.

The authors consider correctly the Probability Limits, But, unfortunately, they wrote "Practically, the parameter σ^2 may be known or unknown then the *probability control limits of the*

control chart are defined as follows:…..".

Notice that in Control Charts (CCs) we use the Control Limits, LCL and UCL, NOT the "*probability control limits of the control chart*"!.

The authors made a lot of simulation and eventually applied their ideas to

> **Real Data Analysis**. *A boring machine is a tool used for making a wide hole in a fixed workpiece. These machines make use of a single steel cutting edge, carbide or diamond, or a small grinding wheel to make the hole cleaner, more accurate, and more specific. Boring machines with multiple spindles are typically used in a manufacturing plant where production is on a large scale. This study uses a real dataset by Hossain et al. that addressed the failure rate of the vertical boring machine. This data set was also considered … to review the optimum maintenance approach for the vertical boring machine. Subsequently, … conducted a detailed statistical investigation to evaluate the distributions, which best fit this data set. They examined models such as exponential, gamma, Maxwell, lognormal, Weibull, and estimated the parameters of these models with the maximum likelihood method. In addition, various information criteria such as Akaike, second-order, Bayesian, and the Kolmogorov–Smirnov test have shown that the Maxwell distribution is the best fitted to this data set: the data set followed the Maxwell distribution (p-value 0.4775) with=1777.86. The failure time data for a vertical boring machine are specified as follows: 2802, 2937, 2136, 4359, 4020, 1781, 2816, 2655, 3886, 2296, 3158, 3695, 4155, 3811, 2380, 376, 2172, 3705, 2848, 4339, 2076, 2672, 3632, 1976, 1700, 1596, 1701, 3575, 3802, 4351, 4291, and 808.*

Excerpt 11. Alshahrani et al., "On Designing … of Boring Machine. Mathematics 2023"

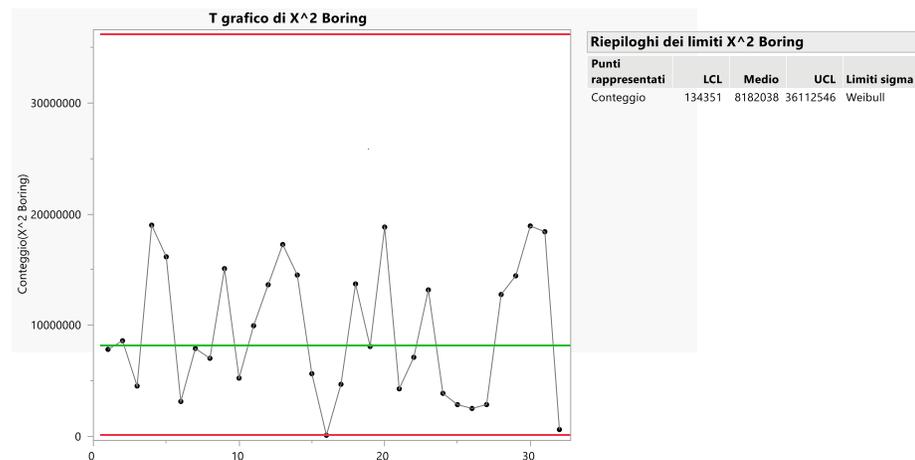Using JMP for the Individual Control Chart on x^2 data we found figure 20

Figure 20: ICC by JMP on x^2 data; the LCL and UCL are not the Control Limits but the Probability Limits
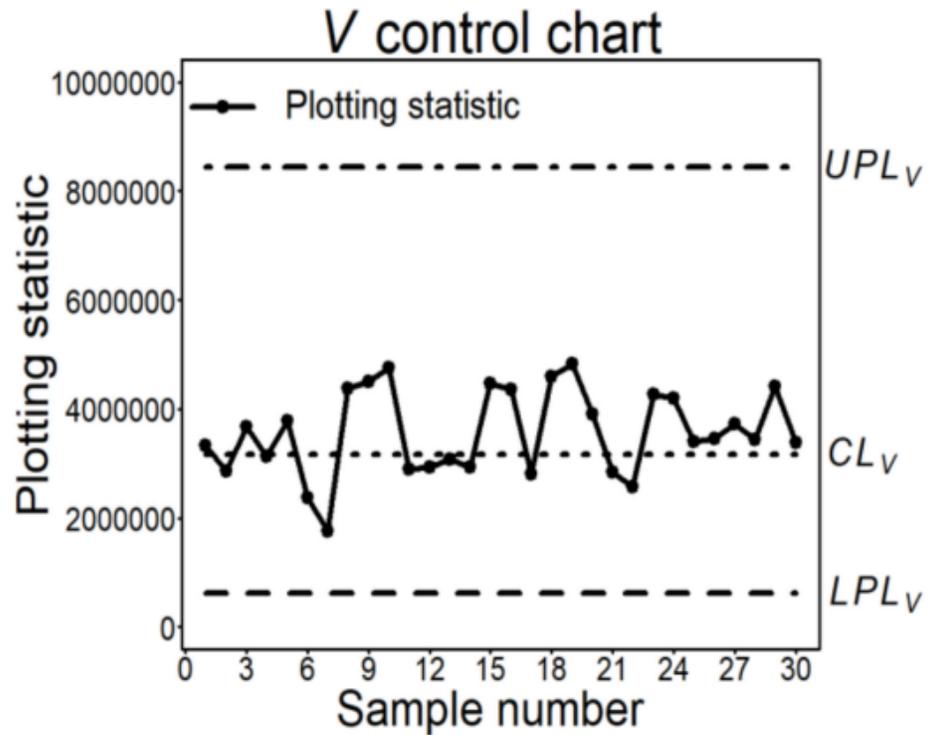
This can be compared to figure 21

Figure 21. ICC by Alshahrani et al., "On Designing … of Boring Machine. Mathematics 2023"; you see the Probability Limits (LPL, UPL) …

To understand the difference between the Control Limits (LCL and UCL) and the Probability Limits (L and U) you have to analyse the figure 22.
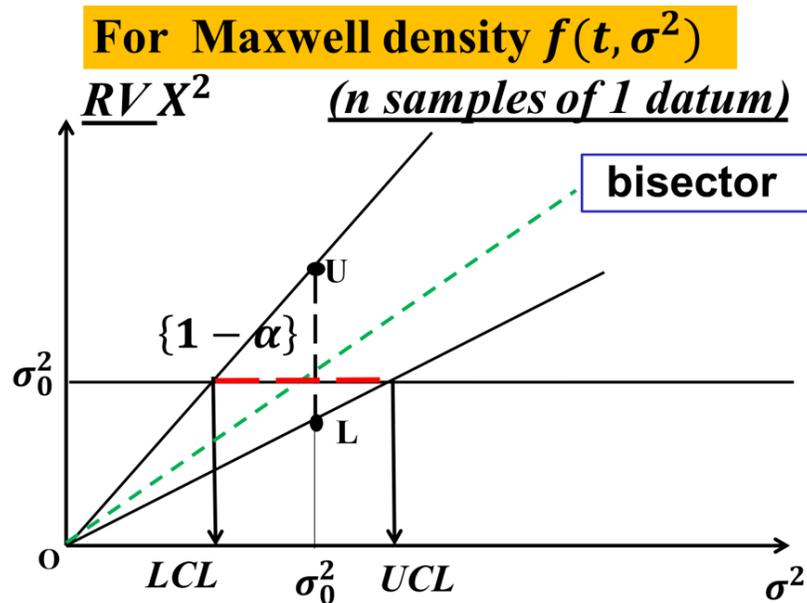

Figure 22. Difference between the Control Limits (LCL and UCL) and the Probability Limits (L and U) by the Theory [6-58].

The application of the Theory [6-58] to the Boring data (Excerpt 11) is in the figure 23.
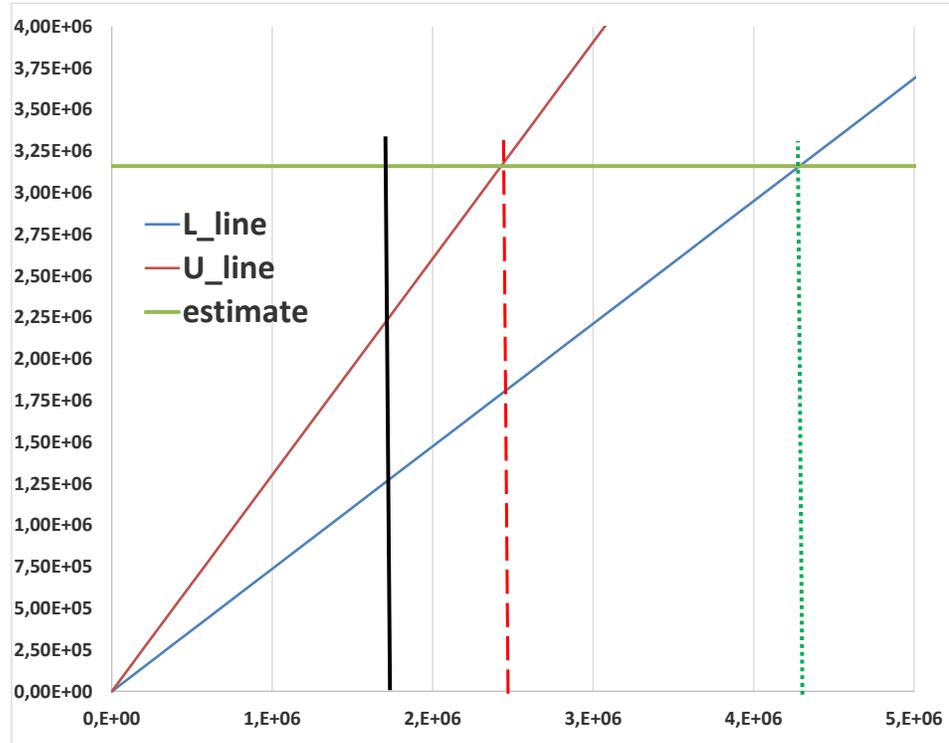

Figure 23: Difference between the Control Limits (LCL and UCL) and the Probability Limits (L and U) of the Boring data, according to the Theory [6-58], with CL=0.95; the vertical lines (red, green) intersect the horizontal line at the points LCL and UCL

The "scientific" Control Chart for the Boring data (Excerpt 11) is in the figure 24: the process is OOC, contrary to the findings in the figure 20; the cause is the use of the Probability Limits (LPL, UPL) instead of the Control Limits (LCL, UCL).
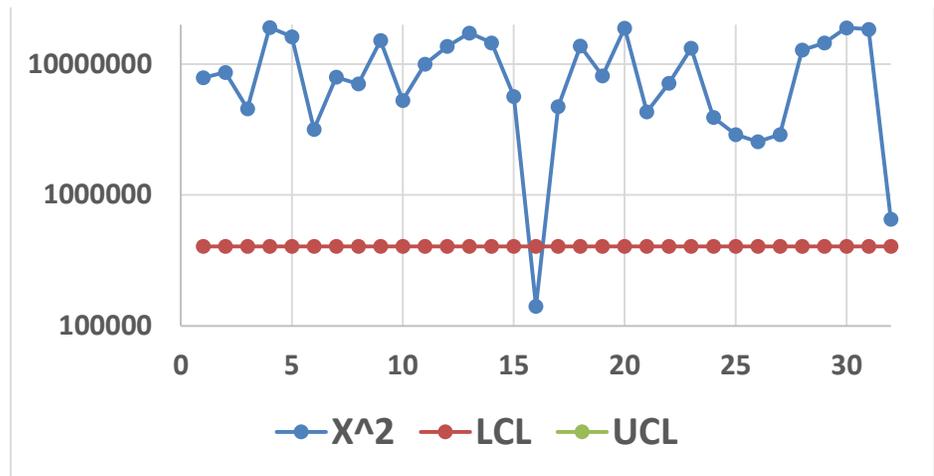

Figure 24: The "scientific" Control Limits (LCL and UCL) of the Boring data, according to the Theory [6-58], with CL=0.9973

The figure 25 shows the sequential Confidence Intervals for the Boring data (Excerpt 11); 13 data are necessary for a Confidence Interval=(2333350, 5733177)

The figure 26 shows the Sequential (Wald) Test Confidence for the Boring data (Excerpt 11); one sees that at the 13rd datum the "step-line" $G(\sum x_i^2)$, number of failures versus the total of squared times, intersects the Acceptance line; the competing Hypotheses are $H_0 = \{3160782\}$ with $\alpha = 0.025$ versus $H_1 = \{1359136\}$ with $\beta = 0.025$.
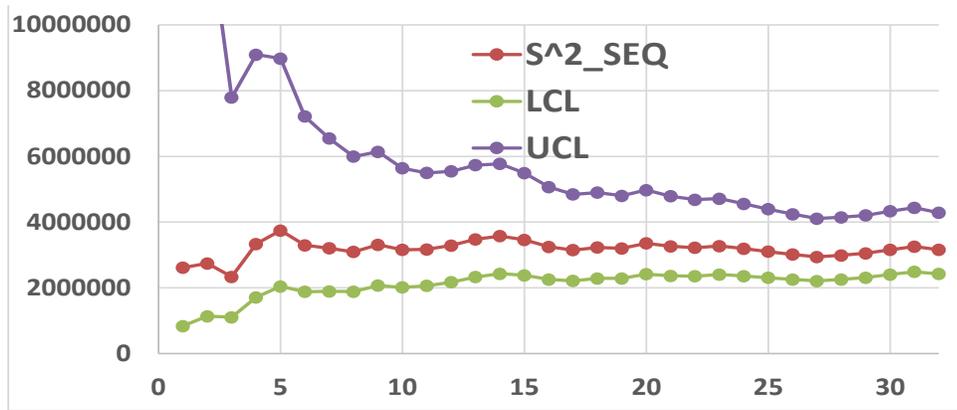
Figure 25: The Sequential Confidence Limits of the Boring Data, with CL=0.95

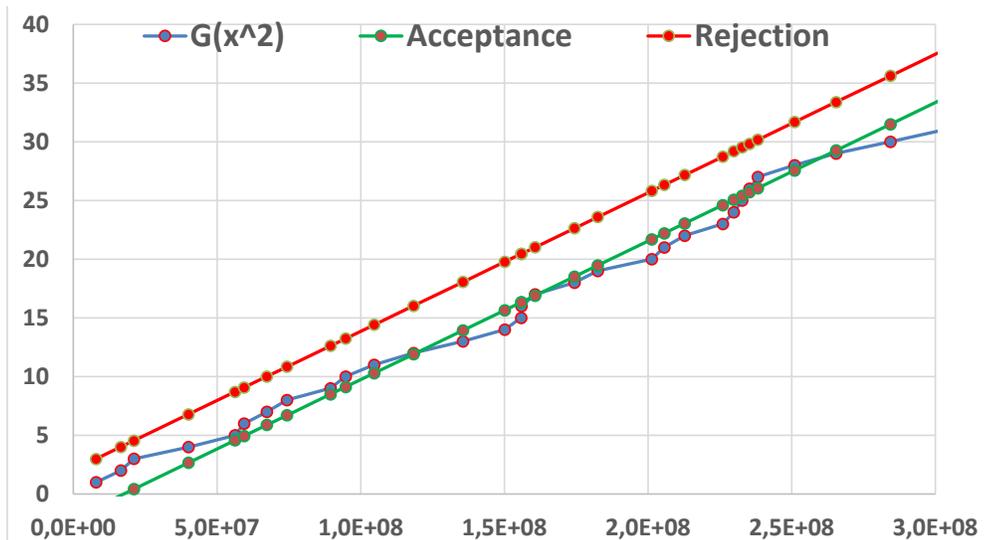**Figure 26:** Sequential Test (Wald) for the Boring Data, with □=0.025 and □=0.025

It is important to remember that the CI=(2427479, 4286847) is computed from all the data with CL=0.95.

All the results are found via RIT (Reliability Integral Theory) [25-33].

4. Discussion

We decided to use the data from the papers [3-5] and the analysis by the authors.

We got different results from those authors: the cause is that they use the Probability Limits of the PI (Probability Interval) as they were the Confidence Limits (Control Limits of the Control Charts).

The proof of the confusion between the intervals L--------U (Probability Interval) and LCL--------UCL (Confidence Interval) in the domain of Control Charts (for Process Management) highlight the importance and novelty of these ideas in the Statistical Theory and in the applications.

For the "location" parameter in the CCs, from the Theory, we know that two mean $\mu_{\bar{X}(t_q)}$ (parameter), q=1,2, …, n, and any other

mean $\mu_{\bar{X}(t_r)}$ (parameter), r=1,2, …, n, are different, with risk α, if their estimates are not both included in their common Confidence Interval as the CI of the grand mean $\mu_{\bar{X}} = \mu$ (parameter) is.

Let's consider the formula (4) and apply it to a "Normal model" (due to CLT, and assuming known variance), sequentially we can write the "real" fixed interval L‾‾‾‾U comprising the RV $\bar{\bar{X}}$ (vertical interval) and the Random Interval comprising the unknown mean μ (horizontal interval) (fig. 14)

$$P\left[L = \mu - \frac{\sigma z_{1-\frac{\alpha}{2}}}{\sqrt{k}} \leq \bar{\bar{X}} \leq \mu + \frac{\sigma z_{1-\frac{\alpha}{2}}}{\sqrt{k}} = U\right] =$$

$$P\left[\bar{\bar{X}} - \frac{\sigma z_{1-\frac{\alpha}{2}}}{\sqrt{k}} \leq \mu \leq \bar{\bar{X}} + \frac{\sigma z_{1-\frac{\alpha}{2}}}{\sqrt{k}}\right] \quad (14)$$

When the RV $\bar{\bar{X}}$ assume its determination (numerical value) $\bar{\bar{x}}$ (grand mean) the Random Interval becomes the Confidence Interval for the parameter μ, with CL=1-α: risk α that the horizontal line does not comprise the "mean" μ.

This is particularly important for the Individual Control Charts for Exponential, Weibull, Inverted Weibull, General Inverted Weibull, Maxwell and Gamma distributed data: this is what Deming calls "*Profound Knowledge* (understanding variation)" [9-10]. In this case, the figures 22, 23, 27 look like the figure 2, where you see the Confidence Interval, the realisation of the horizontal Random Interval.

The case we considered shows clearly that the analyses, in the Process Management, taken so far have been wrong and the decisions have been misleading, when the collected data follow a Non-Normal distribution [24].

Since a lot of papers (related to Exponential, Weibull, Inverted Weibull, General Inverted Weibull, Maxwell and Gamma distributions), *with the same problem as that of "The garden of flowers"* [24], are published in reputed Journals we think that the "*alternative*" title "*History is written by the winners. Reflections on Control Charts for Process Control*" is suitable for this paper: *the authors of the wrong papers* [24] are the winners.
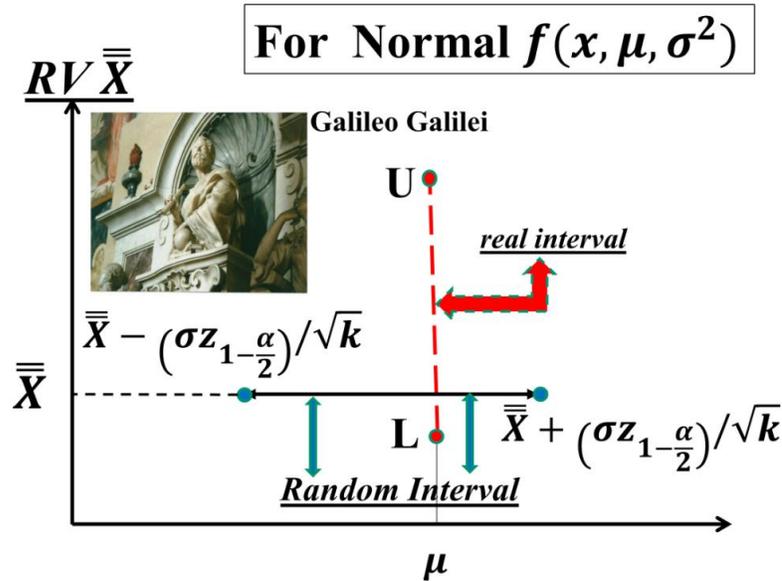

Figure 27: Probability Interval L⁻⁻⁻U (vertical line) versus Random Intervals comprising the "mean" μ (horizontal random variable lines), for Normally distributed RVs $\overline{\overline{X}} \sim N(\mu, \sigma^2)$.

Further studies should consider other distributions which cannot be transformed into the above distributions considered before.

5. Conclusions
With our figures (and the Appendix, that is a short extract from the "*Garden …* [24]*")* we humbly ask the readers to look at the references [1-58] and find how much the author has been fond of Quality and Scientificness in the Quality (Statistics, Mathematics, Thermodynamics, …) Fields.

The errors, in the "*Garden … [24]*", are caused by the lack of knowledge of sound statistical concepts about the properties of the parameters of the parent distribution generating the data, and the related Confidence Intervals. For the I-CC_TBE the computed Control Limits (which are actually the Confidence Intervals), in the literature are wrong due to lack of knowledge of the difference between Probability Intervals (PI) and Confidence Intervals (CI); see the figures 22, 23, 26 and 1). Therefore, the consequent decisions about Process IC and OOC are wrong.

We saw that RIT is able to solve various problems in the estimation (and Confidence Interval evaluation) of the parameters of distributions. The basics of RIT have been given.

We could have shown many other cases (from papers not mentioned here, that you can find in [22-24]) where errors were present due to the lack of knowledge of RIT and sound statistical ideas.

Following the scientific ideas of Galileo Galilei, the author many times tried to compel several scholars to be scientific (Galetto 1981-2025). Only Juran appreciated the author's ideas when he mentioned the paper "Quality of methods for quality is important" at the plenary session of EOQC Conference, Vienna. [1]

For the control charts, it came out that RIT proved that the T Charts, for rare events and TBE (Time Between Events), used in the software Minitab, SixPack, JMP or SAS are wrong [56-58]. So doing the author increased the h-index of the mentioned authors who published wrong papers.

RIT allows the scholars (managers, students, professors) to find sound methods also for the ideas shown by Wheeler in Quality Digest documents.

We informed the authors and the Journals who published wrong papers by writing various letters to the Editors…: no "Corrective Action", a basic activity for Quality has been carried out by them

so far. The same happened for Minitab Management. We attended a JMP forum in the JMP User Community and informed them that their "Control Charts for Rare Events" were wrong: they preferred to stop the discussion, instead to acknowledge the JMP faults [56-58].

So, dis-quality continues to be diffused people and people continue taking wrong decisions…

Deficiencies in products and methods generate huge cost of Dis-quality (poor quality) as highlighted by Deming and Juran. Any book and paper are products (providing methods): their wrong ideas and methods generate huge cost for the Companies using them. The methods given here provide the way to avoid such costs, especially when RIT gives the right way to deal with Preventive Maintenance (risks and costs), Spare Parts Management (cost of unavailability of systems and production losses), Inventory Management, cost of wrong analyses and decisions.
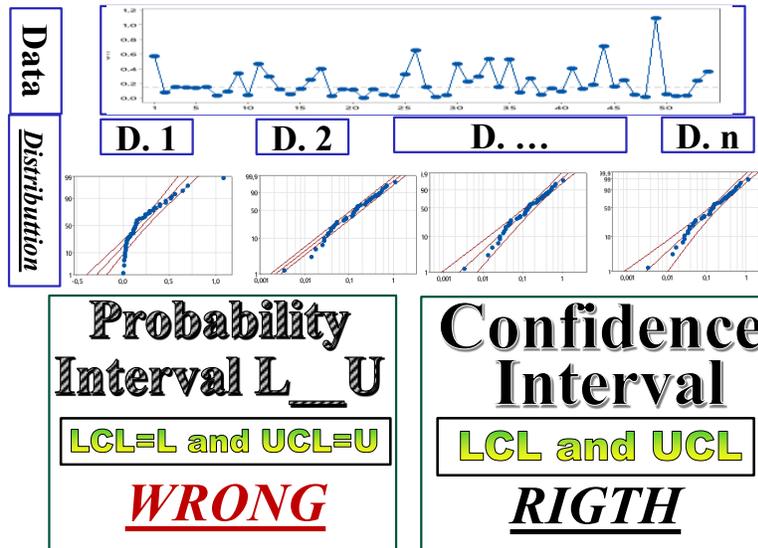

Figure 28: Probability Intervals L⁻⁻⁻⁻⁻U versus Confidence Intervals LCL⁻⁻⁻⁻⁻UCL in Control Charts

We think that we provided the readers with the belief that *Quality of Methods for Quality is important*.

The reader should remember the Deming's statements and the ideas in [6-58].
Unfortunately, many authors do not know Scientifically the role (concept) of Confidence Intervals (Appen-dix B) for Hypothesis Testing.
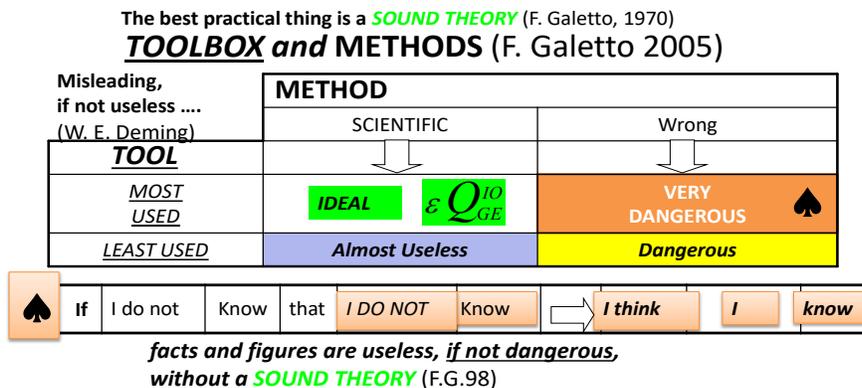

Figure 29: Knowledge versus Ignorance, in Tools and Methods.

Therefore, *they do not extract the maximum information form the data in the Process Control.*

Control Charts are a means to test the hypothesis about the process states, H_0={Process In Control} versus H_1={Process Out Of Control}, with stated risk $\alpha=0.0027$.

We have a big problem about Knowledge: sound Education is needed.

We think that the figure 29 conveys the fundamental ideas about the need of Theory for devising sound Methods, to be used in real applications in order to avoid the Dis-quality Vicious Circle.

Humbly, given our commitment to Quality and our long-life love for it [1-58], we would venture to quote Voltaire:

"*It is dangerous to be right in matters on which the established men are wrong.*" because "Many are destined to reason wrongly; others, not to reason at all; and others, to persecute those who do reason." So, "*The more often a stupidity is repeated, the more it gets the appearance of wisdom.*" and "It is difficult to free fools from the chains they revere."

Let's hope that Logic and Truth prevail and allow our message to be understood (figs. 27, 28).

The objective of collecting and analysing data is to take the right action. The computations are merely a means to characterize the process behaviour. However, it is important to use the right Control Limits take the right action about the process states, i.e., In Control versus Out Of Control.

On July-December 2024 we *again* verified (through several *new* downloaded papers, not shown here) that the *Pandemic Disease about the (wrong) Control Limits, that are actually the Probability Limits of the PI* is still present …

There will be any chance that the *Pandemic Disease* ends? See the Excerpt 12: *notice* the (ignorant) words "***plugging into*** …". The only way out is Knowledge… (fig. 28): *Deming's* [7-8] *Profound Knowledge, Metanoia, Theory.*

As mentioned before, when the parameter is unknown, it is estimated from a phase I sample of size m, say, $Y_1, \ldots, Y_m$ which is collected from an IC process. Recently, Kumar and Jaiswal (2020) used an estimator which is a function of the sample median to show the effect of the presence of outliers in the phase I sample. However, the minimum variance unbiased estimator (MVUE) is a commonly used estimator of the rate parameter which is given by $\hat{\lambda}_0 = \frac{m-1}{T}$, where $T = \sum_{i=1}^{m} Y_i$, the sum of all phase I observations. Thus, the phase II (estimated) control limits of the exponential control chart are given by plugging $\hat{\lambda}$ \(\hat{\lambda}_0\) into the control limits in Equation (1) as follows.

$$\widehat{LCL} = \frac{A_1}{\hat{\lambda}_0} = \frac{A_1 T}{m-1} \quad \text{and} \quad \widehat{UCL} = \frac{A_2}{\hat{\lambda}_0} = \frac{A_2 T}{m-1} \qquad (4)$$

These control limits are known as the conditional control limits conditioned on a given phase I sample (or a given estimated value of λ_0). In order to examine the conditional performance of the estimated control chart, we consider the CRL conditioned on a given phase I sample which follows a geometric distribution with parameter $\hat{\beta}(\delta) = P[X < \widehat{LCL}|\lambda_1] + P[X > \widehat{UCL}|\lambda_1]$.

Excerpt 12. From "Conditional analysis of Phase II exponential chart… an event", Q. Tech. & Quantitative Mgt, '19

We think that we provided the readers with several ideas and methods to be meditated in view of the appli-cations, generating wealth for the companies using them.

The documents [56-58] are very important: ASSURE …

There is no "free lunch": metanoia and study are needed and necessary.

Data Availability Statement: "MDPI Research Data Policies" at https://www.mdpi.com/ethics.

References
1. Galetto, F. (1989). Quality of methods for quality is important. In *EOQC Conference*.
2. Galetto, F. (1999). The golden integral quality approach: From management of quality to quality of management. *Total Quality Management, 10*(1), 17-35.
3. Zhuang, Y., Bapat, S. R., & Wang, W. (2024). Statistical Inference on the Shape Parameter of Inverse Generalized Weibull Distribution. *Mathematics, 12*(24), 3906.
4. Hu, J., Zheng, L., & Alanazi, I. (2025). Sequential Confidence Intervals for Comparing Two Proportions with Applications

in A/B Testing. *Mathematics, 13*(1), 161.

5. Alshahrani, F., Almanjahie, I. M., Khan, M., Anwar, S. M., Rasheed, Z., & Cheema, A. N. (2023). On designing of Bayesian Shewhart-type control charts for Maxwell distributed processes with application of boring machine. *Mathematics, 11*(5), 1126.
6. Belz, M. Statistical Methods in the Process Industry: McMillan; 1973.
7. Casella, Berger, Statistical Inference, 2nd edition: Duxbury Advanced Series; 2002.
8. Cramer, H. Mathematical Methods of Statistics: Princeton University Press; 1961.
9. Deming W. E., Out of the Crisis, Cambridge University Press; 1986.
10. Deming W. E., The new economics for industry, government, education: Cambridge University Press; 1997.
11. Dore, P. (1962). Introduzione al calcolo delle probabilità e alle sue applicazioni ingegneristiche: Fondamenti di analisi statistica per ingegneri. (*No Title*).
12. Juran, J., Quality Control Handbook, 4th, 5th ed.: McGraw-Hill, New York: 1988-98.
13. Kendall, M. G., & Stuart, A. (1969). The Advanced Theory of Statistics. Vol. 3. *Biometrics, 25*(2), 435.
14. Meeker, W. Q., Hahn, G. J., & Escobar, L. A. (2017). *Statistical intervals: a guide for practitioners and researchers*. John Wiley & Sons.
15. Mood, A. M. (1950). Introduction to the Theory of Statistics.
16. Rao, C. R. (1965). *Linear statistical inference and is applications*. John Wiley & Sons, Incorporated.
17. Rozanov, Y., Processus Aleatoire, Editions MIR: Moscow, (traduit du russe); 1975.
18. Ryan, T. P. (2011). *Statistical methods for quality improvement*. John Wiley & Sons.
19. Shewhart W. A., Economic Control of Quality of Manufactured Products: D. Van Nostrand Company; 1931.
20. Shewhart, W. A., & Deming, W. E. (1986). *Statistical method from the viewpoint of quality control*. Courier Corporation.
21. D. J. Wheeler, Various posts, Online available from Quality Digest
22. Galetto, F., (2014), Papers, and Documents of FG, Research Gate
23. Galetto, F., (2015-2024), Papers, and Documents of FG, Academia.edu
24. Galetto, F., (2024), The garden of flowers, Academia.edu
25. Galetto, F., Affidabilità Teoria e Metodi di calcolo: CLEUP editore, Padova (Italy); 1981-94.
26. Galetto, F. (1982). AFFIDABILITÀ vol. 2 Prove di affidabilità: distribuzione incognita, distribuzione esponenziale. *CLEUP editore, Padova, 85*, 94.
27. Galetto, F., Qualità. Alcuni metodi statistici da Manager: CUSL, Torino (Italy; 1995-99).
28. Galetto, F., Gestione Manageriale della Affidabilità: CLUT, Torino (Italy); 2010.
29. Galetto, F., Manutenzione e Affidabilità: CLUT, Torino (Italy); 2015.
30. Galetto, F., Reliability and Maintenance, Scientific Methods, Practical Approach, Vol-1: www.morebooks.de.; 2016.
31. Galetto, F., Reliability and Maintenance, Scientific Methods, Practical Approach, Vol-2: www.morebooks.de.; 2016.
32. Galetto, F., Statistical Process Management, ELIVA press ISBN 9781636482897; 2019
33. Galetto F., Affidabilità per la manutenzione, Manutenzione per la disponibilità: tab edizioni, Roma (Italy), ISBN 978-88-92-95-435-9, www.tabedizioni.it; 2022
34. Galetto, F. (2015). Hope for the Future: Overcoming the DEEP Ignorance on the CI (Confidence Intervals) and on the DOE (Design of Experiments). *Science Journal of Applied Mathematics and Statistics, 3*(3), 99-123.
35. Galetto, F. (2015). Management versus science: peer-reviewers do not know the subject they have to analyse. *Journal of Investment and Management, 4*(6), 319-329.
36. Galetto, F. (2015). The first step to Science Innovation: Down to the Basics. *Science, 3*(6), 81-91.
37. Galetto, F. (2022). Minitab T charts and quality decisions. *Journal of Statistics and Management Systems, 25*(2), 315-345.
38. Galetto, F., (2012) Six Sigma: help or hoax for Quality?, 11th Conference on TQM for HEI, Israel
39. Galetto, F. (2020). Six Sigma: Hoax against Quality Ignorant Professionals fond of money and not of Quality MINITAB wrong T Charts.
40. Galetto, F., (2021) Control Charts for TBE and Quality Decisions, Academia.edu
41. Galetto, F. (2021). Assure: Adopting Statistical Significance for Understanding Research and Engineering. *Journal of Engineering and Applied Sciences Technology. SRC/JEAST-128.*
42. Galetto F. (2023) Control Charts, Scientific Derivation of Control Limits and Average Run Length, International Journal of Latest Engineering Research and Applications (IJLERA) ISSN: 2455-7137 Volume – 08, Issue – 01, January 2023, PP – 11-45
43. Galetto, F., (2006) Quality Education and quality papers, IPSI, Marbella (Spain)
44. Galetto, F., (2006) Quality Education versus Peer Review, IPSI, Montenegro
45. Galetto, F., (2006) Does Peer Review assure Quality of papers and Education? 8th Conference on TQM for HEI, Paisley (Scotland)
46. Galetto, F., (1998), Quality Education on Quality for Future Managers, 1st Conference on TQM for HEI (Higher Education Institutions), Toulon (France)
47. Galetto, F., (2000), Quality Education for Professors teaching Quality to Future Managers, 3rd Conference on TQM for HEI, Derby (UK)
48. Galetto, F., (2001), Looking for Quality in "quality books", 4th Conference on TQM for HEI, Mons (Belgium)
49. Galetto, F., (2001), Quality and Control Carts: Managerial assessment during Product Development and Production Process, AT&T (Society of Automotive Engineers), Barcelona (Spain)
50. Galetto, F., (2001), Quality QFD and control charts,

Conference ATA, Florence (Italy)

51. Galetto, F., (2002), Business excellence Quality and Control Charts", 7th TQM Conference, Verona (Italy)
52. Galetto, F., (2002), Fuzzy Logic and Control Charts, 3rd ICME Conference, Ischia (Italy)
53. Galetto, F., (2002), Analysis of "new" control charts for Quality assessment, 5th Conference on TQM for HEI, Lisbon (Portugal)
54. Galetto, F., (2009), The Pentalogy, VIPSI, Belgrade (Serbia)
55. Galetto, F., (2010),The Pentalogy Beyond, 9th Conference on

TQM for HEI, Verona (Italy)
56. Galetto, F. (2021). Assure: Adopting Statistical Significance for Understanding Research and Engineering. *Journal of Engineering and Applied Sciences Technology. SRC/JEAST-128.*
57. Galetto, F., (2024), News on Control Charts for JMP, Academia.edu
58. Galetto, F., (2024), JMP and Minitab betray Quality, Academia.edu

Appendix A (related to [24])

Typical statement by ALL ...

A uniform model the exponential TBE charts is that the occurrence of events is modelled by a Poisson process, and the time between events X_i (i=1, 2, ...) re independent and identically distributed random variables with pdf $f(x) = \theta^{-1}\exp(-x/\theta)\,for\,x \geq 0$, 0 otherwise, where θ is the "mean time between events". The Control Chart plots the quantity produced before observing an event; The Control Limits can be calculated as

$$LCL = \theta\ln(1 - \alpha/2), \qquad UCL = \theta\ln(\alpha/2)$$

Liu J., Xie M., Sharma P., "A Comparative Study of Exponential Time Between Event Charts", *Quality Technology & Quantitative Management*, 2006 _Issue 3, pp. 347-359

ACTUALLY LCL=L and UCL=U

To construct a t chart, we determine the control limits based on a false alarm rate (α) of 0.0027, equaling that of an individual chart of normal data, and use the median as the centreline". Whenever historical estimates are not available, the scale parameter θ can be estimated using maximum likelihood. …. because both control limits and the centerline are functions of solely θ, by the invariance property of MLEs the estimates are $0.00135\,\bar{t}$, $6.60773\,\bar{t}$, and $\log(2)\,\bar{t}$." .

$$LCL_T = 0.00135\,\bar{t}, \qquad UCL_T = 6.60773\,\bar{t}$$

E. Santiago, J. Smith, Control charts based on the Exponential Distribution, Quality Engineering, Vol. 25, Issue 2, 85-96

ACTUALLY LCL=L and UCL=U

In a subsequent paper "*Improved Shewhart-Type Charts for Monitoring Times Between Events*", *Journal of Quality Technology*, 2016 (*found online, 2024, March*), we find again the *same error [formula (2)]*:

probability control limits *UCL* and *LCL* of a two-sided t_r-chart are respectively given by (see Xie et al. 2002b)

$$P[T_r > UCL] = \alpha_0/2 \text{ and } P[T_r < LCL] = \alpha_0/2.$$

ACTUALLY LCL=L and UCL=U

In another paper we found

Suppose LCL and UCL denote the lower and upper control limits of the Phase II t_r-chart respectively. Then for a given false alarm rate (FAR) α_0, they can be obtained from $P(T_r < LCL|IC) = P(T_r > UCL|IC) = \alpha_0/2$ according to the equal tail probabilities approach. Thus, we have (see also Kumar and Baranwal (2019))

$$\text{LCL} = \frac{\chi^2_{2r,\,\alpha_0/2}}{2\lambda_0} = \frac{A_1}{\lambda_0} \text{ and } \text{UCL} = \frac{\chi^2_{2r,\,1-\alpha_0/2}}{2\lambda_0} = \frac{A_2}{\lambda_0} \qquad (1)$$

where $A_1 = \frac{\chi^2_{2r,\,\alpha_0/2}}{2}$, $A_2 = \frac{\chi^2_{2r,\,1-\alpha_0/2}}{2}$ are the design constants and λ_0 is the known or specified IC rate parameter value. The $\chi^2_{2r,\,a}$ denotes the a-th quantile of the chi-square distribution with $2r$ degrees of freedom. The center line (CL) of the t_r-chart can be considered as the median of the IC distribution of T_r and it is given by $\text{CL} = \frac{\chi^2_{2r,\,0.5}}{2\lambda_0}$.

TBE data (exponential distribution, r=1)

ACTUALLY LCL=L and UCL=U

Chakraborti et. al. (with several papers…)

Excerpt A1. Typical statements in the "Garden …[24]" where the authors name LCL and UCL what **actually** are the Probability Limits L and U. See the figure 9 and the Excerpt 10.

Many other cases, with the same errors, can be found in the "*Garden* …[24], and the Conclusions" where *the authors name LCL and UCL what **actually** are the Probability Limits L and U*.

There is no "free lunch": metanoia and study are needed and necessary.