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Short Communication

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Reflection and Transmission Coefficients of Electromagnetic Wave Equation in Schwarzschild Black Hole

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Abstract
The electromagnetic perturbation to the Schwarzschild metric is studied. The perturbation of the Schwarzschild black hole is described by the general time-dependent Regge-Wheeler equation. We trasnform this equation to usual Schwarzschild coordinates. In this case, it is possible to separate a harmonic time-dependence. Then, the resulting radial equation belong to the class of confluent Heun equation consisting of effective potential. The potential behaves like 1-D potential barrier in elementary quantum mechanics which is used to calculate the reflection and transmission coefficients near the Schwarzschild space-time using black hole boundary condition and the properties of confluent Heun function.

1. Introduction

Before Einstein theory of general theory of relativity, scientists only knew about behavior of wave equation in the absence of gravity, but after general theory of relativity, they try to use wave equation in gravity [1]. Initially, they got the problem about singularity (mainly three singularity, i.e. $r = 1,0,\infty$) in the differential equation of electromagnetic wave in gravity and no perfect idea was obtained. But Heun deduced the method to solve equation containing such singularity in terms of confluent Heun equation [2-6]. Then many scientists developed the idea to solve wave equation near the black-hole and Teukolsky Master equation, Zerilli equation are obtained and the research of wave equation is going forward [7].

Detail understanding of black hole physics can be achieved by propagating wave into it. This is the technique that somehow reveals the quantum phenomena underlying within black hole. Thus, the wave propagation onto the black hole space-time opens a perspective on additional phenomena such as interference effects, scattering of radiation at a black hole, its quasi-normal modes and black hole evaporation [8]. To describe the propagation of waves on Schwarzschild space-time, we use wave equation named as Regge-Wheeler equation, which is analogous to Schrodinger equation [8]. Additionally, Regge-Wheeler equation is time-dependent equation where we use tortoise co-ordinate as well as spin-perspective. Actually, to detect the type of wave propagated, we use spin-perspective [9]. On this basis, Regge-Wheeler categorized the perturbations in the field as tensorial or gravitational perturbation (spin s = 2), vectorial or electromagnetic perturbation (spin s = 1) and scalar or massless perturbation (spin s = 0) [10-13]. After these successive perturbation, Regge-Wheeler equation becomes spherical harmonics whose variable co-ordinate is radial co-ordinate. Regge-Wheeler equation is a Schrodinger type differential equation with a spin-dependent potential barrier. Due to the form of this potential no exact solution is known as long as the tortoise co-ordinate is kept as the radial variable [9].

Keeping in mind that scattering patterns have proven to be very important tools in other branches of physics and yield observational properties of different physical systems, we are strongly devoted to the black-hole scattering properties in this work. First, we have found the Heun solution of wave equation in case of elctromagnetic perturbation at Kerr black hole (vanishing rotation term) and then we have calculated the reflection and transmission coefficients of electromagnetic waves in the presence of a black hole using black hole boundary conditions. Finally, we have plotted these reflection and transmission coefficients to study their nature.

2. Electromagnetic Perturbation

To describe electromagnetic perturbation, let us consider source-free Maxwell equation as

$$\nabla_\mu F^{\mu\nu} = 0 \tag{1}$$

$$e^{\mu\nu\rho\sigma}\nabla_\nu F_{\rho\sigma} = 0 \tag{2}$$

Now, we can use the metric as

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr_*^2 + \frac{r^3(d\vartheta^2 + sin^2\vartheta d\phi^2)}{(r-1)} \tag{3}$$

Metric (3) is conformally equivalent to Schwarzschild metric [14]. Since Debye potential is a scalar function, so equation (1) and (2) can be reduced to Debyeequation. As metric (3) describes space-time, so required Debye-equation is [15].

$$\left[\frac{(r-1)}{r^3}\left(cot\vartheta\frac{\partial}{\partial\vartheta} + \frac{\partial^2}{\partial\vartheta^2} + \frac{1}{sin^2\vartheta}\frac{\partial^2}{\partial\phi^2}\right) + \frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2}\right]\Pi = 0 \tag{4}$$

Then, we have to build four-potential of electromagnetic field, to get the general solution of Maxwell's equation, i.e. [14]

$$A_\mu = (u^\nu v_\mu - v^\nu u_\mu)\partial_\nu \Pi + \epsilon_\mu^{\nu\lambda\sigma} v_\lambda u_\sigma \partial_\nu \Phi \tag{5}$$

where Π and Φ are two independent solution of Debye-equation and $(v^\mu) = (0,1,0,0)$, $(u^\mu) = (1,0,0,0)$. Hence the components of electromagnetic field are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6}$$

In equation (4), the symbol Π represents Debye-potential. As Debye-potential is a scalar function, hence, in terms of spherical harmonics, we can elaborate it as

$$\Pi(t, r_*, \vartheta, \varphi) = \sum_{l=1}^{\infty}\sum_{m=-1}^{\infty} \Psi_l(t, r_*) Y_{lm}(\vartheta, \varphi) \tag{7}$$

By substituting the value of equation (7) into the Debye-equation (4), we distinguish that the function $\Psi_l(t,r^*)$ has to fulfill the wave equation in which potential given by [16]

$$V_{l,s=1}(r) = \frac{\Delta}{r^4}\lambda \tag{8}$$

where $\lambda = l(l+1) - s(s+1)$ is separation constant and $\Delta = r^2 - 2Mr$.

3. 1-D Wave Equation and Its Solution for Spin (s) = +1

To get the confluent type Heun equation which is the essential tool for this work, we strike with 1-D Schrodinger wave equation [16] given as

$$\Lambda^2 Z = VZ \tag{9}$$

Where $\Lambda^2 = \Lambda_+\Lambda_-$

$$= (\frac{d}{dr_*} + iw)(\frac{d}{dr_*} - iw)$$

$$= (\frac{d^2}{dr_*^2} + w^2) \tag{10}$$

Where r_* is the tortoise coordinate [9] and is given by

$$r_* = r + 2Mln\left(\frac{r}{2M} - 1\right)$$

For Schwarzschild black hole, we use $2M \longrightarrow 1$, and hence r_* becomes

$$r_* = r + ln(r-1) \qquad (11)$$

The logical reason behind using tortoise coordinate is to cover entire space of Schwarzschild black hole from $-\infty$ to ∞, as Schwarzschild coordinate r has only the coverage from $2M$ to ∞.

Taking the differentiation of equation (11) with respect to r, we get

$$\begin{aligned}\frac{dr_*}{dr} &= 1 + \frac{1}{r-1}\\ &= \frac{r}{r-1}\\ or, dr_* &= \frac{r}{r-1}dr\\ or, \frac{d}{dr_*} &= (1 - \frac{1}{r})\frac{d}{dr}\end{aligned}$$

and hence

$$\begin{aligned}\frac{d^2}{dr_*^2} &= \frac{d}{dr_*}\left(\frac{d}{dr_*}\right)\\ &= \left(1 - \frac{1}{r}\right)^2 \frac{d^2}{dr^2} + \frac{1}{r^2}\left(1 - \frac{1}{r}\right)\frac{d}{dr}\end{aligned}$$

Now using the value of $\frac{d^2}{dr_*^2}$, equation (10) becomes

$$\Lambda^2 = \left(1 - \frac{1}{r}\right)^2 \frac{d^2}{dr^2} + \frac{1}{r^2}\left(1 - \frac{1}{r}\right)\frac{d}{dr} + w^2 \qquad (12)$$

The striker part of our work is the potential part given below by which we analyze the reflection and transmission coefficients of electromagnetic wave in case of Schwarzschild black hole, i.e.,

$$V(r) = \frac{\Delta}{r^4}\lambda \qquad (13)$$

Where $\lambda = l(l+1) - s(s+1)$ is separation constant. As we are dealing with electromagnetic wave, spin parameter $'s'$ has values +1(up) and −1(down). Here we use $s = +1$ for the case study. Therefore

$$\lambda = l(l+1) - 2$$

As $'l'$ has values 2, 3, 4, ... in this case, so using $l = 2$, and hence

$$\lambda = 4$$

and also in equation (13)

$$\begin{aligned}\Delta &= r^2 - 2Mr\\ &= r^2 - r \qquad ; \quad 2M \rightarrow 1\end{aligned}$$

Therefore the desired potential becomes

$$V(r) = \frac{4}{r^2}\left(1 - \frac{1}{r}\right) \tag{14}$$

Using equation (12) and (14) in (9), we get desired Heun type equation as

$$\left(1 - \frac{1}{r}\right)^2 \frac{d^2}{dr^2}Z(r) + \frac{1}{r^2}\left(1 - \frac{1}{r}\right)\frac{d}{dr}Z(r) + \left[w^2 - \frac{4}{r^2}\left(1 - \frac{1}{r}\right)\right]Z(r) = 0 \tag{15}$$

On solving equation (15), we get

$$Z(r) = C_1(r-1)^{iw}r^2e^{iwr}HeunC(-2iw, 2iw, 2, -2w^2, 2w^2-3, -r+1) + \\ C_2(r-1)^{-iw}r^2e^{iwr}HeunC(-2iw, -2iw, 2, -2w^2, 2w^2-3, -r+1) \tag{16}$$

Where HeunC function has parameters $\alpha = -2iw$, $\beta = 2iw$, $\gamma = 2$, $\delta = -2w^2$, $\eta = 2w^2 - 3$, $z = 1 - r$.

HeunC function has the signicant property of changing the sign of α paramater, i.e.,

$$HeunC(\alpha, \beta, \gamma, \delta, \eta, z) = e^{-\alpha z}HeunC(-\alpha, \beta, \gamma, \delta, \eta, z)$$

Applying this property in 2nd part of R.H.S. of equation (16) results into

$$Z(r) = C_1(r-1)^{iw}r^2e^{iwr}HeunC(-2iw, 2iw, 2, -2w^2, 2w^2-3, -r+1) + \\ C_2(r-1)^{-iw}r^2e^{-iwr}HeunC(2iw, -2iw, 2, -2w^2, 2w^2-3, -r+1) \tag{17}$$

4. Reflection and Transmission Coefficients for Spin (s) = +1

Now, we move towards plotting the potential part, given in equation (14).

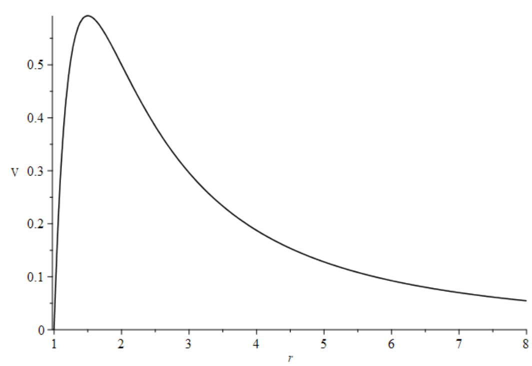


Figure 1: The Regge-Wheeler Potential as a Function of Radial Coordinate '*r*' for *l* = 2 and Spin *s* = 1.

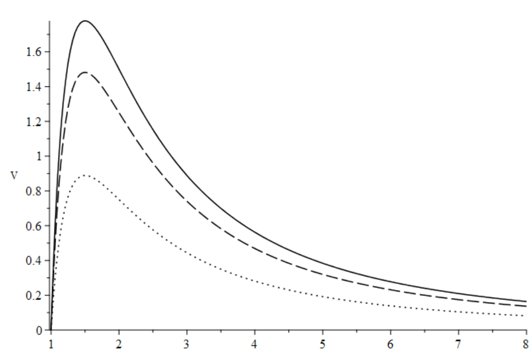


Figure 2: The Regge-Wheeler Potential as a Function of Radial Coordinate '*r*' for *l* = 3 and Spin *s* = 0 (Solid), *s* = 1 (Dashed), *s* = 2 (Dotted).

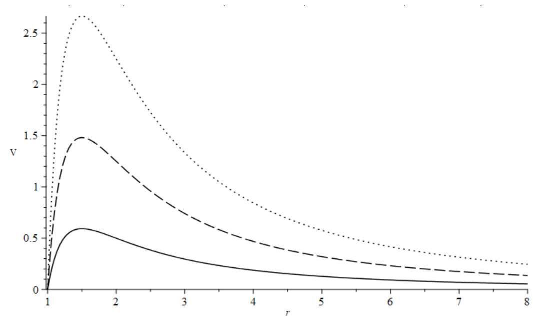


Figure 3: The Regge-Wheeler Potential as a Function of Radial Coordinate 'r' for Spin $s = 1$ and $l = 2$ (Solid), $l = 3$ (Dashed), $l = 4$ (Dotted).

The figure 2 represents the shape of potential for three spin values and a angular momentum index $l = 3$. The potential near the radius $r = \frac{3}{2}$ shows the maxima. Although the maximum for $s = 1$ is always located at that radius, for $s = 2$ it is found slightly below and for s = 2 slightly above, but in any case it coincides with the radius $r = \frac{3}{2}$ in the limit $l \rightarrow \infty$. Similarly, figure 3 shows the l-dependence potential for $s = 1$ and it shows potential increases with the angular momentum index 'l'. The potential decreses expontentially near the horizon (r $\rightarrow -\infty$) and behaves like r^{-2} for $r \rightarrow +\infty$.

After plotting figure 1, we get 1-D potential barrier through which we assume electromagnetic (em) wave gets strike on it.

As this em-wave gets incident on 1-D potential barrier from ∞, some part of wave gets reflected. So the wavefunction towards the side of ∞ becomes the combination of incident and reflected wave. We consider this region as 'I' in graph (1). And the radial wavefunction for region 'I', given in equation (17) becomes Region 'I'

$$\begin{aligned} Z_I(r) = C_1(r-1)^{iw} r^2 e^{iwr} HeunC(-2iw, 2iw, 2, -2w^2, 2w^2-3, -r+1) + \\ C_2(r-1)^{-iw} r^2 e^{-iwr} HeunC(2iw, -2iw, 2, -2w^2, 2w^2-3, -r+1) \end{aligned} \tag{18}$$

Here, second part of $Z_I(r)$ belongs to incident part and first part of ZI(r) belongs to reflected part.

Also, as some part of the em-wave penetrates the barrier and gets transmitted out, we get the transmission region denoted by region 'II'. From the graph above, this region is beyond 2M, i.e., left side of 2M. As this region consists of only transmitted part, the radial wave function, given in equation (17), becomes Region 'II':

$$Z_{II}(r) = C_3(r-1)^{-iw} r^2 e^{-iwr} HeunC(2iw, -2iw, 2, -2w^2, 2w^2-3, -r+1) \tag{19}$$

For region 'I', at asymptotic, i.e., at $z \rightarrow \infty$, HeunC function becomes as follows [17]

$$HeunC(\alpha, \beta, \gamma, \delta, \eta, z) \sim z^{-(\frac{\beta+\gamma+2}{2}+\frac{\delta}{\alpha})} \sum_{\nu \geq 0} \frac{a_\nu(\alpha)}{z^\nu}$$

On using $v = 0$, $a_0 = 1$, $z^0 = 1$, and therefore

$$HeunC(\alpha, \beta, \gamma, \delta, \eta, z) \sim z^{-(\frac{\beta+\gamma+2}{2}+\frac{\delta}{\alpha})}$$

This is the desired *HeunC* function for v exactly equal to 0.
For reflected part of region 'I', we get *HeunC* function as

$$\begin{aligned} HeunC(-2iw, 2iw, 2, -2w^2, 2w^2-3, -r+1) &\sim (1-r)^{-(\frac{2iw+2+2}{2}+\frac{-2w^2}{-2iw})} \\ &\sim (1-r)^{-(iw+2-iw)} \\ &\sim (1-r)^{-2} \\ &\sim \frac{1}{(1-r)^2} \end{aligned} \tag{20}$$

And for incident part of region 'I', we get the *HeunC* function as

$$\begin{aligned} HeunC(2iw, -2iw, 2, -2w^2, 2w^2-3, -r+1) &\sim (1-r)^{-\left(\frac{-2iw+2+2}{2}+\frac{-2w^2}{2iw}\right)} \\ &\sim (1-r)^{-(-iw+2+iw)} \\ &\sim \frac{1}{(1-r)^2} \end{aligned} \tag{21}$$

Substituting this incident and reflected parts of *HeunC* function, i.e., (20) and (21) in the radial part of wave function in region 'I', i.e., in equation (18)

$$Z_I(r) = C_1(r-1)^{iw}\frac{r^2}{(1-r)^2}e^{iwr} + C_2(r-1)^{-iw}\frac{r^2}{(1-r)^2}e^{-iwr} \tag{22}$$

For region 'II', i.e., for transmission region, the boundary condition is $r \to 2M$ i.e. $z \to 0$. So the property of *HeunC* function for this boundary condition becomes

$$\begin{aligned} Z_{II}(r) &= C_3(r-1)^{-iw}e^{-iwr} \\ &= C_3r^2(r-1)^{-iw}e^{-iwr} \end{aligned} \tag{23}$$

For solving equations (22) and (23), we use boundary condition at $r \to 1.5$ which we used by scaling the peak of the potential plot of Figure 1.

At $r \to 1.5$

$$Z_I(r) = Z_{II}(r)$$

$$\begin{aligned} or, C_1\frac{1.5^2}{(-0.5)^2}(0.5)^{iw}e^{1.5iw} + C_2\frac{1.5^2}{(-0.5)^2}(0.5)^{-iw}e^{-1.5iw} &= C_3(1.5)^2(0.5)^{-iw}e^{-1.5iw} \\ or, C_1(0.5)^{iw}e^{1.5iw} + C_2(0.5)^{-iw}e^{-1.5iw} &= 0.25C_3(0.5)^{-iw}e^{-1.5iw} \\ or, C_1(0.5)^{iw}e^{1.5iw} &= 0.25C_3(0.5)^{-iw}e^{-1.5iw} - C_2(0.5)^{-iw}e^{-1.5iw} \end{aligned} \tag{24}$$

Again at $r \to 1.5$

$$\frac{d}{dr}Z_I(r) = \frac{d}{dr}Z_{II}(r)$$

$$\begin{aligned} & or, -24C_1(0.5)^{iw}e^{1.5iw} + 27iwC_1(0.5)^{iw}e^{1.5iw} - 24C_2(0.5)^{-iw}e^{-1.5iw} - \\ & 27iwC_2(0.5)^{-iw}e^{-1.5iw} = 3C_3(0.5)^{-iw}e^{-1.5iw} - 6.75iwC_3(0.5)^{-iw}e^{-1.5iw} \\ & or, -24[0.25C_3(0.5)^{-iw}e^{-1.5iw} - C_2(0.5)^{-iw}e^{-1.5iw}] + 27iw[0.25C_3(0.5)^{-iw} \\ & e^{-1.5iw} - C_2(0.5)^{-iw}e^{-1.5iw}] - 24C_2(0.5)^{-iw}e^{-1.5iw} - 27iwC_2(0.5)^{-iw}e^{-1.5iw} \\ & = 3C_3(0.5)^{-iw}e^{-1.5iw} - 6.75iwC_3(0.5)^{-iw}e^{-1.5iw} \\ & or, -6C_3(0.5)^{-iw}e^{-1.5iw} + 24C_2(0.5)^{-iw}e^{-1.5iw} + 6.75iw(0.5)^{-iw}e^{-1.5iw} - \\ & 27iwC_2(0.5)^{-iw}e^{-1.5iw} - 24C_2(0.5)^{-iw}e^{-1.5iw} - 27iw(0.5)^{-iw}e^{-1.5iw} = \\ & 3C_3(0.5)^{-iw}e^{-1.5iw} - 6.75iw(0.5)^{-iw}e^{-1.5iw} \\ & or, [24 - 27iw - 24 - 27iw]C_2(0.5)^{-iw}e^{-1.5iw} = [3 - 6.75iw + 6 - 6.75iw]C_3 \\ & (0.5)^{-iw}e^{-1.5iw} \\ & or, -54iwC_2(0.5)^{-iw}e^{-1.5iw} = (9 - 13.5iw)C_3(0.5)^{-iw}e^{-1.5iw} \\ & \implies C_2 = \left(\frac{9 - 13.5iw}{-54iw}\right)C_3 \end{aligned} \tag{25}$$

Substituting C_2 on (24), we get

$$C_1(0.5)^{iw}e^{1.5iw} = 0.25C_3(0.5)^{-iw}e^{-1.5iw} + \left(\frac{9-13.5iw}{54iw}\right)C_3(0.5)^{-iw} e^{-1.5iw}$$
$$= \left(0.25 + \frac{9-13.5iw}{54iw}\right)C_3(0.5)^{-iw}e^{-1.5iw}$$
$$= \left(\frac{13.5iw+9-13.5iw}{54iw}\right)C_3(0.5)^{-iw}e^{-1.5iw}$$
$$\implies C_1 = \left(\frac{9}{54iw}\right)C_3(0.5)^{-2iw}e^{-3iw} \tag{26}$$

In non-relativistic quantum mechanics, the transmission coefficient and reflection coefficient are used to describe the bahavior of waves incident on a barrier. The transmission coefficient represents the probability flux of the transmitted wave relative to that of the incident wave. It is often used to describe the probability of a particle tunneling through a barrier. So, the transmission coefficient is given by

$$T = \left|\frac{C_3}{C_2}\right|^2$$
$$= \frac{|C_3|^2}{\left|\frac{9-13.5iw}{-54iw}\right|^2|C_3|^2}$$
$$= \frac{1}{\frac{(9+13.5iw)(9-13.5iw)}{(-54iw)(54iw)}}$$
$$= \frac{2916w^2}{81-182.25i^2w^2}$$
$$\implies T = \frac{2916w^2}{81+182.25w^2} \tag{27}$$

Similarly, the reflection coefficient is given by

$$R = \left|\frac{C_1}{C_2}\right|^2$$
$$= \frac{\left(\frac{9}{-54iw}\right)\left(\frac{9}{54iw}\right)|C_3|^2(0.25)^{2iw}e^{3iw}(0.25)^{-2iw}e^{-3iw}}{\left(\frac{9+13.5iw}{54iw}\right)\left(\frac{9-13.5iw}{-54iw}\right)|C_3|^2}$$
$$= \frac{\left(\frac{81}{2916w^2}\right)}{\left(\frac{81+182.25w^2}{2916w^2}\right)}$$
$$\implies R = \frac{81}{81+182.25w^2} \tag{28}$$

Now, we know that

$$|incidentwave|^2 = |reflectedwave|^2 + |transmittedwave|^2$$
$$or, \left|\frac{C_2r^2(r-1)^{-iw}e^{-iwr}}{(1-r)^2}\right|^2 = \left|\frac{C_1r^2(r-1)^{-iw}e^{-iwr}}{(1-r)^2}\right|^2 + |C_3r^2(r-1)^{-iw}e^{-iwr}|^2$$
$$or, \frac{|C_2|^2}{(1-r)^4} = \frac{|C_1|^2}{(1-r)^4} + |C_3|^2$$
$$or, |C_2|^2 = |C_1|^2 + (1-r)^4|C_3|^2$$

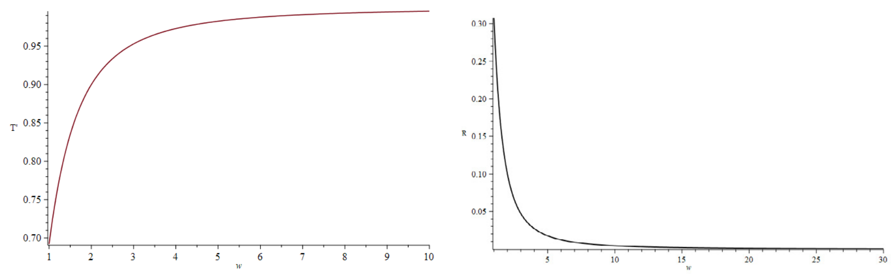
At $r \to 1.5$:

$$
\begin{aligned}
|C_2|^2 &= |C_1|^2 + (1-1.5)^4 |C_3|^2 \\
&= |C_1|^2 + 0.0625 |C_3|^2 \\
or, 1 &= \frac{|C_1|^2}{|C_2|^2} + 0.0625 \frac{|C_3|^2}{|C_2|^2} \\
\implies 1 &= R + 0.0625T \qquad (29)
\end{aligned}
$$

So to make $R + T = 1$, we need to multiply the transmission coefficient by 0.0625. Hence, multiplying equation (27) by 0.0625, we get the new transmission coefficient as

$$
\begin{aligned}
T^{'} &= 0.0625T \\
&= 0.0625 \left(\frac{2916w^2}{81 + 182.25w^2} \right) \\
\implies T^{'} &= \frac{182.25w^2}{81 + 182.25w^2} \qquad (30)
\end{aligned}
$$

The plot of transmission and reflection coefficient are shown below:



(a) Transmission coefficient as a function of 'w' for $s = +1$.

(b) Reflection coefficient as a function of 'w' for $s = +1$.

Figure 4: Transmission and Reflection Coefficients as a Function of 'ω' for s=+1.

5. 1-D Wave Equation and Its Solution for s = -1

From equation (9),

$$\Lambda^2 Z(r) = V(r)\, Z(r)$$

Using the value of Λ^2 from equation (12), above equation becomes

$$\left(1 - \frac{1}{r}\right)^2 \frac{d^2 Z(r)}{dr^2} + \frac{1}{r^2}\left(1 - \frac{1}{r}\right)\frac{dZ(r)}{dr} + (w^2 - V(r))Z(r) = 0 \qquad (31)$$

Where potential $V(r)$ is given by

$$
\begin{aligned}
V(r) &= \frac{\Delta}{r^4}\lambda \\
&= \frac{r^2 - r}{r^4}[l(l+1) - s(s+1)] \\
&= \frac{1}{r^2}\left(1 - \frac{1}{r}\right)[l(l+1) - s(s+1)]
\end{aligned}
$$

Using $l = 2$ and $s = -1$ in above expression, we get

$$V(r) = \frac{1}{r^2}\left(1-\frac{1}{r}\right)[2(2+1)+s(-1+1)]$$
$$\implies V(r) = \frac{6}{r^2}\left(1-\frac{1}{r}\right) \quad (32)$$

Using the value of $V(r)$ from equation (32) in equation (31), we get

$$\left(1-\frac{1}{r}\right)^2 \frac{d^2Z(r)}{dr^2} + \frac{1}{r^2}\left(1-\frac{1}{r}\right)\frac{dZ(r)}{dr} + \left(w^2 - \frac{6}{r^2}\left(1-\frac{1}{r}\right)\right)Z(r) = 0 \quad (33)$$

On solving equation (33), we get

$$Z(r) = C_1(r-1)^{iw} r^2 e^{iwr} HeunC(-2iw, 2iw, 2, -2w^2, 2w^2-5, -r+1) + C_2(r-1)^{-iw} r^2 e^{iwr} HeunC(-2iw, -2iw, 2, -2w^2, 2w^2-5, -r+1) \quad (34)$$

Where *HeunC* function has parameters $\alpha = -2iw$, $\beta = 2iw$, $\gamma = 2$, $\delta = -2w^2$, $\eta = 2w^2 - 5$, $z = 1 - r$.

HeunC function has the signicant property of changing the sign of α parameter, i.e., $HeunC(\alpha, \beta, \gamma, \delta, \eta, z) = e^{-\alpha z} HeunC(-\alpha, \beta, \gamma, \delta, \eta, z)$

Applying this property in second part of R.H.S. of equation (34), we get

$$Z(r) = C_1(r-1)^{iw} r^2 e^{iwr} HeunC(-2iw, 2iw, 2, -2w^2, 2w^2-5, -r+1) + C_2(r-1)^{-iw} r^2 e^{-iwr} HeunC(2iw, -2iw, 2, -2w^2, 2w^2-5, -r+1) \quad (35)$$

6. Reflection and Transmission Coefficients for s = -1

Now, we move towards plotting the potential part, given in equation (32),

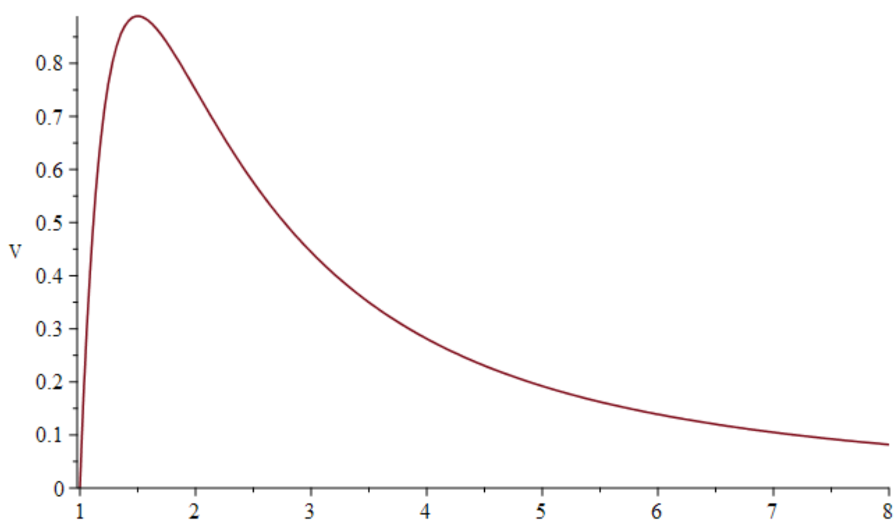


Figure 5: The Potential as a Function of Radial Coordinate *'r'* For $l = 2$ and Spin $s = -1$.

After plotting, we get 1-D potential barrier through which we assume em-wave gets strike on it.
As this em-wave gets incident on 1-D potential barrier from ∞, some part of wave gets reflected. So the total wave function becomes the combination of incident and reflected wave. We consider this region as (I) in graph. And the radial wave function for region-I, given in equation (35), becomes Region-I:

$$Z_I(r) = C_1(r-1)^{iw} r^2 e^{iwr} HeunC(-2iw, 2iw, 2, -2w^2, 2w^2-5, -r+1)+ \\ C_2(r-1)^{-iw} r^2 e^{-iwr} HeunC(2iw, -2iw, 2, -2w^2, 2w^2-5, -r+1) \quad (36)$$

In equation (36), first and second part of R.H.S. represent reflected and incident waves respectively.
Also, as some part of the em-wave penetrates the barrier and gets transmitted out, we get the transmission region denoted by region II. From the graph above, this region is beyond 2M, i.e., left side of 2M. As this region consists of only transmitted part, the radial wave function, given in equation (35), becomes Region-II:

$$Z_{II}(r) = C_3(r-1)^{-iw} r^2 e^{-iwr} HeunC(2iw, 2iw, 2, -2w^2, 2w^2-5, -r+1) \quad (37)$$

At region-I, i.e. at $z \to \infty$, *HeunC* function becomes

$$HeunC(\alpha, \beta, \gamma, \delta, \eta, z) \sim z^{-(\frac{\beta+\gamma+2}{2}+\frac{\delta}{\alpha})} \sum_{\nu \geq 0} \frac{a_\nu(\alpha)}{z^\nu}$$

On using $v = 0$, $a_0 = 1$, $z^0 = 1$, and therefore

$$HeunC(\alpha, \beta, \gamma, \delta, \eta, z) \sim z^{-(\frac{\beta+\gamma+2}{2}+\frac{\delta}{\alpha})}$$

Which is the desired *HeunC* function for $z \to \infty$ and v exactly equal to zero.

For the reflected and incident part of equation (36), we get the *HeunC* function as

$$HeunC(-2iw, 2iw, 2, -2w^2, 2w^2-5, -r+1) = HeunC(2iw, -2iw, 2, -2w^2, \\ 2w^2-5, -r+1) \\ = \frac{1}{1-r^2}$$

Substituting these incident and reflected parts of *HeunC* function in equation (36), we get

$$Z_I(r) = \frac{C_1 r^2 (r-1)^{iw}}{(1-r)^2} e^{iwr} + \frac{C_2 r^2 (r-1)^{-iw}}{(1-r)^2} e^{-iwr} \quad (38)$$

Now for the transmitted part, the *HeunC* function at $z \to 0$, i.e. at $r \to 2M$, becomes

$$HeunC(\alpha, \beta, \gamma, \delta, \eta, 0) = 1$$

Therefore radial wave-function for region-II, given in equation (37), becomes

$$Z_{II}(r) = C_3 r^2 (r-1)^{-iw} e^{-iwr} \quad (39)$$

When we solve equations (38) and (39), using boundary condition $r \to 1.5$, we get the same transmission and reflection coefficients as in case of s=+1, i.e.,

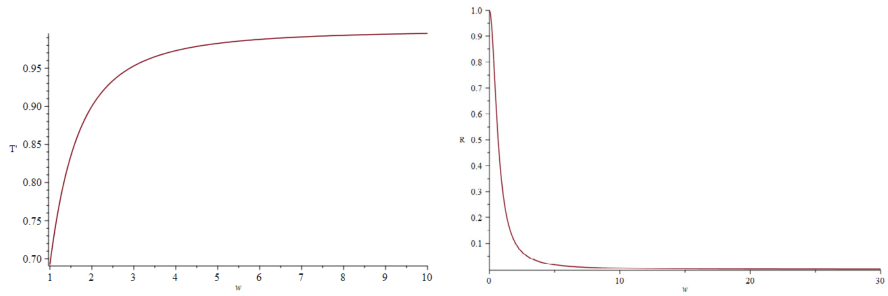
$$Transmission - cofficient(T) = \frac{2916w^2}{81 + 182.25w^2} \quad (40)$$

$$Reflection - coefficient(R) = \frac{81}{81 + 182.25w^2} \quad (41)$$

And the corrected form of transmission coefficients is also the same in case of s=+1, i.e.,

$$T' = \frac{182.25w^2}{81 + 182.25w^2} \quad (42)$$

The plot of so-obtained transmission and reflection coefficients are shown in figure below:



(a) Transmission coefficient as a function of 'w' for $s = -1$.

(b) Reflection coefficient as a function of 'w' for $s = -1$.

Figure 6: Transmission and Reflection Coefficients as a Function of 'ω' for $s = -1$.

7. Discussion and Conclusion

We investigated the transmission and reflection coefficients of em-wave equation in Schwarzschild black hole from beginning to end of this research. To do this, we took the 1-D wave equation with the potential which is in Kerr-geometry (but taking the rotation term tends to zero). Then, we splitted the wave equation into two parts by using different potentials (i.e.one with spin $s = +1$ and another with $s = -1$). First of all, we plotted the nature of potentials for the case of $s = +1$ and s = −1 against 'r'. When we plotted the potential against 'r' by varying 's', we found that the peak of the potential is located at r = 1.5 for s = 1 and s = 0 it is slightly below r = 1.5 and for $s = 2$ slightly above. But in the limit $l \to \infty$, all three cases coincides with the radius $r = 1.5$. From this, we concluded that the potential peak near the Schwarzschild black hole is found maximum for em-wave. Similarly, when we drew the plot of potential against 'r' by varying l (taking $s = +1$), we found that potential is increased with the increase of l-value. Also, the potential decreases exponentially near the horizon ($r \to -\infty$) and behaves like r^{-2} for $r \to +\infty$. After investigating potential nature, we transformed the wave equation into confluent Heun type equation which gave the Heun type solution.

Then, we took the three region by considering Schwarzschild black hole as a potential barrier. Then, we used the black hole boundary condition and some of the confluent Heun properties to get the transmission and reflection probabilities. When we found the value of transmission and reflection probabilities. we amazed that these values do not depend on l-values and depend only on the energy parameter 'w'. When 'w' increases, then the transmission probability also increases up to certain value and remains approximately constant after that value. It concluded that the high energy particles are more trapped when they pass through the Schwarzschild black hole. Similarly, when we saw the graph of reflection coefficients against 'w', we found that the reflection probability decreases with the increase of 'w' upto certain limit and it remains unchanged after that limit. From this, we concluded that the high energy particles are less reflected when they pass through the Schwarzschild black hole. We found the same nature of reflection and transmission probabilities results in case of $s = -1$ also.

Finally, we came in the conclusion that this work is very helpful to study the quantum nature of black hole like scattering, penetrating, interference effects, quasi-normal modes of black hole, black hole evaporation, etc., and we hope that it also helps to study other branches of Physics where confluent Heun type solution occurs.

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