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Ribbonness of Kervaire's Sphere-Link in Homotopy 4-Sphere and its Consequences to 2-Complexes

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Abstract

M. A. Kervaire showed that every group of deficiency d and weight d is the fundamental group of a smooth sphere-link of d components in a smooth homotopy 4-sphere. In the use of the smooth unknotting conjecture and the smooth 4D Poincar´e conjecture, any such sphere-link is shown to be a sublink of a free ribbon sphere-link in the 4-sphere. Since every ribbon sphere-link in the 4-sphere is also shown to be a sublink of a free ribbon sphere-link in the 4-sphere, Kervaire's sphere-link and the ribbon sphere-link are equivalent con- cepts. By applying this result to a ribbon disk-link in the 4-disk, it is shown that the compact complement of every ribbon disk-link in the 4-disk is aspherical. By this property, a ribbon disk-link presentation for every contractible finite 2-complex is introduced. By using this presentation, it is shown that every connected subcomplex of a contractible finite 2-complex is aspherical (meaning partially yes for Whitehead aspherical conjecture).

Keywords: Kervaire's Sphere-Link, Ribbon Sphere-Link, 2-Complex, Whitehead Aspherical Conjecture.

1. Introduction

A group with finite presentation $< x_1, x_2, ..., x_n | r_1, r_2, ..., r_{n-d} >$ is called a group of *deficiency* d. A group G has *weight d* if there are d elements $w_1,w_2, ...,w_d$ in G whose normal closure is equal to G, where the system of elements $w_1,w_2, ...,w_d$ is called a *weight system* of G. Let X be a closed connected oriented smooth 4-manifold. A *sphere-link* or an S^2*-link* in X is a disjoint sphere system smoothly embedded in X. A *surgery of X along a loop system* k_i $(i = 1, 2, . . . , n)$ is the operation replacing a normal D^3-bundle system $k_i \times D^3$ $(i = 1, 2, . . . , n)$ of k_i $(i = 1, 2, . . . , n)$ in X by a normal D^2-bundle system $D^2_i \times S^2$ $(i = 1, 2, . . . , n)$ of the 2-sphere system $K_i = 0_i \times S^2$ $(i = 1, 2, . . . , n)$ under the identifications that $\partial D^2_i = k_i$ $(i = 1, 2, . . . , n)$ and $\partial D^3 = S^2$. Let X' be the smooth 4-manifold resulting from X by this surgery. The spheres K_i $(i = 1, 2, . . . , n)$ form an S^2-link K in X'. The 4-manifold X' is said to be *obtained from the 4-manifold X by surgery along a loop system* k_i $(i = 1, 2, . . . , n)$ in X, and conversely the 4-manifold X is said to be *obtained from the 4-manifold X′ by surgery along a sphere system K in X′.* Note that there are canonical fundamental group isomorphisms

$$\pi_1(X, v) \cong \pi_1(X \setminus k, v) \cong \pi_1(X' \setminus K, v)$$

by general position. The *closed 4D handlebody of genus n* is the 4-manifold

$$Y^S = S^4 \#_{i=1}^n S^1 \times S^3_i$$

which is the connected sum of S^4 and n copies $S^1 \times S^3_i$ $(i = 1, 2, . . . , n)$ of the closed 4D handle $S^1 \times S^3$. A *legged loop system with base point v* in X is a graph ωk of legged loops $\omega_i k_i$ $(i = 1, 2, . . . , d)$ embedded in X consisting of a disjoint simple loop system k_i $(i = 1, 2, . . . , d)$ and a leg system (=embedded path system) ω_i $(i = 1, 2, . . . , d)$ such that ω_i connects the base point v and a point $p_i \in k_i$ for every i and the legs ω_i for all i are made disjoint except for the base point v. The fundamental group $\pi_1(Y^S, v^S)$ is identified with the free group $< x_1, x_2, . . . , x_n >$ with basis $x_1, x_2, . . . , x_n$ represented by the *standard* legged loop system $\omega^S x$ of legged loops $\omega_i k_i$ $(i = 1, 2, . . . , n)$ with base point v^S in Y^S using the standard loop $k_i = S^1 \times 1_i$ of $S^1 \times S^3_i$ and a leg ω_i joining point v^S in Y^S using the standard loop $k_i = S^1 \times 1_i$ of $S^1 \times S^3_i$ and a leg ω_i joining v^S and the point $(1, 1_i) \in 1 \times S^3_i$ not meeting $1 \times (S^3_i \setminus \{1_i\})$, for every i. A smooth homotopy 4-sphere is a smooth 4-manifold M homotopy equivalent to the 4-sphere S^4. A meridian system of an S^2-link K with k components in M is a legged loop system ωm with base point v in $M \setminus K$ whose loop system m consists of a meridian loop of every component of K. Kervaire showed the following theorem in [1].

(The condition that $H_1(G)=G/[G,G]$ is a free abelian group of rank d is omitted since every group G of deficiency d and weight d has this condition.)

Kervaire's Theorem

For every group G of deficiency d and weight d, there is an S^2-link K with d components in a smooth homotopy 4-sphere M such that there is an isomorphism $G \cong \pi_1(M \setminus K, v)$ sending the weight system to a meridian system of K.

The construction of an S^2-link in this theorem is explained as follows.

Construction of Kervaire's S^2-link. Let $< x_1, x_2, ..., x_n | r_1, r_2, ..., r_{n-d} >$ be a finite presentation of G of deficiency d, and $w_1, w_2, ..., w_d$ a weight system of G. Let $G(n; n-d, d)$ be the *triple system* of the free group $< x_1, x_2, ..., x_n >$, the relator system $r_1, r_2, ..., r_{n-d}$ written as words in $x_1, x_2, ..., x_n$ and a weight system $w_1, w_2, ..., w_d$ written as words in $x_1, x_2, ..., x_n$. Identify the free group $< x_1, x_2, ..., x_n >$ with the fundamental group $\pi_1(Y^S, v^S)$ of the 4D closed handlebody Y^S. Let X be a 4-manifold obtained from Y^S by surgery along a loop system $k(r_1), k(r_2), . . . , k(r_{n-d})$ in Y^S representing the words $r_1, r_2, . . . , r_{n-d}$ in $\pi_1(Y^S, v^S)$. The fundamental group $\pi_1(X, v^S)$ has the presentation $< x_1, x_2, ..., x_n | r_1, r_2, ..., r_{n-d} >$ by Seifert-van Kampen theorem. Let M be the 4-manifold obtained by surgery along a loop system $k(w_1), k(w_2), . . . , k(w_d)$ in X representing the weight system $w_1, w_2, ..., w_d$ of $\pi_1(X, v^S)$. The manifold M is a smooth homotopy 4-sphere by Seifert-van Kampen theorem. The S^2-link K of d components in M is given by the core spheres $K_i = 0_i \times \partial D^3$ ($i = 1, 2, . . . , d$) of $D^2_i \times \partial D^3$ replacing $k(w_i) \times D^3$ ($i = 1, 2, . . . , d$). The fundamental group $\pi_1(M \setminus K, v)$ is isomorphic to $\pi_1(X, v) \cong G$ by an isomorphism sending a meridian system of K in M to the weight system $w_1, w_2, ..., w_d$. This completes the construction of Kervaire's S^2-link.

Kervaire's S^2-link K in this construction is uniquely determined by the triple system $G(n; n-d, d)$ of the free group $< x_1, x_2, ..., x_n >$, the relator system $r_1, r_2, ..., r_{n-d}$ and the weight system $w_1, w_2, ..., w_d$, which is called Kervaire's S^2-link of type $G(n; n-d, d)$ or simply an S^2-link of *type* $G(n; n-d, d)$. For a smooth surface-link L in S^4, the fundamental group $\pi_1(S^4 \setminus L, v)$ is a *meridian-based free group* if $\pi_1(S^4 \setminus L, v)$ is a free group with a basis represented by a meridian system of L with base point v. A smooth surface-link L in S^4 is a trivial surface-link if the components of L bound disjoint handlebodies smoothly embedded in S^4. In this paper, *Kervaire's* S^2-link is studied by using Smooth 4D Poincaré Conjecture and Smooth Unknotting Conjecture for a surface-link stated as follows:

Smooth 4D Poincaré Conjecture. Every 4D smooth homotopy 4-sphere M is diffeomorphic to S^4.

Smooth Unknotting Conjecture. Every smooth surface-link F in S^4 with a meridian-based free fundamental group $\pi_1(S^4 \setminus F, v)$ is a trivial surface-link.

The positive proofs of these conjectures are in [2] and [3-5], respectively. *From now on, every smooth homotopy 4-sphere M is identified with the 4-sphere S^4.* An S^2-link L in S^4 is a *ribbon* S^2-link if L is an S^2-link obtained from a trivial S^2-link O in S^4 by surgery along embedded 1-handles on O. See [8, II], [6] for earlier concept of a ribbon surface-link. An S^2-link L in S^4 is a *free S^2-link of rank n* if the fundamental group $\pi_1(S^4 \setminus L, v)$ is a (not necessarily meridian based) free group of rank n. The following theorem is the first result of this paper.

Theorem 1.1. The following three statements on an S^2-link K with d components in the 4-sphere S^4 are mutually equivalent:

(1) The S^2-link K is an S^2-link of type $G(n; n-d, d)$ for some n.
(2) The S^2-link K is a sublink with d components of a free ribbon S^2-link of some rank n.
(3) The S^2-link K is a ribbon S^2-link with d components.

By combining Kervaire's Theorem with Theorem 1.1, the following characterization of the fundamental group $\pi_1(S^4 \setminus K, v)$ of a ribbon S^2-link K in S^4 is obtained.

Corollary 1.2. A group G is a group of deficiency d and weight d if and only if G is isomorphic to the group $\pi_1(S^4 \setminus K, v)$ of a ribbon S^2-link K of d components in S^4 by an isomorphism sending the weight system of G to a meridian system of K.

In the proof of Theorem 1.1, the claim that every free S^2-link is a free ribbon S^2-link is needed whose proof was done in [11]. For completeness of the present argument, this claim is moved to Appendix of this paper as *Free Ribbon Lemma* together with the proof. The proof of Theorem 1.1 is done in Section 2 by assuming Free Ribbon Lemma. A *trivial proper disk system* in the 4-disk D^4 is a disjoint proper disk system D_i ($i = 1, 2, . . . , n$) in D^4 obtained by an interior push of a disjoint disk system D_i^0 (i=1,2,...,n) in the 3-sphere $S^3 = \partial D^4$. A *ribbon disk-link* of d components is a disjoint proper disk system L^D in D^4 which is obtained by an interior push of a disjoint disk system that is the union of a trivial proper disk system D_i ($i = 1, 2, . . . , n$) in D^4 for some n and a disjoint band system b^0_j ($j = 1, 2, . . . , n-d$) in S^3 spanning the trivial link ∂D_i ($i = 1, 2, . . . , n$) in S^3. The link ∂L^D in S^3 is called a *classical ribbon link.* By construction, the double of a ribbon disk-link L^D of k components in D^4 is a ribbon S^2-link L of k components in S^4. It is a standard fact that every ribbon S^2-link (S^4,L) is considered as the double $(D^4 \cup -D^4, L^D \cup -L^D)$ of a ribbon disk-link (D^4,L^D) and its copy $(-D^4,-L^D)$, namely $(S^4,L) = (\partial(D^4 \times I), \partial(L^D \times I))$, $I = [-1, 1]$. To construct a ribbon disk-link (D^4,L^D) from a ribbon S^2-link (S^4,L), it is noted that a trivial S^2-link O and embedded 1-handles to construct L are always isotopically deformed into a symmetric position with respect to the equatorial 3-sphere $S^3 = \partial D^4 = \partial(-D^4)$ in $S^4 = D^4 \cup -D^4$ (see [8, II]). A *free ribbon disk-link of rank n* is a ribbon disk-link L^D in D^4 such that the fundamental group $\pi_1(D^4 \setminus L^D, v)$ is a free group of rank n. In Lemma 3.1, it is shown that the inclusion $(D^4,L^D) \to (S^4,L)$ induces an isomorphism

$$\pi_1(D^4 \setminus L^D, v) \to \pi_1(S^4 \setminus L, v).$$

Thus, the S^2-link L is a free ribbon S^2-link in S^4 if and only if the ribbon disk-link L^D is a free ribbon disk-link in D^4. The *compact complement* of a ribbon disk-link L^D in the 4-disk D^4 is the compact 4-manifold $E(L^D) = cl(D^4 \setminus N(L^D))$ for a normal disk-bundle $N(L^D) = L^D \times D^2$ of L^D in D^4. By Theorem 1.1, every ribbon S^2-link K is a sublink of a free ribbon S^2-link L of some rank n, so that every ribbon disk-link K^D is a sublink of a free ribbon disk-link L^D of some rank n by Lemma 3.1. A connected complex is understood as a cell complex P obtained from a bouquet of loops, called the *1-skelton* P^1 of P, by adding $q(\geq 2)$-cells to P^1. A connected complex is aspherical if the universal covering space is contractible. A connected 2-complex P is aspherical if and only if the second homotopy group $\pi_2(P, v) = 0$.

For a subcomplex P' of a cell complex P, a *deformation retract* from P to P' is a map $r : P \to P'$ such that the composite map $ir : P \to P$ for the inclusion $i : P' \subset P$ is homotopic to the identity $1 : P \to P$, where if the homotopy is further relative to P', then the map r is called a *strong deformation retract* from P to P' (see [9]). It is shown in Lemma 3.2 that for every free ribbon disk-link L^D in D^4, there is a strong deformation retract

$$r : E(L^D) \to \omega x$$

from the compact complement $E(L^D)$ to a legged n-loop system ωx in $E(L^D)$ representing the free group $\pi_1(E(L^D), v) = < x_1, x_2, \ldots, x_n >$. Section 3 is devoted to explanations of Lemmas 3.1 and 3.2 on ribbon disk-links. In Section 4, a decomposition of the 4-disk D^4 into a normal disk-bundle $N(L^D) = L^D \times D^2$ of a free ribbon disk-link L^D and the compact complement $E(L^D)$ is considered. Let $Q(L^D) = E(L^D) \cup N(L^D)$ denote this decomposition of D^4. For a disk-link L^D of n components, let $p_* = \{p_i | i = 1, 2, \ldots, n\}$ be a set of n points, one point taken from each component of L^D. The strong deformation retract $N(L^D) \to p_* \times D^2$ shrinking L^D into p_* and the strong deformation retract $r : E(L^D) \to \omega x$ in Lemma 3.2 define a map

$$\rho : Q(L^D) \to P(L^D)$$

with $P(L^D)$ a finite 2-complex consisting of the 1-skelton $P(L^D)^1 = \omega x$ and the 2-cells $p_* \times D^2$ attached by the attaching map $p_* \times \partial D^2 \to \omega x$ defined by r. The map ρ is called a *ribbon disk-link presentation* for the finite 2-complex $P(L^D)$. A *1-full* subcomplex of a cell complex P is a subcomplex P' of P such that the 1-skelton $(P')^1$ of P' is equal to the 1-skelton P^1 of P. For a sublink K^D of L^D, let $N(K^D) = K^D \times D^2$ be the subbundle of the disk-bundle $N(L^D)$. Then the union $Q(L^D, K^D) = E(L^D) \cup N(K^D)$ is a decomposition of the compact complement $E(L^D \setminus K^D)$ of the sublink $L^D \setminus K^D$ of L^D in D^4 and the ribbon disk-link presentation $\rho : Q(L^D) \to P(L^D)$ for $P(L^D)$ sends $Q(L^D, K^D)$ to a 1-full 2-subcomplex $P(L^D, K^D)$ of $P(L^D)$. Further, every 1-full 2-subcomplex of $P(L^D)$ is obtained from a sublink K^D of L^D in this way. The following theorem is shown in Section 4.

Theorem 1.3. For every free ribbon disk-link L^D in the 4-disk D^4, the ribbon disk-link presentation $\rho : Q(L^D) \to P(L^D)$ for the finite 2-complex $P(L^D)$ induces a homotopy equivalence $Q(L^D, K^D) \to P(L^D, K^D)$ for every sublink K^D of L^D including $K^D = \emptyset$ and $K^D = L^D$. In particular, the finite 2-complex $P(L^D)$ is contractible. Further, every contractible finite 2-complex P is taken as $P = P(L^D)$ for a free ribbon disk-link L^D in the 4-disk D^4.

In Section 5, the following theorem is shown by using Theorem 1.3.

Theorem 1.4. The compact complement $E(K^D)$ of every ribbon disk-link K^D in the 4-disk D^4 is aspherical.

The asphericity of the compact complement of a ribbon disk-knot in D^4 has been conjectured by Howie [10] after having found some gaps on the arguments of Yanagawa [11] and Asano, Marumoto, Yanagawa [12]. Since the fundamental group of an aspherical complex is torsion-free, the following corollary is obtained from Lemma 3.1 and Theorem 1.4.

Corollary 1.5. The fundamental group $\pi_1(S^4 \setminus L, v)$ of every ribbon S^2-link in the 4-sphere S^4 is torsion-free.

This result gives the positive answer to the author's old question in [8, II(pp.57-58)]. The following corollary is obtained from Theorems 1.3 and 1.4, because if a connected subcomplex P' of a contractible finite 2-complex P is not 1-full, then a 1-full subcomplex P'' of P is constructed from P' by adding a bouquet of some loops in the 1-skelton P^1 of P to P', and P'' is aspherical if and only if P' is aspherical.

Corollary 1.6. Every connected subcomplex of every contractible finite 2-complex is aspherical.

This result is a partial positive confirmation of Whitehead aspherical conjecture claiming that every connected subcomplex of an aspherical 2-complex is aspherical [13].

2. Proof of Theorem 1.1

The following lemma is a standard result obtained as a corollary of Smooth 4D Poincaré Conjecture and Smooth Unknotting Conjecture and shown in [2, Corollary 1.5] without a mention of a legged loop system.

Lemma 2.1. Every closed connected orientable smooth 4-manifold Y with $\pi_1(Y, v)$ a free group and $H_2(Y; Z) = 0$ is diffeomorphic to the closed 4D handlebody Y^S by a diffeomorphism $f : Y \to Y^S$ sending any given a legged loop system ωx with base point v representing a basis $x_1, x_2, \ldots, x_n$ of $\pi_1(Y, v)$ to a standard legged loop system $\omega^S x$ of Y^S. For any given spin structures on Y and Y^S, the diffeomorphism f can be taken spin-structure-preserving.

Proof of Lemma 2.1. Let M be the 4-manifold obtained from Y by surgery along the loop system $k(\omega x)$ of ωx, which is identified with S^4 by Smooth 4D Poincaré Conjecture since it is a smooth homotopy 4-sphere by the van Kampen theorem and a homological argument. Let L be the S^2-link in S^4 obtained from $k(\omega x)$ by the surgery. Then $\pi_1(S^4 \setminus L, v) = < x_1, x_2, \ldots, x_n >$ and the legged loop system ωx with base point v in Y is a meridian system of L in S^4 representing the basis $x_1, x_2, \ldots, x_n$. By Smooth Unknotting Conjecture for an S^2-link, the S^2-link L bounds disjoint 3-balls smoothly embedded in S^4 so that each 3-ball meets ωx with just one transverse intersection point in the loop system $k(\omega x)$ (see [5]). By the back surgery from (M, L) to $(Y, k(\omega x))$, there is an orientation-preserving diffeomorphism $f : Y \to Y^S$ with $f(\omega x) = \omega^S x$. Given any spin structures on Y and Y^S, note that there is an orientation-preserving spin-structure-changing diffeomorphism : $S^1 \times S^3 \to S^1 \times S^3$ (see [14] for a similar diffeomorphism on $S^1 \times S^2$). Thus, by composing f with the orientation-preserving spin-structure-changing diffeomorphisms on some connected summands of Y^S which are copies of $S^1 \times S^3$, the diffeomorphism $f : Y \to Y^S$ is modified into an orientation-preserving spin-structure-preserving diffeomorphism.This completes the proof of Lemma 2.1.

The proof of Theorem 1.1 is done as follows.

2.2 Proof of Theorem 1.1.

Proof of (1)→(2). Assume that an S^2-link K of type $G(n; n-d, d)$ in S^4 for any n is constructed from the triple system $G(n; n-d, d)$ consisting of the free basis x_i ($i = 1, 2, \dots, n$), the relator system r_i ($i = 1, 2, \dots, n-d$) written as words in x_i ($i = 1, 2, \dots, n$) and a weight system w_j ($j = 1, 2, \dots, d$) written as words in x_i ($i = 1, 2, \dots, n$). The fundamental group $\pi_1(Y^S, v^S)$ of Y^S of rank n is identified with the free group $< x_1, x_2, ..., x_n >$. Note that the elements r_i, w_j ($i = 1, 2, \dots, n-d; j = 1, 2, \dots, d$) form a weight system of the free group $\pi_1(Y^S, v^S)$. Represent the elements $r_i, w_j \in \pi_1(Y^S, v^S)$ ($i = 1, 2, \dots, n-d; j = 1, 2, \dots, d$) by a disjoint simple loop system $k(r_i)$, $k(w_j)$ ($i = 1, 2, \dots, n-d; j = 1, 2, \dots, d$) in Y^S. The 4-manifold M obtained from Y^S by surgery along the loop system $k(r_i)$, $k(w_j)$ ($i = 1, 2, \dots, n-d; j = 1, 2, \dots, d$) is a smooth homotopy 4-sphere identified with S^4. Let L be the S^2-link in S^4 of the sphere system $K(r_i), K(w_j)$ ($i = 1, 2, \dots, n-d; j = 1, 2, \dots, d$) occurring from the loop system $k(r_i)$, $k(w_j)$ ($i = 1, 2, \dots, n-d; j = 1, 2, \dots, d$) by the surgery. The fundamental group $\pi_1(S^4 \setminus L, v)$ is isomorphic to the free group $< x_1, x_2, ..., x_n >$ by an isomorphism sending a meridian system of L to the weight system r_i, w_j ($i = 1, 2, \dots, n-d; j = 1, 2, \dots, d$). By Free Ribbon Lemma of Appendix, the S^2-link L is a free ribbon S^2-link in S^4 of rank n. The sublink of L consisting of the components $K(w_j)$ ($j = 1, 2, \dots, d$) is is just the S^2-link K of type $G(n; n-d, d)$, which is a sublink of the free ribbon S^2-link L in S^4. This shows (1)→(2).

Proof of (2)→(1). Let K be a sublink of d components of a free ribbon S^2-link L of n components in S^4 of rank n. Let $\pi_1(S^4 \setminus L, v) =< x_1, x_2, \dots, x_n >$. Let Y be the 4-manifold obtained from S^4 by surgery along L. By Lemma 2.1, Y is identified with Y^S of genus n such that $\pi_1(S^4 \setminus L, v) =< x_1, x_2, \dots, x_n >$ is identified with $\pi_1(Y^S, v^S)$ by an isomorphism sending a meridian system of L in S^4 to a weight system of $\pi_1(Y^S, v^S)$. This means that the ribbon S^2-link K is nothing but an S^2-link of type $G(n; n-d, d)$ for the triple system $G(n; n-d, d)$ consisting of the free group $\pi_1(Y^S, v) =< x_1, x_2, \dots, x_n >$, a relator system $r_1, r_2, ..., r_{n-d}$ coming from the meridian system of $L \setminus K$, and a weight system $w_1, w_2, ..., w_d$ coming from the meridian system of K. This shows (2)→(1).

Proof of (2)→(3). This proof is trivial.

Proof of (3)→(2). By definition, assume that a ribbon S^2-link K of d components in S^4 is obtained from a trivial S^2-link O of n components in S^4 by surgery along a 1-handle system h on O. Let $O \times [0, 1]$ be a collar of O in S^4 where the 1-handle system h meets only to $O \times 0$, and $W = O \times [0, 1] \cup h$ a d-component compact 3-manifold bounded by $K \cup O \times 1$. Let K_i ($i = 1, 2, \dots, d$) be the components of K. Let O' be a sublink of $O \times 1$ of $n - d$ components obtained by removing any one component of $O \times 1$ from the boundary of the component of W containing the component K_i for every i. Then there are isomorphisms

$$\pi_1(S^4 \setminus W, v) \to \pi_1(S^4 \setminus (K \cup O'), v) \quad \text{and} \quad \pi_1(S^4 \setminus W, v) \to \pi_1(S^4 \setminus O, v).$$

This is because there are deformation retracts from W to a 2-complex consisting of $K \cup O'$ and some spanning arcs and from W to a 2-complex consisting of O and some spanning arcs, and the spanning arcs do not affect the fundamental group. Since $\pi_1(S^4 \setminus O, v)$. is a free group of rank n, the S^2-link $L = K \cup O'$ of n components is a free ribbon S^2-link of rank n in S^4 containing K as a sublink. This shows (3)→(2).

This completes the proof of Theorem 1.1.

3. Basic Lemmas of ribbon disk-links

For a ribbon disk-link (D^4, L^D) of a ribbon S^2-link (S^4, L), let α be the reflection of (S^4, L) exchanging (D^4, L^D) and the other copy $(-D^4, -L^D)$ in (S^4, L). Although the following lemma may be more or less known (cf. [11]), the proof is given here for convenience.

Lemma 3.1. For a ribbon disk-link L^D in D^4 of a ribbon S^2-link L in S^4, the inclusion $(D^4, L^D) \to (S^4, L)$ induces an isomorphism

$$\pi_1(D^4 \setminus L^D, v) \to \pi_1(S^4 \setminus L, v).$$

Proof of Lemma 3.1. Use the retraction $S^4 \setminus L \to D^4 \setminus L^D$ induced from the quotient by the reflection α. Then the canonical homomorphism $\pi_1(D^4 \setminus L^D, v) \to \pi_1(S^4 \setminus L, v)$ is shown to be a monomorphism. On the other hand, for the copy $(-D^4, -L^D)$ of (D^4, L^D), the inclusion $(\partial(-D^4), \partial(-L^D)) \to (-D^4, -L^D)$ induces an epimorphism $\pi_1(\partial(-D^4) \setminus \partial(-L^D), v) \to \pi_1(-D^4 \setminus -L^D, v)$ by the definition of ribbon disk-link and Seifert-van Kampen theorem. This means that the canonical monomorphism $\pi_1(D^4 \setminus L^D, v) \to \pi_1(S^4 \setminus L, v)$ is also an epimorphism and thus, an isomorphism.

The *4D handlebody of genus n* is the 4-manifold

$$Y^D = D^4{}_{\partial}\#_{i=1}^{n} S^1 \times D_i^3$$

which is the boundary connected sum of D^4 and n copies $S^1 \times D_i^3$ ($i = 1, 2, \dots, n$) of the 4D handle $S^1 \times D^3$. By using the asphericity of Y^D, the following lemma is obtained.

Lemma 3.2. For every free ribbon disk-link L^D of rank n in D^4, there is a strong deformation retract

$$r : E(L^D) \to \omega x$$

from the compact complement $E(L^D)$ to a legged n-loop system ωx with base point v in $E(L^D)$ representing any basis $x_1, x_2, \dots, x_n$ of the free group $\pi_1(E(L^D), v)$.

Proof of Lemma 3.2. Let L be the free ribbon S^2-link of rank n in S^4 obtained by taking the double of (D^4, L^D). Note that the double

$$Y = \partial(E(L^D) \times I) = E(L^D) \times \{-1\} \cup (\partial E(L^D)) \times I \cup E(L^D) \times \{1\}$$

of $E(L^D)$ is diffeomorphic to the 4-manifold Y' obtained from S^4 by surgery along L. Since there is a canonical isomorphism $\pi_1(S^4 \setminus L, v) =< x_1, x_2, \ldots, x_n > \to \pi_1(Y', v)$ and $H_2(Y'; \mathbf{Z}) = 0$, the 4-manifold Y' is identified with Y^S under the canonical identities $\pi_1(E(L^D, v) = \pi_1(Y^S, v) =< x_1, x_2, \ldots, x_n >$ by Lemmas 2.1 and 3.1. Let ωx be a legged n-loop system in $E(L^D)$, and $-\omega x$ a copy of ωx in the copy $-E(L^D)$ of $E(L^D)$ in $Y' = Y^S$. Note that $\pm\, \omega x$ are isotopically deformed into the standard n-loop system in Y^S. Let $N(\omega x)$ be a regular neighborhood of ωx in $E(L^D)$, and $N(-\omega x)$ the copy of $N(\omega x)$ in the copy $-E(L^D)$. Since $N(\omega x)$ is diffeomorphic to the 4D handlebody Y^D of genus n, it is shown that the compact complement $E(L^D)^+ = \mathrm{cl}(Y^S \setminus N(-\omega x))$ is diffeomorphic to Y^D and the compact complement $H = \mathrm{cl}(Y^S \setminus N(\omega x) \cup N(-\omega x))$ is diffeomorphic to the product $Z^S \times I$ for the closed 3D handlebody $Z^S = S^3 \#_{i=1}^{n} S^1 \times S^2_i$ of genus n. Note that the reflection α in Y^S exchanging $E(L^D)$ and $-E(L^D)$ induces a reflection in H whose fixed point set is the boundary $Z(\partial L^D) = \partial E(L^D)$ of $E(L^D)$. Let H' be one of the two 3-manifolds obtained by splitting H along $Z(\partial L^D)$ such that $E(L^D)^+ = E(L^D) \cup H'$. Then $H = H' \cup \alpha(H')$. By [15], the 3-manifold $Z(\partial L^D)$ is an imitation of Z^S which has the property that the inclusion homomorphism $\pi_1(Z^S, v) \to \pi_1(H', v)$ is an isomorphism and any covering triad $(\tilde{H}'; \tilde{Z}(\partial L^D), \tilde{Z}^S)$ of the triad $(H'; Z(\partial L^D), Z^S)$ is a homology cobordism. This means that the inclusion $i : E(L^D) \to E(L^D)^+$ is a homotopy equivalence by Seifert-van Kampen theorem and the universal covering lift $\tilde{i} : \tilde{E}(L^D) \to \tilde{E}(L^D)^+$ induces an isomorphism $\tilde{i}_* : H_*(\tilde{E}(L^D); \mathbf{Z}) \to H_*(\tilde{E}(L^D)^+; \mathbf{Z})$ because

$$H_*(\tilde{E}(L^D)^+, \tilde{E}(L^D); \mathbf{Z}) \cong H_*(\tilde{H}', \tilde{Z}(\partial L^D); \mathbf{Z}) = 0$$

by the excision isomorphism. Thus, $E(L^D)$ is homotopy equivalent to the legged n-loop system ωx. For a polyhedral pair (P, P'), if the inclusion $i : P' \subset P$ is homotopy equivalent, then there is a strong deformation retract $r : P \to P'$ (see [9, p. 31]). Thus, there is a strong deformation retract $r : E(L^D) \to \omega x$.

In Lemma 3.2, note that in general the compact complement $E(L^D)$ of a free ribbon disk-link L^D in D^4 is not diffeomorphic to Y^D. For example, the Kinoshita-Terasaka knot k_{KT} in S^3 bounds a free ribbon-disk knot K^D of rank one in D^4. Since the 3-manifold $Z(\partial K^D)$ which is the 0-surgery manifold of k_{KT} is not diffeomorphic to $Z^S = S^1 \times S^2$ by the solution of property R conjecture (see [16]), the compact complement $E(K^D)$ is not diffeomorphic to Y^D (see [15]).

4. Proof of Theorem 1.3

The proof of Theorem 1.3 is done as follows.

4.1: Proof of Theorem 1.3. Identifications

$$\pi_1(E(L^D), v) = \pi_1(\omega x) =< x_1, x_2, \ldots, x_n >$$

are fixed by the strong deformation retract $r : E(L^D) \to \omega x$. The ribbon disk-link presentation $\rho : Q(L^D) \to P(L^D)$ for $P(L^D)$ induces an isomorphism $\rho_\# : \pi_1(Q(L^D, K^D), v) \to \pi_1(P(L^D, K^D), v)$ for every sublink K^D of L^D including $K^D = \emptyset$ and $K^D = L^D$ by Seifert-van Kampen theorem, because the strong deformation retract $r : E(L^D) \to \omega x$ induces the identical word system $r_* = \{r_1, r_2, \ldots, r_n\}$ of the loop system $p_* \times \partial D^2$ in $< x_1, x_2, \ldots, x_n >$ by the attaching map $r : p_* \times \partial D^2 \to \omega x$ of the 2-cell system $p_* \times D^2$. In particular, $\pi_1(P(L^D), v) =< x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n >= \{1\}$. Let $\tilde{\rho} : \tilde{Q}(L^D, K^D) \to \tilde{P}(L^D, K^D)$ be the universal covering lift of $\rho : Q(L^D, K^D) \to P(L^D, K^D)$. By Mayer-Vietoris homology sequence, $H_m(\tilde{Q}(L^D, K^D); \mathbf{Z}) = 0$ for all $m \geq 3$ and ρ induces an isomorphism $\tilde{\rho}_* : H_2(Q(L^D, K^D); \mathbf{Z}) \to H_2(P(L^D, K^D); \mathbf{Z})$ for every sublink K^D of L^D including $K^D = \emptyset$ and $L^D = K^D$. Thus, $\rho : Q(L^D, K^D) \to P(L^D, K^D)$ is a homotopy equivalence for every sublink K^D of L^D including $K^D = \emptyset$ and $K^D = L^D$. In particular, $P(L^D)$ is a finite contractible 2-complex. Let P be a contractible finite 2-complex obtained from the 1-skelton $P^1 = \omega x$, a legged n loop system with base point v, so that $\pi_1(P^1, v) =< x_1, x_2, \ldots., x_n >$. Assume that P is obtained from P^1 by attaching 2-cells $e_1, e_2, \ldots, e_n$. Since $\pi_1(P, v) = 1$, the 2-complex P provides the triple system $G(n; 0, n)$ in the construction of Kervaire's 2-link which consists of the free group $< x_1, x_2, \ldots., x_n >$, the empty relator set and the weight system $w_1, w_2, \ldots, w_n$ given by the attaching data of $e_1, e_2, \ldots, e_n$ to P^1. By Theorem 1.1, there is a free ribbon S^2-link (S^4, L) with an isomorphism $\pi_1(S^4 \setminus L, v) = < x_1, x_2, \ldots, x_n >$ sending a meridian system of L to the weight system $w_1, w_2, \ldots, w_n$. By Lemma 3.1, there is a free ribbon disk-link (D^4, L^D) with an isomorphism $\pi_1(D^4 \setminus L^D, v) = < x_1, x_2, \ldots, x_n >$ sending a meridian system of L^D in D^4 to the weight system $w_1, w_2, \ldots, w_n$. By Lemma 3.2, there is a strong deformation retract $r : E(L^D) \to P^1 = \omega x$, which induces a ribbon-disk presentation $\rho : Q(L^D) \to P(L^D)$ for $P(L^D) = P$ because the loop system $p_* \times \partial D^2$ is just the meridian system of L^D.

5. Proof of Theorem 1.4

The proof of Theorem 1.4 is done as follows.

5.1: Proof of Theorem 1.4. Let K^D be a ribbon disk-link in D^4 of d components, and $S(*)$ any immersed 2-sphere in $E(K^D)$. It suffices to show that there is a free ribbon disk-link L^D in D^4 of some rank n which contains K^D as a sublink and is disjoint from $S(*)$. This is because $S(*) \subset E(L^D) \subset E(K^D)$ meaning that $S(*)$ is null-homotopic in $E(L^D)$ and hence in $E(K^D)$ since $\pi_2(E(L^D), v) = 0$ by Lemma 3.2, so that $\pi_2(E(K^D), v) = 0$ meaning that $E(K^D)$ is aspherical, for $E(K^D)$ is homotopy equivalent to a 2-complex by Theorem 1.3.

The pair (D^4, S^3) is considered as the one-point compactification of the pair $(\mathbf{R}^3[0,+\infty), \mathbf{R}^3)$ of the upper-half 4-space

$$\mathbf{R}^3[0, +\infty) = \{(x_1, x_2, x_3, t) | -\infty < x_i < +\infty\, (i = 1, 2, 3), t \geq 0\}$$

and the 3-space

$$\mathbf{R}^3 = \{(x_1, x_2, x_3) | -\infty < x_i < +\infty\, (i = 1, 2, 3)\}.$$

Also, K^D and $S(*)$ are considered in $\mathbf{R}^3[0,+\infty)$. By the motion picture method [8, I], assume that a normal form of the disk-link K^D in $(\boldsymbol{R}^3[0,+\infty)$ is given as follows:

$$K^D \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 2, \\ d_*[t], & \text{for } t = 2, \\ o_*[t], & \text{for } 1 < t < 2, \\ (o_* \cup b_*)[t], & \text{for } t = 1, \\ k^D[t], & \text{for } 0 \le t < 1, \end{cases}$$

where d_* is a disjoint trivial disk system of m disks d_i ($i = 1, 2, \dots, m$) for some m in R^3 with $o_* = \partial d_*$, b_* is a disjoint band system of $m-d$ bands b_j ($j = 1, 2, \dots, m-d$) in $\mathbf{R}^3$ spanning the trivial loop system o_* used for a fusion operation, and k^D is a ribbon link in $\mathbf{R}^3$ of d-components obtained from o_* by surgery along the band system b_* as a fusion. By the proof of Theorem 1.1 and Lemma 3.1, there is a free ribbon disk-link L^D in $\mathbf{R}^3[0,+\infty)$ of some rank n such that $L^D = K^D \cup C^D$ for a trivial disk system C^D in $\mathbf{R}^3[0,+\infty)$ whose normal form is given as follows by extending the normal form of K^D:

$$L^D \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 2, \\ (d_* \cup d^C)[t], & \text{for } t = 2, \\ (o_* \cup o^C)[t], & \text{for } 1 < t < 2, \\ (o_* \cup b_* \cup o^C)[t], & \text{for } t = 1, \\ (k^D \cup o^C)[t], & \text{for } 0 \le t < 1, \end{cases}$$

where d^C is a disjoint disk system in R^3 with $o^C = \partial d^C$. Note that the disk systems d_* and d^C are disjoint, but in general the band system b_* meets the interior of d^C in a disjoint arc system. By pulling down a neighborhood of every double point of $S(*)$ into $\mathbf{R}^3[0]$, the immersed 2-sphere $S(*)$ is changed into a non-immersed singular 2-sphere in $\mathbf{R}^3[0,+\infty)$, but a normal form of the union $K^D \cup S(*)$ in $\mathbf{R}^3[0,+\infty)$ extending the normal form of K^D is given as follows (see [8, I]):

$$(K^D \cup S(*)) \cap \mathbf{R}^3[t] = \begin{cases} \emptyset, & \text{for } t > 2, \\ (d_* \cup d^{S(*)})[t], & \text{for } t = 2, \\ (o_* \cup o^{S(*)})[t], & \text{for } 1 < t < 2, \\ (o_* \cup b_* \cup c^{S(*)} \cup b^{S(*)})[t], & \text{for } t = 1, \\ (k^D \cup c^{S(*)})[t], & \text{for } 0 < t < 1, \\ (k^D \cup e^{S(*)})[t], & \text{for } t = 0, \end{cases}$$

where $d^{S(*)}$ is a disjoint disk system in $\mathbf{R}^3$ with $o^{S(*)} = \partial d^{S(*)}$, $b^{S(*)}$ is a disjoint band system spanning $o^{S(*)}$ in R^3, $c^{S(*)}$ is a split union of a split Hopf link system $c^{H(*)}$ and a trivial link system $c^{o(*)}$ in $\mathbf{R}^3$ obtained from $o^{S(*)}$ by surgery along $b^{S(*)}$, and $e^{S(*)}$ is a split union of a disjoint Hopf disk pair system bounded by $c^{H(*)}$ and a disjoint disk system bounded by $c^{o(*)}$ in $\mathbf{R}^3$, where a Hopf disk pair means a disk pair with a clasp singularity in $\mathbf{R}^3$ bounded by a Hopf link. By construction, note that $e^{S(*)}$ is split from k^D. Since d_* and d^C are disjoint, $b_* \cap d^C$ is an arc system in the interior of d^C and $e^{S(*)} \cup b^{S(*)}$ has a graph spine, there is an isotopic move of d^C in $\mathbf{R}^3$ keeping $d_* \cup b_*$ fixed such that

$$d^C \cap (d_* \cup e^{S(*)} \cup b^{S(*)}) = \emptyset.$$

Then the link $o_* \cup o^{S(*)} \cup o^C$ is a trivial link in $\mathbf{R}^3$. In general the disk system d^C meets the interior of the disk system $d^{S(*)}$. However, by Horibe-Yanagawa lemma in [8, I], even if the disk systems d_*, $d^{S(*)}$, d^C are replaced by any disjoint disk systems bounded by the trivial link $o_* \cup o^{S(*)} \cup o^C$ in $\mathbf{R}^3$, the union $K^D \cup S(*)$ and the free ribbon disk-link L^D do not change up to ambient isotopies (with compact supports) of $\mathbf{R}^3[0,+\infty)$ keeping $\mathbf{R}^3[0]$ fixed. This means that the disjoint union $K^D \cup S(*)$ extends to a disjoint union $L^D \cup S(*)$ for a free ribbon disk-link L^D, so that $S(*) \subset E(L^D) \subset E(K^D)$, and thus, $E(K^D)$ is aspherical. This completes the proof of Theorem 1.4.

Appendix: Free Ribbon Lemma

The purpose of this appendix is to prove the following lemma.

Free Ribbon Lemma. Every free S^2-link L in S^4 is a ribbon S^2-link.

Proof of Free Ribbon Lemma. The following claim is used to determine a ribbon S^2-link.

(A.1) Let $(S^3_i)^{(1+m_i)}$ ($i = 1, 2, \dots, n$) be a system of mutually disjoint compact $(1 + m_i)$-punctured 3-spheres in S^4 such that the boundary $\partial(S^3_i)^{(1+m_i)}$ is the union of the component K_i and an S^2-link O_i of m_i components. If the union $O = \cup^n_{i=1} O_i$ is a trivial S^2-link in S^4, then the S^2-link $L = \cup^n_{i=1} K_i$ is a ribbon S^2-link in S^4.

Proof of (A.1). Let K'_i be a 2-sphere obtained from O_i by surgery along mutually disjoint 1-handles h_i ($i = 1, 2, \dots, m_i - 1$) in $(S^3_i)^{(1+m_i)}$, whose closed complement is diffeomorphic to the spherical shell $S^2 \times [0, 1]$. This means that the component K_i with reversed orientation is isotopic to the 2-sphere K'_i in $(S^3_i)^{(1+m_i)}$. This shows that $L = \cup^n_{i=1} K_i$ is a ribbon S^2-link in S^4, completing the proof of (A.1).

Let K_i ($i = 1, 2, \dots, n$) be the components of a free S^2-link L in S^4. Let Y be the 4-manifold obtained from S^4 by surgery along L. Let k_i ($i = 1, 2, \dots, n$) be the loop system in Y produced from K_i ($i = 1, 2, \dots, n$) by the surgery. Since the fundamental group $\pi_1(Y, v)$ is a free group and $H_2(Y; \mathbf{Z}) = 0$, the 4-manifold Y is identified with Y^S by Lemma 2.1. The 3-sphere $1 \times S^3_i$ of the connected summand $S^1 \times S^3_i$ of Y^S is fixed and denoted by S^3_i. Let x_i ($i = 1, 2, \dots, n$) be the basis of $\pi_1(Y^S, v)$ represented by a standard legged loop system $\omega^S x$ with vertex $v = v^S$. Let $k(\omega^S x) = \{k^S_i |\, i = 1, 2, \dots, n\}$ be the loop system of $\omega^S x$. Let $\omega m = \{\omega_i m_i |\, i = 1, 2, \dots, n\}$ be a meridian system with vertex v of the components K_i ($i = 1, 2, \dots, n$) of L in S^4. The meridian system ωm is taken in Y^S as a legged loop system with loop system $k(\omega m) = \{m_j \,|\, j = 1, 2, \dots, n\}$ parallel to the loop system k_i ($i = 1, 2, \dots, n$) in Y^S. Assume that the meridian system ωm in Y^S is made disjoint from ωx except for the vertex v and meets S^3_i ($i = 1, 2, \dots, n$) only in the loop system $k(\omega m)$ transversely. Let y_i ($i = 1, 2, \dots, n$) be the elements of $\pi_1(Y^S, v)$ represented by $\omega_i m_i$ ($i = 1, 2, \dots, n$). By Nielsen transformations of the basis x_i ($i = 1, 2, \dots, n$), assume that the product $x_i^{-1} y_i$ is in the commutator subgroup $[\pi_1(Y^S, v), \pi_1(Y^S, v)]$ of $\pi_1(Y^S, v)$ for every i (see [18]). For the 3-sphere S^3_i, consider all the loops m_j with $m_j \cap S^3_i \neq \emptyset$. For a point $p \in m_j \cap S^3_t$ ($t \neq i$), let $I(p)$ be an arc neighborhood of p in a parallel $k^S_t(p)$ of k^S and then replace the arc $I(p)$ with the arc $\mathrm{cl}(k^S_t(p) \setminus I(p))$. Let $\tilde{m}_j$ be a loop obtained from m_j by doing this operation on m_j for every t ($t \neq i$) and every point $p \in m_j \cap S^3_t$. For every i ($i = 1, 2, \dots, n$), let $m(S^3_i)$ be the system of the loops $\tilde{m}_j$ in Y^S obtained from all the loops m_j with $m_j \cap S^3_i \neq \emptyset$, where the loops m_j with $m_j \cap S^3_i = \emptyset$ are discarded. There is a smoothly embedded annulus A_i with $\partial A_i = (-k^S_i) \cup \tilde{m}_i$ in the open 4-manifold

$$Y^S_{)i(} = Y^S \setminus \cup_{1 \leq t(\neq i) \leq n} S^3_t$$

because the fundamental group $\pi_1(Y^S_{)i(}, v)$ is an infinite cyclic group and the loop $\tilde{m}_i$ is homotopic to k^S_i in $Y^S_{)i(}$. The annulus A_i meets S^3_i transversely with disjoint simple loops and simple arcs. Let α_{is} $(s = 1, 2, \ldots, n_i)$ be the arc system of the intersection $A_i \cap S^3_i$ where α_{i1} joins the point $p^S_i = k^S_i \cap S^3_i$ to a point of the loop $\tilde{m}_i$ and the arc α_{is} with $s > 1$ joins two points of $\tilde{m}_i$. For j with $j \neq i$, the loop $\tilde{m}_j$ is null-homotopic in $Y^S_{)i(}$ and hence bounds a disk D_{ji} in $Y_{)i(}$ which meets S^3_i transversely with disjoint simple loops and simple arcs. Let α_{jis} $(s = 1, 2, \ldots, n_{ji})$ be the arc system of the intersection $D_{ji} \cap S^3_i$ each of which joints two points of $\tilde{m}_j$.The annulus A_i and the disk D_{ji} with $i \neq j$ are made disjoint while fixing the intersection with S^3_i in Y^S for all i, j by doing double point cancellations using free boundary arcs while fixing the intersection with S^3_i for $\tilde{m}_j$. The following observation helps clarify the relationship between the point system $m(S^3_i) \cap S^3_i$ and the arc system $(A_i \cup D_{ji}) \cap S^3_i$ for all j with $j \neq i$.

Observation (A.2) Let $\partial_{\alpha is} = \{q_s, q'_s\}$ $(s = 1, 2, \ldots, n_i)$ with $q_1 = p^S_i$ for the arc system α_{is} $(s = 1, 2, \ldots, n_i)$ of $A_i \cap S^3_i$. Then the open arc of $\tilde{m}_i$ that is separated by any couple $\{q_s, q'_s\}$ with $s > 1$ and does not contain the point q'_1 meets S^3_i with intersection number 0. Conversely, let $\{q_s, q'_s\}$ $(s = 1, 2, \ldots, n_i)$ be any system of couples of distinct points with $q_1 = p^S_i$ such that the union of these points matches the set $(k^S_i \cup \tilde{m}_i) \cap S^3_i$ and the open arc of $\tilde{m}_i$ that is divided by any couple $\{q_s, q'_s\}$ with s > 1 and does not contain the point q'_1 meets S^3_i with intersection number 0. Then $\{q_s, q'_s\}$ $(s = 1, 2, \ldots, n_i)$ is realized by $\partial\alpha_{is} = \{q_s, q'_s\}$ $(s = 1, 2, \ldots, n_i)$ of the arc system α_{is} $(s = 1, 2, \ldots, n_i)$ of $A_i \cap S^3_i$ for an annulus A_i with $\partial A_i = (-k^S_i) \cup \tilde{m}_i$ in $Y^S_{)i(}$. Let $\partial\alpha_{jis} = \{q_s, q'_s\}$ $(s = 1, 2, \ldots, n_{ji})$ for the arc system α_{jis} $(s = 1, 2, \ldots, n_{ji})$ of $D_{ji} \cap S^3_i$. Then every open arc of $\tilde{m}_j$ divided by any couple $\{q_s, q'_s\}$ meets S^3_i with intersection number 0. Conversely, let $\{q_s, q'_s\}$ $(s = 1, 2, \ldots, n_{ji})$ be any system of couples of distinct points such that the union of these points matches the set $\tilde{m}_j \cap S^3_i$ and every open arc of $\tilde{m}_j$ which is divided by any couple $\{q_s, q'_s\}$ meets S^3_i with intersection number 0. Then $\{q_s, q'_s\}$ $(s = 1, 2, \ldots, n_{ji})$ is realized by $\partial\alpha_{jis} = \{q_s, q'_s\}$ $(s = 1, 2, \ldots, n_{ji})$ of the arc system α_{jis} $(s = 1, 2, \ldots, n_{ji})$ of $D_{ji} \cap S^3_i$ for a disk D_{ji} with $\partial D_{ji} = \tilde{m}_j$ in $Y^S_{)i(}$.

Let $B(\alpha_{is})$ $(s = 1, 2, \ldots, n_i)$ be disjoint 3-ball neighborhoods of the arcs α_{is} $(s = 1, 2, \ldots, n_i)$ in S^3_i, and $B(\alpha_{jis})$ $(s = 1, 2, \ldots, n_{ji})$ disjoint 3-ball neighborhoods of the arcs α_{jis} $(s = 1, 2, \ldots, n_{ji})$ in S^3_i. Let $S(\alpha_{is}) = \partial B(\alpha_{is})$ $(s = 1, 2, \ldots, n_i)$ and $S(\alpha_{jis}) = \partial B(\alpha_{jis})$ $(s = 1, 2, \ldots, n_{ji})$ be the boundary 2-spheres of them. The S^2-link L in S^4 with meridian system ωm is recovered from Y^S by the back surgery along the loop system k_i $(i = 1, 2, \ldots, n)$ in Y^S. Since the 2-spheres $S(\alpha_{is})$ $(s = 1, 2, \ldots, n_i)$ and $S(\alpha_{jis})$ $(s = 1, 2, \ldots, n_{ji})$ in Y^S are disjoint from the loop system k_i $(i = 1, 2, \ldots, n)$, the 2-spheres $S(\alpha_{is})$ $(s = 1, 2, \ldots, n_i)$ and $S(\alpha_{jis})$ $(s = 1, 2, \ldots, n_{ji})$ are considered in S^4. The 2-sphere $S(\alpha_{i1})$ is identified with K_i in S^4 for all i $(i = 1, 2, \ldots, n)$. The following claim is shown.

(A.3) The 2-spheres $S(\alpha_{is})$ $(i = 1, 2, \ldots, n;\ s = 2, 3, \ldots, n_i)$ and $S(\alpha_{jis})$ $(i, j = 1, 2, \ldots, n, j \neq i;\ s = 1, 2, \ldots, n_{ji})$ form a trivial S^2-link in S^4.

By (A.1) and (A.3), the S^2-link $L = \overset{i=1}{\cup^{n}} K_i$ is shown to be a ribbon S^2-link in S^4.

Proof of (A.3). The loops k^S_t $(t = 1, 2, \ldots, n)$ in S^4 bound disjoint disks D^S_t $(i = 1, 2, \ldots, n)$ in S^4. Hence the loop k^S_t in S^4 is isotopic to a band sum k'_t of some parallel links $P_t(m_i)$ $(i = 1, 2, \ldots, n)$ of the meridian loops m_i $(i = 1, 2, \ldots, n)$ of K_i $(i = 1, 2, \ldots, n)$ in S^4. For a parallel k^{S+}_t of k^S_t in S^4, let D^{S+}_t be a move of D^S_t with $\partial D^{S+}_t = k^{S+}_t$ in S^4 so that the disk D^{S+}_t is disjoint from the annuli A_i $(i = 1, 2, \ldots, n)$ and the disks D_{ji} $(i, j = 1, 2, \ldots, n;\ j \neq i)$. The 2-spheres $S(\alpha_{is})$ $(i = 1, 2, \ldots, n;\ s = 1, 2, \ldots, n_i)$ and $S(\alpha_{jis})$ $(i, j = 1, 2, \ldots, n, j \neq i;\ s = 1, 2, \ldots, n_{ji})$ may be disjoint from the disk D^{S+}_t in S^4. By passing through a thickening $D^{S+}_t \times I$ of the disk D^{S+}_t for every $t(\neq i)$ in S^4, the annulus A_i and the disk D_{ji} in Y^S extend respectively in S^4 to an annulus $\bar{A}_i$ with $\partial\bar{A}_i = (-k^S_i) \cup m_i$ and a disk $\bar{D}_{ji}$ with $\partial \bar{D}_{ji} = m_j$. The annuli $\bar{A}_i$ $(i = 1, 2, \ldots, n)$ and the disks $\bar{D}_{ji}$ $(i, j = 1, 2, \ldots, n;\ j \neq i)$ should be disjoint in S^4. For s ≥ 2, let $S(\partial\alpha_{is})$ be the two sphere union which is the boundary of a regular neighborhood $B(\partial\alpha_{is})$ of the two point set $\partial\alpha_{is}$ in $B(\alpha_{is})$. The 2-sphere $S(\alpha_{is})$ can be replaced by the 2-sphere obtained from $S(\partial\alpha_{is})$ by surgery along a 1-handle attaching to $S(\partial\alpha_{is})$ whose core is a subarc α'_{is} of α_{is} in $B(\alpha_{is})$. The following observation (whose proof is obvious) is used.

Observation A.4 In the spherical shell $S^3 \times [0,1]$, the 2-sphere S' obtained from the 2-spheres $S^2 \times \{0, 1\}$ by surgery along a 1-handle h' thickening the arc $p \times [0, 1]$ $(p \in S^2)$ bounds the unique 3-ball $B' = \mathrm{cl}(S^2 \times [0, 1] \setminus h')$. Further, let S'' be the 2-sphere obtained from the 2-spheres $S^2 \times \{1/4 , 3/4\}$ by surgery along a 1-handle h″ thickening the arc $p \times [\ [1/4,3/4]$, and $B'' = \mathrm{cl}(S^2 \times [1/4.3/4] \setminus h'')$ the 3-ball bounded by S''. If the 1-handle h' is thinner than the 1-handle h'', then the 3-ball B'' is in the interior of the 3-ball B'.

Assume that the arc α_{is} cuts an innermost disk δ from the annulus $\bar{A}_i$. Then the arc α'_{is} is ∂-relatively isotopic to an arc J in m_i through the disk δ, so that the arc α'_{is} joining the two sphere union $S(\partial\alpha_{is})$ is ∂-relatively isotopic to an arc J joining the boundary $(\partial J) \times K_i$ of a spherical shell $J \times K_i$ of the circle bundle $\partial D^2 \times K_i$ with $J \subset \partial D^2$ for a normal disk bundle $D^2 \times L$ in S^4. Thus, the 2-sphere $S(\alpha_{is})$ is isotopic to the boundary 2-sphere $\partial\Delta(\alpha_{is})$ of a 3-ball $\Delta(\alpha_{is})$ in the spherical shell $J \times K_i$ (see [17]). Note that the 3-ball $\Delta(\alpha_{is})$ does not meet the S^2-link L although the trace of this isotopy may meet L since the disk δ may meet L. By continuing this process, it is seen from Observation A.4 that the 2-spheres $S(\alpha_{is})$ (s = 2, 3, . . . , n_i) are isotopic to the disjoint boundary 2-spheres $\partial\Delta(\alpha_{is})$ (s = 2, 3, . . . , $n_{i)}$ of an inclusive 3-ball family $\Delta(\alpha_{is})$ $(s = 2, 3, \ldots, n_i)$ in $D^2 \times K_i$, where an *inclusive 3-ball family is a family* of finite number of 3-balls such that any two members B_1 and B_2 have the property

$$B_1 \subset \mathrm{Int}(B_2), \quad B_2 \subset \mathrm{Int}(B_1), \quad \text{or} \quad B_1 \cap B_2 = \emptyset.$$

For the disk $\bar{D}_{ji}$, the same argument above can be applied to see that the 2-spheres $S(\alpha_{jis})$ ($s = 1, 2, \dots, n_{ji}$) are isotopic to the disjoint boundary 2-spheres $\partial\Delta(\alpha_{jis})$ ($s = 1, 2, \dots, n_{ji}$) of an inclusive 3-ball family $\Delta(\alpha_{jis})$ ($s = 1, 2, \dots, n_{ji}$) in $D^2 \times K_j$ with $j \neq i$. Thus, for every i, the 2-spheres $S(\alpha_{is})$ ($s = 2, 3, \dots, n_i$) and $S(\alpha_{jis})$ ($s = 1, 2, \dots, n_{ji}$) form a trivial S^2-link in S^4. Since the annuli $\bar{A}_i$ ($i = 1, 2, \dots, n$) and the disks $\bar{D}_{ji}$ ($i, j = 1, 2, \dots, n; j \neq i$) are disjoint, it can be seen that the 2-spheres $S(\alpha_{is})$ ($i = 1, 2, \dots, n; s = 2, 3, \dots, n_i$) and $S(\alpha_{jis})$ ($i, j = 1, 2, \dots, n, j \neq i; s = 1, 2, \dots, n_{ji}$) form a trivial S^2-link in S^4 by varying the radius of the disk D of the normal disk bundle $D \times L$ of L for every i. This completes the proof of (A.3).

This completes the proof of Free Ribbon Lemma.

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References

1. Kervaire, M. A. (1965, December). On higher dimensional knots. In Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse) (pp. 105-119). Princeton, NJ: Princeton Univ. Press.
2. A, Kawauchi. (2023). Smooth homotopy 4-sphere, WSEAS Transactions on Mathematics , (Vol. 22, pp. 690-701).
3. Kawauchi, A. (2021). Ribbonness of a stable-ribbon surface-link, I. A stably trivial surface-link. Topology and its Applications, *Topology and its Applications*, (Vol. 301, 107522 (16 pp.))
4. Kawauchi, A. (2023). Uniqueness of an orthogonal 2-handle pair on a surface-link. *Contemporary Mathematics*, (Vol. 4, pp. 182-188). Universal Wiser Publisher.
5. A, Kawauchi. (2023). Triviality of a surface-link with meridian-based free *fundamental group. Transnational Journal of Mathematical Analysis and Applications*, (Vol.11, pp. 19-27).
6. Yanagawa, T. (1969). On ribbon 2-knots. I. The 3-manifold bounded by the 2-knot, Osaka Journal of Mathematics, (Vol. 6, pp.447-464).
7. Kawauchi, A. (2021). Classical Poincar {\'e} conjecture via 4D topology. arXiv preprint arXiv:2103.16001.
8. A, Kawauchi., T, Shibuya., and S, Suzuki. (1982). Descriptions on Surfaces in Four-Space I Normal Forms. In Mathematics seminar notes, (Vol. 10, pp. 75-125). (1983). Descriptions on Surfaces in Four-Space II Singularities and Cross-Sectional Links. In Mathematics seminar notes, (Vol. 11, pp. 31-69).
9. E, H. Spanier. (1966). Algebraic topology, McGraw-Hill
10. Howie, J. (1983). Some remarks on a problem of JHC Whitehead. *Topology*, (Vol. 22, NNo. 4, pp. 474-485).
11. Yanagawa, T. (1969). On ribbon 2-knots. II. The second homotopy group of the complementary domain , Osaka Journal of Mathematics (Vol. 6, pp.447-464).
12. Asano, K., Marumoto, Y., & Yanagawa, T. (1981). Ribbon knots and ribbon disks , Osaka Journal of Mathematics (Vol. 18, pp.161-174).
13. Whitehead, J. H. C. (1941). On adding relations to homotopy groups. *Annals of Mathematics,* (Vol. 42, pp. 409-428).
14. Gluck, H. (1962). The embedding of two-spheres in the four-sphere. *Transactions of the American Mathematical Society, (Vol. 104,* No. 2, pp. 308-333).
15. Kawauchi, A. (1989). An imitation theory of manifolds.
16. Gabai, D. (1987). Foliations and the topology of 3-manifolds. III. *Journal of Differential Geometry, (Vol. 26* No. 3, pp. 479-536).
17. Hosokawa, F., & Kawauchi, A. (1979). Proposals for unknotted surfaces in four-spaces Osaka Journal of Mathematics, (Vol. 16, pp. 233-248).
18. Magnus, W., Karrass, A., & Solitar, D. (2004). *Combinatorial group theory: Presentations of groups in terms of generators and relations*. Courier Corporation.