

Riemann Sums of sinx and cosx

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Abstract

Riemann integrals of the trigonometric functions sinx and cosx have been computed directly from the appropriate Riemann sums.

1. Introduction

Let us remind the two fundamental theorems of integral calculus. Let $F(x)$ be the area under the plot of $f(x)$ between the line $x = a$, x coordinate line and the Ox axis. By looking at the Figure 1. we can see that

$$\lim_{\Delta x \rightarrow 0} F(x + \Delta x) - F(x) = \lim_{\Delta x \rightarrow 0} f(x + \Delta x) \Delta x \quad (1)$$

Dividing the above equation by Δx we arrive at

$$F'(x) = \lim_{\Delta x \rightarrow 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x) \quad (2)$$

$F'(x)$ is the derivative of $F(x)$. $F(x)$ is being called the primary function of $F'(x) = f(x)$.

The first theorem of integral calculus states that the area derivative $F'(x)$ is equal to the function $f(x)$ under the plot of which the area $F(x)$ is being computed.

From Figure 2. we see that

$$\begin{aligned} F(x_1) - F(x_0 = a) &= f(x_1) \Delta x_1 \\ F(x_2) - F(x_1) &= f(x_2) \Delta x_2 \end{aligned} \quad (3)$$

$$\begin{aligned} F(x_{N-1}) - F(x_{N-2}) &= f(x_{N-1}) \Delta x_{N-1} \\ F(x_N = b) - F(x_{N-1}) &= f(x_N) \Delta x_N \end{aligned}$$

where $\Delta x_k = x_k - x_{k-1}$ for $k = 1, 2, \dots, N$.

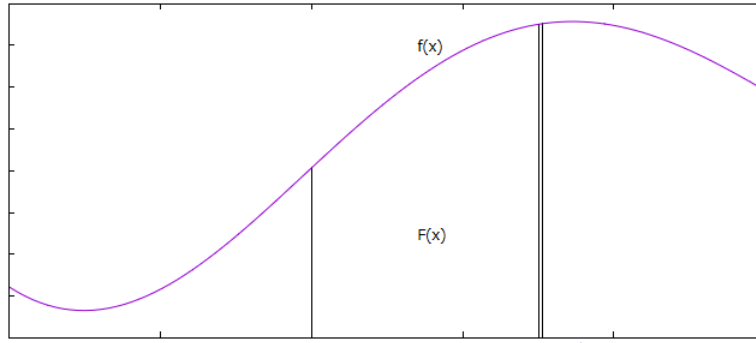


Figure 1: $F(x)$ is the measure of area under the plot of $f(x)$

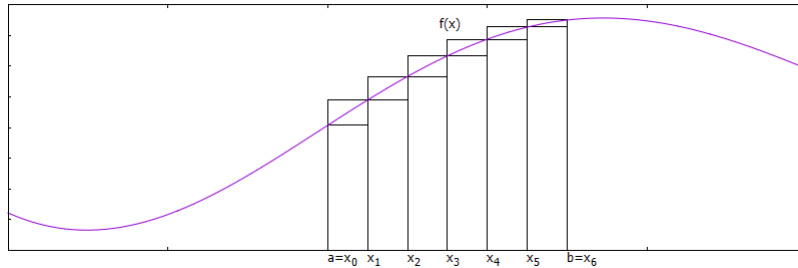


Figure 2: Illustration of the Riemann Integral Computation

Adding all above equations by sides we notice that certain terms cancel out and we arrive at the second theorem of integral calculus

$$F(b) - F(a) = \lim_{\Delta x_k \rightarrow 0} \sum_{k=1}^N f(x_k) \Delta x_k \quad (4)$$

The above equation on the right hand side has the (upper) Riemann sum for the function $f(x)$.

2. Computations

The Riemann sum of $\sin(x)$ and $\cos(x)$ we compute for

$$\Delta x_k = \Delta x = (b - a)/N \quad (5)$$

for $k = 1, 2, \dots, N$ and choose it as

$$\lim_{\Delta x_k \rightarrow 0} \sum_{k=1}^N f(x_k) \Delta x_k = \lim_{\Delta x \rightarrow 0} \sum_{k=1}^N f(k\Delta x) \Delta x \quad (6)$$

with $x_k = k\Delta x$ for $k = 1, 2, \dots, N$ so we have

$$F(b) - F(a) = F(N\Delta x) - F(0) \quad (7)$$

In order to compute the Riemann sums of $\sin x$ and $\cos x$ we use the formula for the sum of the geometric series as presented in [1].

$$\begin{aligned} \sum_{k=1}^N e^{ik\phi} &= \sum_{k=1}^N \cos k\phi + i \sin k\phi \\ &= \cos((N+1)\phi/2) \frac{\sin N\phi/2}{\sin \phi/2} + i \sin((N+1)\phi/2) \frac{\sin N\phi/2}{\sin \phi/2} \end{aligned} \quad (8)$$

where $i^2 = -1$.
We notice that

$$\cos \alpha \sin \beta = \frac{1}{2}(\sin(\alpha + \beta) - \sin(\alpha - \beta)) \quad (9)$$

$$\sin \alpha \sin \beta = -\frac{1}{2}(\cos(\alpha + \beta) - \cos(\alpha - \beta)) \quad (10)$$

and we can substitute

$$\alpha = (N + 1)\phi/2 \quad \beta = N\phi/2 \quad (11)$$

$$\alpha + \beta = N\phi + \phi/2 \quad \alpha - \beta = \phi/2 \quad (12)$$

After computations we arrive at

$$\sum_{k=1}^N \sin k\phi = -\frac{1}{2}(\cos(N\phi + \phi/2) - \cos \phi/2) \frac{1}{\sin \phi/2} \quad (13)$$

$$\sum_{k=1}^N \cos k\phi = \frac{1}{2}(\sin(N\phi + \phi/2) - \sin \phi/2) \frac{1}{\sin \phi/2} \quad (14)$$

Now we calculate the Riemann sums of $\sin x$ and $\cos x$

$$\sum_{k=1}^N \sin k\Delta x \cdot \Delta x = -\frac{\cos(N\Delta x + \Delta x/2)}{\sin \Delta x/2} \frac{\Delta x}{2} + \frac{\cos \Delta x/2}{\sin \Delta x/2} \frac{\Delta x}{2} \quad (15)$$

$$\sum_{k=1}^N \cos k\Delta x \cdot \Delta x = \frac{\sin(N\Delta x + \Delta x/2)}{\sin \Delta x/2} \frac{\Delta x}{2} - \frac{\sin \Delta x/2}{\sin \Delta x/2} \frac{\Delta x}{2} \quad (16)$$

We obtain

$$\lim_{\Delta x \rightarrow 0} \sum_{k=1}^N \sin k\Delta x \cdot \Delta x = \lim_{\Delta x \rightarrow 0} -\frac{\cos(N\Delta x + \Delta x/2)}{\frac{\sin \Delta x/2}{\Delta x/2}} + \frac{\cos \Delta x/2}{\frac{\sin \Delta x/2}{\Delta x/2}} \quad (17)$$

$$\lim_{\Delta x \rightarrow 0} \sum_{k=1}^N \cos k\Delta x \cdot \Delta x = \lim_{\Delta x \rightarrow 0} \frac{\sin(N\Delta x + \Delta x/2)}{\frac{\sin \Delta x/2}{\Delta x/2}} - \frac{\sin \Delta x/2}{\frac{\sin \Delta x/2}{\Delta x/2}} \quad (18)$$

3. Conclusion

The limits of the expressions

$$\lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x/2}{\Delta x/2} = 1 \quad (19)$$

and the Riemann sums for $\sin x$ and $\cos x$ are

$$\lim_{\Delta x \rightarrow 0} \sum_{k=1}^N \sin k\Delta x \cdot \Delta x = \lim_{\Delta x \rightarrow 0} [-\cos x]_{x=0}^{x=N\Delta x} \quad (20)$$

$$\lim_{\Delta x \rightarrow 0} \sum_{k=1}^N \cos k\Delta x \cdot \Delta x = \lim_{\Delta x \rightarrow 0} [\sin x]_{x=0}^{x=N\Delta x} \quad (21)$$

We can check that our results are correct. The derivative of $-\cos x$ is equal to the primary function $\sin x$ and the derivative of $\sin x$ is equal to the primary function $\cos x$.

References

1. Hirst, E. K. (1995) Modular Mathematics Numbers, Sequences and Series Butterworth-Heinemann, *Oxford*.

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