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# Recovering the Standard Model Gauge Group from Loop Quantum Gravity: a Unified Relative-Entropy, Flux-Gauge Network, and Geometric-Engineering Construction

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**Abstract**

*We give a single, self-contained construction in which the Standard Model gauge group SU(3)×SU(2)×U(1) arises alongside gravity from an enlarged connection built on the Ashtekar variables of loop quantum gravity (LQG). The paper unifies, into one logical development, three previously separate lines of work by the author: (I) a relative-entropy renormalization-group (RG) toolkit, built from the data-processing inequality, that is unconditionally monotone under lattice and spin-network coarse-graining; (II) a flux-gauge network model in which a confining gauge connection is the sole microscopic substrate, geometry and the graviton are coarse-grained derived quantities, and SU(N) matter content is generated at ADE-orbifold degenerations of an internal fiber; and (III) an explicit application of both toolkits to the Ashtekar connection, in which the matter sector is fixed to su(5) by a rank-4 A4 singularity and broken to the Standard Model by the network's own adjoint vertex-Higgs mechanism. Every group-theoretic step of part (III) is proved as a theorem with complete proof and verified constants: a conservativity theorem showing the enlargement leaves the LQG kinematical Hilbert space an exact tensor product with the unchanged geometric sector; the Borel–de Siebenthal classification identifying su(3) ⊕su(2) ⊕u(1) as the unique maximal-rank subalgebra of su(5) with the Standard Model rank profile; the exact adjoint branching with hypercharges from root data and the resulting sin2θW = 3/8 matching relation; the Li vacuum-alignment theorem fixing the open region of quartic couplings for which the network's vertex scalar breaks SU(5) to the Standard Model rather than to SU(4)×U(1); the resulting X/Y boson mass; and the global form of the unbroken group, S(U(3)×U(2)) ∼= (SU(3)×SU(2)×U(1))/Z6, from which Standard Model hypercharge and electric-charge quantization follow as corollaries rather than inputs. We report the numerical tests carried out for parts (I)–(II) — exact confirmation of the relative-entropy monotonicity under block-summing coarse-graining, an independently measured percolation threshold consistent with the known three-dimensional value, and a suggestive (not conclusive) correlation between the growth of the half-dissipation scale and the percolation crossover — and we state precisely, in a unified status ledger, what the whole construction proves, what it imports from established results elsewhere in the literature, and what remains open: the dynamical selection of the singularity rank and the sign of the alignment coupling, and, entirely outside this paper's scope, fermion content, chirality, generation number, Yukawa structure, and unification phenomenology.*

## 1. Introduction

### 1.1. Two tasks

Any claim to recover the Standard Model gauge group from loop quantum gravity must distinguish two tasks. The first is *structural:* exhibit a connection, a structure group, and a symmetry-breaking mechanism from which $SU(3) \times SU(2) \times U(1)$ — with its correct rank, root system, hypercharge normalization, global form, and massive-sector spectrum — emerges by computation rather than fiat. The second is *dynamical*: derive why that structure group and breaking pattern, and no other, is selected by quantum gravity, together with the chiral fermion content, three generations, and Yukawa textures that make the result physically the Standard Model. This paper performs the first task at the level of rigor of a mathematical-physics article and states plainly that it does not perform the second. Every substantive claim is labeled as a definition, a proved theorem, an imported (established-elsewhere) result, or an open conjecture, and a single unified status ledger (Section 7) collects all such labels at the end.

### 1.2. What is Unified here, and why

This paper combines, into one self-contained development, three constructions previously developed by the author as separate manuscripts, none of which had appeared in a journal at the time of this writing. Rather than cite unpublished work externally, we fold the necessary content directly into the present paper as internally cross-referenced Parts, so that the whole argument — from the data-processing inequality to the Standard Model's hypercharge quantization — is verifiable from this document alone.

**Part I** develops a relative-entropy renormalization-group toolkit: a functional built from the data-processing inequality that is unconditionally monotone under admissible coarsegraining, first for lattice gauge link variables and then, in Part III, for gauge-invariant spin networks. This toolkit was originally developed in the context of a research program on the four-dimensional Yang–Mills mass gap; we reproduce only the functional-analytic core reused later (monotonicity, dissipation identity, budget equation) and summarize, without full reproduction, the mass-gap-specific machinery that is not needed for the Standard Model construction, since including it in full would roughly triple the length of this paper without contributing to its stated goal.

**Part II** develops a flux-gauge network model of emergent spacetime, in which a confining gauge connection is the sole microscopic substrate and geometry, the metric, and the graviton are coarse-grained statistical properties of its curvature. We reproduce in full the two pieces of this model that Part IV depends on: the ADE-orbifold mechanism generating *SU(N)* matter content at singular nodes, and the adjoint vertex-Higgs mechanism that freezes symmetrybroken directions of an enlarged connection. We summarize, without full reproduction, the metric-emergence and holographic machinery that is not needed for the gauge-sector construction of Part IV.

**Part III** transplants the Part I toolkit onto the kinematical Hilbert space of gauge-invariant spin networks and reports numerical tests, including a corrected and extended version of the original numerics (a quartic-stabilized single-link ansatz reaching the percolation-relevant regime, with a numerical-precision issue identified, diagnosed against arbitrary-precision arithmetic, and corrected).

**Part IV** is the paper's culmination and its one genuinely new mathematical content relative to Parts I–III individually: it enlarges the Ashtekar connection of canonical LQG by a directsum su(5) matter sector, generates that sector's gauge content at a rank-4 singularity via the Part II mechanism, and proves — as a complete chain of theorems — that the Part II vertex-Higgs mechanism breaks it to the Standard Model with the correct hypercharge normalization, quantization pattern, and global structure.

### 1.3. Conventions

Throughout, G denotes a compact connected Lie group and g its Lie algebra. Fundamental *SU(5)* generators $T^A$, $A = 1,...,24$, are normalized by $\mathrm{Tr}(T^A T^B) = \frac{1}{2}\delta^{AB}$ in the defining representation. Relative entropy (Kullback–Leibler divergence) between probability measures μ,ν on a common space is D(μ‖ν) = $\int \log \frac{d\mu}{d\nu}\, d\mu$ when $\mu \ll \nu$, and +∞ otherwise.

## 2. Part I: A Relative-Entropy Renormalization-Group Toolkit

**Setup: lattice gauge link variables and admissible blocking**

On a hypercubic lattice $a\mathbb{Z}^4$, a link $\ell = (x, x + a\hat{\mu})$ carries $U_\ell \in SU(N)$,and the Wilson measure at coupling $\beta$ is the Gibbs measure $\mu\beta$ ∝ e−SW[U] λ⊗ with λ Haar measure on *SU(N)* and *SW* the Wilson plaquette action.

**Lemma 2.1** (Single-link marginals are Haar). Let $\mu$ be *any probability measure on link configurations invariant under lattice gauge transformations* $U_\ell \mapsto g_x U_\ell g_{x+\hat{\mu}}^{-1}$ *(in particular, the Wilson measure at any coupling). Then the marginal of every single link* $U_\ell$ *under* $\mu$ *is Haar measure λ.*

Proof. Fix $\ell$ = *(x, x + aμ ˆ)* and act with a gauge transformation supported at *x* alone, $U_\ell$ →7 g$U_\ell$, g ∈ *SU*(N) arbitrary. Invariance of $\mu$ forces the marginal $\rho$ of $U_\ell$ to satisfy $\rho = g{*}\rho$ for every $g$, i.e. $\rho$ is left-invariant. The unique left-invariant probability measure on a compact group is Haar measure.

**Definition 2.2** (Admissible blocking kernel). *A blocking kernel Tk with factor b is admissible if: (i) locality — it assigns to each blocked link a variable built by a fixed local rule from fine links within a fixed block, blocks pairwise disjoint as link sets; (ii) gauge covariance — fine-lattice gauge transformations act on blocked variables as blocked-lattice gauge transformations at block corners; (iii) Haar invariance — λ⊗Tk = λ⊗ on the blocked lattice.*

**Lemma 2.3** (Path-product blockings are admissible). *Let* $T_k$ *assign to each blocked link the ordered product of fine links along a fixed simple path through its block. Then* $T_k$ *is admissible.*

*Proof.* Locality and gauge covariance are immediate. For Haar invariance: if $U_1,...,U_n$ are independent Haar variables on a compact group, the product $U_1 \cdots U_n$ is Haar (condition on U2,...,Un and use left invariance), and products over disjoint paths draw on disjoint, independent link sets, hence are independent. By Lemma , the Haar product measure $\lambda\otimes$ on links is simultaneously the canonical maximalignorance reference and the exact product of the true single-link marginals of $\mu\beta$, at every coupling and every admissible blocking level [1,2].

### 2.1. The Confinement Relative-Entropy Density

**Definition 2.4** (Relative-entropy density). *For blocking level k and fixed physical volume V ,*

$$\kappa_k^{(V)} := \frac{1}{V} D(\mu_k \,\|\, \lambda^{\otimes})$$

**Theorem 2.5** (Unconditional RG monotonicity). *For every admissible kernel Tk, every fixed physical volume V , and every k ≥ 1,*

$$D(\mu_k \| \lambda^{\otimes}) \leq D(\mu_{k-1} \| \lambda^{\otimes}), \qquad hence \qquad \kappa_k^{(V)} \leq \kappa_{k-1}^{(V)}.$$

*Proof.* By the data-processing inequality (Appendix A, Lemma A.1), $D(\mu_{k-1} T_k \| \lambda^{\otimes} T_k) \leq D(\mu_{k-1} \| \lambda^{\otimes})$. By admissibility (iii), $\lambda\otimes T_k = \lambda\otimes$, and by definition $\mu k{-}1 T_k = \mu k$. Both sides are evaluated on the same fixed $V$ ; divide by $V$.

**Theorem 2.6** (Dissipation identity). *Let $T_k$ be admissible, π the joint law of (fine, coarse) configuration under μk−1 and $T_k$, and $\pi^\lambda$ the analogous joint law under $\lambda^\otimes$. Then*

$$D(\mu_{k-1} \| \lambda^{\otimes}) - D(\mu_k \| \lambda^{\otimes}) = \mathbb{E}_{y \sim \mu_k} \Big[ D\big(\pi(\cdot|y) \,\|\, \pi^{\lambda}(\cdot|y)\big) \Big] =: V \delta_k \geq 0$$

*Proof.* Apply the chain rule for relative entropy (Appendix A, Lemma A.2) to $(\pi,\pi^\lambda)$ in the two conditioning orders; the fine-given-coarse order and admissibility (iii) identify the two decompositions.

**Corollary 2.7** (Entropy budget equation). *Telescoping,* $\kappa_{k^*}^{(V)} = \kappa_0^{(V)} - \sum_{k \leq k^*} \delta_k, \; \delta_k \geq 0$ .

**Definition 2.8** (Half-dissipation scale)$k_{1/2} := \min\{k \geq 0 : \kappa_k \leq \frac{1}{2}\kappa_0\}$ *(finite or $+\infty$),* (finite or +∞), and $\ell_{1/2} := b^{k_{1/2}} a$.

**Remark 2.9** *(Scope: the Yang–Mills mass-gap program). The functional κk above, together with an exact thermodynamic identity $\kappa_k^{(V)} = \frac{1}{V}(-\mathbb{E}_{\mu_k} S_k - \log Z_k)$ and a susceptibility representation proving strict positivity at every coupling, is the central object of a separate research program addressing the four-dimensional Yang–Mills mass gap, organized around a structural analogy with Perelman's entropy-monotonicity method for Ricci flow , and resting on reflection positivity of the Wilson lattice action at every cutoff [17,18]. In that program, κk is interpreted as a confinement order parameter and its persistence at the physical blocking scale is conjecturally tied to the mass gap via a Peierls-type argument and Osterwalder–Schrader reconstruction. That program proves unconditional monotonicity, existence and finiteness of the thermodynamic density, an exact dissipation identity, and a localization of the persistence conjecture to a single measurable plaquette-expectation condition (verified as a theorem in the strong-coupling regime); it states explicitly that it does not solve the Clay Millennium Problem, isolating the d = 4 scaling window as the sole remaining open frontier. None of this mass-gap-specific content is needed below; only Theorem 2.5, Theorem 2.6, Corollary 2.7, and Definition 2.8 are reused, in Part III, on the different (spin-network) Hilbert space relevant to gauge-structure selection.*

## 3. Part II: Emergent Gauge Content from a Flux-Gauge Network

### 3.1. The Microscopic Model

We work on a dynamical graph whose links carry a compact gauge connection $U_{ij} \in SU(N)$, the lattice holonomy of a Yang–Mills connection, together with an integer winding label $n_{ij} \geq 0$ counting elementary flux quanta in the strong-coupling representation of $U_{ij}$. The effective Hamiltonian for the winding labels retains a tension, density, and reconnection term,

$$H[n] = \sigma \sum_{\langle ij \rangle} n_{ij} \; - \; \mu \sum_{\langle ij \rangle} n_{ij}^2 \; + \; g \sum_{i} \sum_{j,k \in \mathrm{nn}(i)} n_{ij} n_{ik},$$

with $\sigma > 0$ the bare string tension, $\mu$ a density coupling favoring nontrivial holonomy, and $g$ a reconnection coupling encoding the non-Abelian self-interaction of $U_{ij}$ in the strong-coupling image of $[A_\mu, A_\nu]$.

**Definition 3.1** (Gauge-connection phases). *Let $P_\infty$ be the relative size of the largest connected cluster of active links ($n_{ij} > 0$). The ensemble* *is subcritical ($P_\infty$ = 0, no large-scale coherent curvature, no geometry), critical (scale-invariant curvature fluctuations, identified with spacetime foam), or supercritical ($P_\infty$ = O(1), a spanning cluster whose curvature supports an emergent Cartan pair and metric).*

**Remark 3.2** (Scope: metric emergence). *The supercritical phase supports, via an enlarged so(3,2)valued connection and a MacDowell–Mansouri/Cartan mechanism, an emergent vierbein, metric, and (by double copy of the connection's own gauge-boson fluctuation ) a massless spintwo graviton, with Newton's constant a computed gauge-coupling functional and the Einstein equations recovered thermodynamically from gauge-link entanglement entropy via a bit-thread (maxflow/min-cut) construction [11-15]. This machinery underlies the gravitational sector of the enlarged connection used in Part IV (Postulate 5.1) but is not itself re-derived here; only the block-link renormalization-group flow to an interacting fixed point (needed for the continuum gauge-connection limit assumed throughout) and the two mechanisms below — ADE gauge-content generation and the vertex Higgs field — are used in what follows.*

### 3.2. ADE Geometric Engineering of SU(N) Matter Content

**Definition 3.3** (Rank-($N$−1) singular node). *A node xi is singular of rank N−1 if the connection's internal-fiber data on its incident links drives N−1 two-cycles of an internal fiber $X_i$ simultaneously to zero volume, so that $X_i \to C^2/ZN$, an $A_{N-1}$ ADE orbifold singularity.*

**Result 3.4** (Katz–Vafa mechanism, imported). *At a rank-(N − 1) singular node, wrapped-brane states massive away from the singularity, with mass proportional to the collapsing cycle volume, become exactly massless at the degeneration point. The N −1 vanishing cycles supply N −1 Cartan generators; states supported on unions of collapsed cycles supply the AN−1 root system; the full massless spectrum at the node fills out the adjoint of SU(N). This is an established mechanism in the string-theoretic geometric-engineering literature (Katz and Vafa 1997 ; Bershadsky, Intriligator, Kachru, Morrison, Sadov, and Vafa 1996 ), imported here, not re-derived; the N = 5 case used below is the same singularity data underlying F-theory constructions of SU(5) grand unification, though we do not adopt the F-theory setting itself* [7-10].

### 3.3. The Adjoint Vertex-Higgs Mechanism

**Postulate 3.5** (Vertex Higgs field). *A scalar field $\Phi_i$, transforming in the adjoint representation of the local structure group $G_{loc}$ carried at node $x_i$, is attached to every vertex, with a gaugeinvariant quartic potential $V(\Phi_i)$, and a nearest-neighbor hopping term coupling $\Phi_i$ to the incident link variables. The absolute minimum of V lies on an orbit $\langle\Phi_i\rangle \neq 0$ whose stabilizer is a proper subgroup $H \subset G_{loc}$ .*

**Result 3.6** (Symmetry breaking via the vertex field). *At the minimum of $V(\Phi_i)$, the local gauge symmetry is broken $G_{loc} \to H$; the link variable's holonomy decomposes, via the exact Haar-measure fibration of the coset $G_{loc}/H$, into a dynamical H-holonomy piece and a coset piece that survives as (pseudo-)Goldstone content, with mass gap set by the curvature of V at its minimum.* This mechanism was originally used, with $G_{loc}$an (A)dS-type extension of the structure group and $H = SO(3,1)\times SU(N)$, to freeze the gravitational soldering-form sector of an enlarged connection. Part IV applies the identical mechanism, with $G_{loc} = SU(5)$ and $H = SU(3)\times SU(2)\times U(1)$, to the matter sector generated by Result 3.4.

### 3.4. Numerical Evidence for the Percolation/Geometric Transition

Direct Monte Carlo simulation of the winding-number ensemble on a three-dimensional cubic lattice confirms a sharp percolation transition at $p_c \approx 0.320$–$0.325$, consistent with the known threedimensional site-percolation value$p_c \simeq 0.3116$ under finite-size rounding, and a boundarycut entropy area law $N_{cut} = \gamma$ Area + const with $\gamma \approx 0.098$–$0.099$ and $R^2 \simeq 0.9996$ [16]. A baseline simulation using explicit compact $SU(2)$ link matrices confirms that the percolation order parameter tracks the true non-Abelian plaquette curvature $\langle\kappa\rangle = \langle 1 - \frac{1}{2}\Re\,\mathrm{Tr}\,U_p\rangle$ directly directly (Pearson correlation $r \approx 0.78$–$0.89$ across the simulated range), rather than only in the strong-coupling winding-number proxy. These results establish that the underlying lattice supports a genuine geometric/percolation transition with the mechanism of Definition 3.1; they do not, by themselves, test any claim about the Standard Model and are reported here as the empirical foundation reused in Part III.

## 4. Part III: Relative-Entropy Selection of Gauge Structure on Spin Networks

### 4.1. Spin Networks and Admissible Coarse-Graining

A spin network is a graph γ with edges labeled by irreducible representations $j_e$ of a compact gauge group $G$ and vertices labeled by intertwiners; the kinematical Hilbert space is $H_\gamma = L^2(G^{|E(\gamma)|}/G^{|V(\gamma)|})$, the quotient enforcing gauge invariance at every vertex.

**Definition 4.1** (Admissible spin-network blocking). *A coarse-graining map $T_k : H_{\gamma k}-1 \to H_{\gamma k}$ is admissible if it satisfies the direct analogues of Definition 2.2(i)–(iii), with reference invariance now requiring the maximal-ignorance reference state — Haar measure on holonomies, or equivalently the dimension-weighted (2j +1) measure on representation labels induced by the Peter–Weyl decomposition — to push forward correctly under $T_k$. By Lemma 2.1, applied unchanged to any gauge-invariant measure on spin-network link configurations, this reference is canonical.*

**Definition 4.2** (Spin-network relative-entropy functional) $\kappa_k^{(V)} := \frac{1}{V} D(\mu_k \| \nu_k)$, with $\nu_k$ *the maximalignorance reference of Definition 4.1.*

**Theorem 4.3** (Unconditional monotonicity, transplanted). *For every admissible spin-network kernel and fixed* $V$, $\kappa_k^{(V)}$ *is non-increasing in k. Proof.* Identical to Theorem 2.5, with $\lambda^{\otimes}$ replaced by $v_k$ throughout; the proof uses only the dataprocessing inequality and reference invariance, both of which hold by construction.

**Theorem 4.4** (Dissipation identity, transplanted). *The direct analogue of Theorem 2.6 holds for spin-network coarse-graining.*

### 4.2.The Attractor-Classification Conjecture

**Conjecture 4.5** (Attractor classification). *Let* $G_{UV}$ *be the structure group carried by spin-network edges in the deep ultraviolet. The set of subgroups H ⊆ GUV for which* $D_\gamma[H] := D(\mu_\gamma \| \nu_\gamma^H) \to 0$ *under iterated admissible coarse-graining forms a discrete set determined by the branching rules of GUV-representations under H, together with the UV graph connectivity.*

**Proposition 4.6.** *For each fixed* $H$, $(D_{\gamma_k}[H])_k$ *is non-increasing (Theorem 4.3 applied to H), hence converges to D∞[H] ≥ 0; Conjecture 4.5 is exactly the assertion that D∞[H] = 0 for a discrete, branching-rule-determined set of H.*

Conjecture 4.5 is the direct information-theoretic counterpart of Result 3.4: both replace an assumed algebraic answer for the surviving gauge group with a computable criterion attached to an independently motivated dynamical process (geometric degeneration in one case, RG coarsegraining in the other). Whether the two criteria select the same subgroups on a common network is the selection problem formulated precisely in Section 6.

### 4.3. Numerical Tests

#### 4.3.1. A Quartic-Stabilized Single-Link Ansatz

We test Theorem 4.3 on a mean-field ansatz in which each edge carries an i.i.d. occupation number $n \in \{0,\ldots,n_{\max}\}$ drawn from $w(n) \propto \exp[-\beta(\sigma n - \mu n2 + \lambda n^4)]$, the quartic term added — exactly as in Part II's winding-number ensemble — to prevent boundary pileup at the truncation cutoff for large $\mu$, with reference $r(n) \propto (2n+1)e^{-\beta\sigma n}$ (the dimension-weighted maximal-ignorance measure). Coarse-graining by block-summing $b$ edges gives an exact, checkable instance of Theorem 4.3: $\kappa k := b^{-k}D(w*^{bk}\|r*^{bk})$ is guaranteed non-increasing by the data-processing inequality applied to the (lossy) b-fold convolution channel. An initial FFT-based implementation produced spurious violations of this monotonicity at $\mu \gtrsim 0.45$, traced to floating-point round-off in spectral convolution of distributions with different tail behavior; cross-checking D($w{/\!/}r$) and $D(w^{*4}\|r^{*4})$ at $\mu = 0.45$ against 50-digit arbitrary-precision arithmetic confirmed the theorem ($\kappa_0$ = 1.43296 → κ1 = 1.04044, a clean decrease) and identified the FFT path as the source of the artifact; we replaced it with exact (non-FFT) convolution throughout.

| *μ* | *pactive* | *κ0* | *κ1* | *κ2* | *κ3* | *κ4* |
|---|---|---|---|---|---|---|
| 0.050 | 0.052 | 0.04058 | 0.04022 | 0.04010 | 0.04006 | 0.04005 |
| 0.250 | 0.067 | 0.02757 | 0.02500 | 0.02385 | 0.02350 | 0.02340 |
| 0.400 | 0.091 | 0.09652 | 0.05628 | 0.02231 | 0.00651 | 0.00171 |
| 0.440 | 0.139 | 0.76991 | 0.53070 | 0.33437 | 0.26518 | 0.24919 |
| 0.450 | 0.175 | 1.43296 | 1.04044 | 0.76318 | 0.68810 | 0.67226 |
| 0.460 | 0.238 | 2.68297 | 2.06246 | 1.73226 | 1.67096 | 1.65793 |
| 0.462 | 0.254 | 3.03581 | 2.36208 | 2.03047 | 1.97412 | 1.95533 |

**Table 1: Relative-entropy functional $\kappa k = D(w^{*4k}\|r^{*4k})/4k$ under repeated block-summing coarsegraining ($\beta = 1, \sigma = 3, \lambda = 0.002, n_{\max} = 30, b = 4$), monotone decrease confirmed by exact convolution to $10^{-9}$. $\mu = 0.462$ is the last row before $p_{\text{active}}$ crosses the independently measured percolation threshold.**

#### 4.3.2. Half-Dissipation Scale near the Perolation Crossover

An independent bond-percolation simulation on a three-dimensional periodic cubic lattice ($L = 24$, union–find) locates the threshold at $p_c \approx 0.25$, consistent with the known bond-percolation value $p_c \simeq 0.2488$ [16]. The half-dissipation scale $k_I/2(\mu)$ (Definition 2.8) is 2 for $\mu \in [0.35,0.44]$, rises to 3 at $\mu = 0.45$, and is unresolved within $k \leq 4$ for $\mu \geq 0.455$ — including μ = 0.462, where pactive has already crossed pc. The half-dissipation scale thus grows and becomes numerically unresolvable, at the coarse-graining depths reached, in the same narrow parameter window where the independently measured percolation threshold is crossed.

**Remark 4.7** *(What this does and does not show). This is the qualitative signature Conjecture 4.5 would predict — distinguishability from the reference dissipates more slowly as a genuine transition is approached — but it is not a proof. A numerical-precision floor at the sixth coarse-graining level prevented resolving whether κk resumes decreasing beyond k = 4, and pactive(μ) and pc were measured on* *two different, uncorrelated ensembles rather than self-consistently on the same configurations. A correlated (non-mean-field) simulation measuring both quantities together remains an open numerical target.*

## 5. Part IV: Recovering the Standard Model from the Ashtekar Connection

### 5.1. The Ashtekar–Barbero Phase Space

On a spatial slice Σ, canonical general relativity in connection variables is coordinatized by a densitized triad $E_i^{\,a}$ and connection $A_i^{\,a} = \Gamma^{\,i}_{\;a} + \gamma K^{\,i}_{\;a} \in su(2)$ [1, 2], with $\Gamma^{\,i}_{\;a}$ the triad-compatible spin connection, the extrinsic curvature, and $\gamma$ the Barbero–Immirzi parameter. Holonomies of  along edges and fluxes of $E$ through dual surfaces generate the kinematical Hilbert space $\mathrm{H}_\gamma^{\mathrm{grav}}$ of spin networks used in Part III, on which the area operator has spectrum $8\pi G_\gamma{}^{P}{}_{p}{}^{p} j_p(j_p + 1)$ [3].

### 5.2. Enlargement As A Direct Sum, and its Conservativity

Postulate 5.1 (Enlarged connection). *The connection on every edge is valued in gUV* = so(3,2)⊕ su(5),

$$A_\mu = \underbrace{\omega_\mu^{ab} M_{ab} + \tfrac{1}{\ell} e_\mu^a P_a}_{\textit{gravity: Part II mechanism}} \quad \underbrace{+ B_\mu^A T^A}_{\textit{matter}},$$

*with $M_{ab}$, $P_a$ the so(3,2) generators of Part II's pure-connection gravity construction (whose Lorentz part contains the self-dual su(2) of the Ashtekar formulation), and $T^A$ the su(5) generators normalized as in Section 1.3.*

**Lemma 5.2** (Holonomy factorization). *For* $\mathrm{g} = \mathrm{g}_1 \oplus \mathrm{g}^2$ with $\mathrm{G} = G_1 \times G_2$ and $A = A_1 + A_2$ valued correspondingly,

$$\mathcal{P}\exp \textstyle\int_e A = \left(\mathcal{P}\exp \int_e A_1\right) \times \left(\mathcal{P}\exp \int_e A_2\right) \text{ for every path } e.$$

*Proof.* $[A_1(s), A_2(s')] = 0$ for all $s, s'$ since the summands commute in $\mathrm{g}_1 \oplus \mathrm{g}_2$; both sides solve $\frac{d}{ds} h(s) = (A_1(s) + A_2(s))h(s),\ h(0) = ⊮$, checked by differentiating the product using $[A_1, h_2] = 0$,
and solutions are unique.

**Theorem 5.3** (The LQG sector is exactly unchanged). *With Postulate 5.1, the kinematical Hilbert space on any graph γ factorizes as $\mathcal{H}_\gamma = \mathcal{H}_\gamma^{\mathrm{grav}} \otimes \mathcal{H}_\gamma^{\mathrm{su}(5)}$, gauge invariance imposed independently in each factor, and every LQG operator built from gravity-sector holonomies and fluxes — in particular area and volume — acts as (operator)⊗⊮, unchanged.*

*Proof.* By Lemma 5.2 the configuration space factorizes as $(G_1 \times G_2)^{|E|} \cong G_1^{|E|} \times G_2^{|E|}$, and Haar measure on a product group is the product of Haar measures, so $L^2((G_1 \times G_2)^{|E|}) = L^2(G_1^{|E|}) \otimes L^2(G_2^{|E|})$. Gauge transformations act factor-wise, so the invariant subspace is the tensor product of invariant subspaces. Fluxes conjugate to $A_1$ act only on the first factor, so the area and volume operators act as stated.

**Remark 5.4.** *This theorem is the precise sense in which the construction extends, rather than deforms, loop quantum gravity: every kinematical LQG result survives verbatim, because the matter sector is attached by tensor product. It also disciplines Part III's coarse-graining kernels, which act on $\mathrm{H}^{grav} \oplus \mathrm{H}^{su}(5)$ factor-wise or jointly, with the relative-entropy functional of Conjecture 4.5 well defined on the matter factor alone.*

### 5.3. Geometric Engineering of the Matter Sector

**Definition 5.5** (Rank-4 singular node). *Specializing Definition 3.3 to N = 5: a node is rank-4 singular if four two-cycles of its internal fiber collapse, $X_i \to C^2/Z5$, an A4 singularity.*

By Result 3.4 at $N = 5$, the massless spectrum at such a node fills out the adjoint of $SU(5)$, dimsu(5) = 24. As emphasized in Part II, this mechanism is imported, not derived; what is specific to this paper is the choice $N = 5$, made to match the target embedding below and not derived from any dynamical principle internal to Parts I–III (see Section 6).

### 5.4. Root Data, Branching, and Normalization

We use standard conventions and results for simple Lie algebras and their representation branchings throughout this subsection [20, 21]. Fix the Cartan subalgebra of diagonal traceless matrices, simple roots $\alpha_i = e_i - e_i + 1$, $i = 1,\ldots,4$, lowest root $\alpha_0 = e_5 - e_1$.

**Theorem 5.6** (Borel–de Siebenthal, specialized to $A_4$). *The proper maximal-rank regular subalgebras of su(5) are obtained by deleting one node of the extended Dynkin diagram $A_{e4}$ (a pentagon) [6]: su*(4) ⊕ u(1) (split 5 = 1 + 4) or su(3) ⊕ su(2) ⊕ *u*(1) (*split* 5 = 2 + 3), *and no others at the first step.*

*Proof.* Removing one node of the 5-cycle Ae4 leaves either $A_4$ itself (removing $\alpha_0$) or a disjoint pair of chains $A_{k-1} \sqcup A_{4-k}$; rank is restored to 4 by an extra u(1) since the subalgebra is regular of full rank [6]. The splits k = 1,...,4 give the two isomorphism classes.

**Proposition 5.7** (Uniqueness of the Standard Model profile). $su(3) \oplus su(2) \oplus u(1)$ *is the unique maximal-rank regular subalgebra of su(5) containing simple summands of rank 2 and 1, realized by the block-diagonal split 5 =* 3 + 2,

$(u,v,e^{i\theta})$ 7−→ $\mathrm{diag}(e^{-2i\theta}u,\ e^{3i\theta}v)$, $\qquad$ $u \in SU(3),\ v \in SU(2)$,*with* u(1) *generator* $Y = \mathrm{diag}(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$.

Proof. By Theorem 5.6, su(4)⊕u(1) has a rank-3 simple summand, not 2+1, so su(3)⊕su(2)⊕u(1) is the only candidate. The centralizer of su(3) ⊕ su(2) in su(5) is one-dimensional (Schur's lemma blockwise, tracelessness fixing the ratio of block scalars), spanned by Y ; the exponentiated phases ) are the minimal integers making the map well defined on U(1) = R/2πZ.

**Theorem 5.8** (Exact adjoint decomposition). **24** = (**8**,**1**)0 ⊕ (**1**,**3**)0 ⊕ (**1**,**1**)0 ⊕ (**3**,**2**)−5/6 ⊕ (**3**,**2**)+5/6, dimensions 8 + 3 + 1 + 6 + 6 = 24.

*Proof.* Decompose a traceless Hermitian 5 × 5 matrix $M = \begin{pmatrix} M_{33} & M_{32} \\ M_{32}^{\dagger} & M_{22} \end{pmatrix}$. The traceless parts of $M_{33}, M_{22}$ give (**8**,**1**)0 and (**1**,**3**)0 (block-diagonal, commuting with $Y$). The remaining diagonal direction, spanned by $Y$, gives (**1**,**1**)0. The off-diagonal block transforms as (**3**,**2**) under $M_{32}$ 7⟶ $uM_{32v}{}^{\dagger}$, with $[Y,M]_{32} = (Y_{(3)} - Y_{(2)})M_{32} = -\frac{5}{6}M_{32}$; its conjugate carries $+\frac{5}{6}$. Real dimensions sum to 8 + 3 + 1 + 12 = 24.

**Lemma 5.9** (Canonical normalization). $\mathrm{Tr}(Y^2) = 3 \cdot \frac{1}{9} + 2 \cdot \frac{1}{4} = \frac{5}{6}$, *so* $T^{24}$ = p3/5$Y$ *obeys*$\mathrm{Tr}((T^{24})^2) = \frac{1}{2}$.

**Corollary 5.10**($\sin^2\theta_W = 3/8$ at matching)**.** *If* $g_3 = g_2 = g_5$ *and* $g'$ = p3/5$g_5$ at the matching scale, then $\sin^2\theta_W = g'^2/(g_2^2 + g'^2) = (3/5)/(8/5) = 3/8$

## 5.5.Vacuum Alignment

**Postulate 5.11** (Adjoint vertex scalar, specialized). Specializing Postulate 3.5 to $G_{loc} = SU(5)$: *each rank-4 node carries a Hermitian traceless* $\mathbf{\Sigma}_i \in$ **24** *with potential* $V(\Sigma) = -\frac{1}{2}m^2\,\mathrm{Tr}\Sigma^2 + \frac{1}{4}a(\mathrm{Tr}\Sigma^2)^2 + \frac{1}{4}b\,\mathrm{Tr}\Sigma^4$*,* $m^2 > 0$(cubic invariant excluded by Σ → −Σ).

**Lemma 5.12** (Reduction to eigenvalue optimization). Every extremum orbit has a diagonal representative $\mathrm{diag}(\phi_1,...,\phi_5)$, $P\phi_i = 0$; at any extremum the $\phi_i$ take at most three distinct values, and at a minimum with b≠ 0, at most two.

**Theorem 5.13** (Li vacuum alignment). For $b > 0$ *(and* $a > -\frac{7}{30}b$*, boundedness), the global minimum is* $\langle\Sigma\rangle = V_0\,\mathrm{diag}(2, 2, 2, -3, -3)$, $V_0^2 = m^2/(30a + 7b)$, with stabilizer exactly su(3) ⊕ su(2) ⊕ u(1) (diag(2,2,2,−3,−3) = −6Y ). For $b < 0$ *(and* $a + \frac{13}{20}b > 0$*),* the minimum is ∝ diag(1,1,1,1,−4), *stabilizer su*(4) ⊕ u(1).

*Proof.* By Lemma 5.12, extrema have eigenvalue pattern ($k$ copies of $x$,(5−$k$) copies of $y$), kx+(5− k)$y$ = 0. Parametrizing $x = (5-k)$t, y = $-kt$: $\mathrm{Tr}\Sigma^2{}_k = 5k(5-k)t^2$, $\mathrm{Tr}\Sigma_k{}^4 = k(5-k)[k^3+(5-k)^3]t^4$, so the scale-invariant anisotropy is $c(k) = \mathrm{Tr}\Sigma^4{}_k/(\mathrm{Tr}\Sigma^2{}_k)^2 = [k^3 + (5-k)^3]/[25k(5-k)]$,giving $c(1) = c(4) = 13/20$ and $c(2) = c(3) = 7/30$. Minimizing $V$ at fixed pattern over the overall scale gives $V_{\min}(k) = -m^4/[4(a + bc(k))]$; for $b > 0$ the smallest $c(k)$ (balanced split $k = 3$) gives the deepest minimum, since 7/30 < 13/20; for $b < 0$ the largest $c(k)$ ($k = 4$) wins. Inserting $\mathrm{Tr}\Sigma^2 = 30V_0^2$, $\mathrm{Tr}\Sigma^4 = 210V_0^4$ into the stationarity equation for the k = 3 pattern fixes the scale to the stated . Both constants were verified by independent symbolic computation.

**Proposition 5.14** (Gauge boson masses). *With kinetic term* $\mathrm{Tr}(D_\mu\Sigma D^\mu\Sigma)$, $D_\mu\Sigma = \partial_\mu\Sigma - ig_5[A_\mu,\Sigma]$, the (3,2)±5/6 sector acquires common mass $M_X^2 = 25\,g_5^2V_0^2$*; the* (**8**,**1**)0 ⊕ (**1**,**3**)0 ⊕ (**1**,**1**)0 sector remains massless.

*Proof.* For $T$ in the stabilizer, $[T,\langle\Sigma\rangle] = 0$. For the broken generator $T^X = \frac{1}{2}(E + E^{\dagger})$, $E$ a matrix unit in the 3 × 2 block (obeying $\mathrm{Tr}((T^X)^2) = \frac{1}{2})$, $[E, \langle\Sigma\rangle] = -5V_0E$, giving $[T^X,\langle\Sigma\rangle] = \frac{5V_0}{2}(E^{\dagger} - E)$ and mass term $-g_5^2\,\mathrm{Tr}([A,\langle\Sigma\rangle]^2)|_{A=A^XT^X} = \frac{25}{2}g_5^2V_0^2 (A^X)^2 = \frac{1}{2}M_X^2(A^X)^2$, i.e. $M_X{}^2 = 25g_5^2V_0^2$ , for each of the twelve broken directions (24−12 = 12), verified numerically by direct trace evaluation.

## 5.6. Global Structure and Charge Quantization

Theorem 5.15 (Global form of the unbroken group). *The stabilizer of* ⟨Σ⟩ *in* $SU(5)$ *is* $G_{\mathrm{SM}} = S(U(3) \times U(2))$ ~= $(SU(3) \times SU(2) \times U(1))/$Z6*, the* Z6 *generated by* (ω⊮3,−⊮2,$e^{i\pi/3}$).

*Proof*. $g \in SU(5)$ commutes with the two-eigenvalue matrix $\langle\Sigma\rangle$ iff block-diagonal, $g = \mathrm{diag}(A,B)$, $A \in U(3), B \in U(2)$, with $\det A \det B = 1$ from $g \in SU(5)$: this is $S(U(3) \times U(2))$. The covering map of Proposition 5.7 has kernel $\{(u,v,e^{i\theta}) : e^{-2i\theta}u = \mathbb{1}_3, e^{3i\theta}v = \mathbb{1}_2\}$; $u = e^{2i\theta}\mathbb{1}_3 \in SU(3)$ forces $e^{6i\theta} = 1$, giving a cyclic kernel of order 6.

**Corollary 5.16** (Charge quantization). *Hypercharge is quantized in units of* $\frac{1}{6}$ *correlated with triality and duality by* $6Y \equiv 3d + 2t \pmod 6$*; electric charge* $Q = T3 + Y$ *is quantized with the observed pattern (e.g.* $Y = \frac{1}{6}$ *for a color triplet, weak doublet).*

**Remark 5.17.** *That the true global gauge group of the Standard Model is S(U(3)×U(2)) rather than the naive product SU(3) × SU(2) × U(1), with exactly this consequence for charge quantization, is established independently of any embedding in SU(5) or in loop quantum gravity by directly analyzing the Standard Model's own representation content and line-operator spectrum [19]. Theorem 5.15 shows that this global form is additionally the automatic output of the symmetry-breaking chain constructed here, rather than an independent input.*

### 5.7. What Part IV has and has not Shown

(1) *Proved:* Theorem 5.3 (conservativity); given Postulate 5.11 with $b > 0$, Theorems 5.13, 5.15; the exact branching (Theorem 5.8), normalization and $\sin^2 \theta_W = 3/8$ (Lemma 5.9, Corollary 5.10); the *X/Y* mass (Proposition 5.14); charge quantization (Corollary 5.16).

(2) *Imported:* Result 3.4 ($A_4 \Rightarrow SU(5)$), from Katz–Vafa [7] and Bershadsky–Intriligator–Kachru– Morrison–Sadov–Vafa [8]; the vacuum-classification method of Theorem 5.13, from Li (1974) and the original Georgi–Glashow (1974) breaking [4,5].

(3) *Chosen, not derived*: the rank ($N = 5$) and the sign $b > 0$. Nothing in Parts I–III selects either dynamically; Section 6 states this as the paper's central open problem.

(4) *Not addressed*: fermion content and its LQG realization, chirality, generation number, Yukawa couplings, coupling running below the matching scale, proton decay phenomenology, and the fate of the Z6 quotient's line-operator spectrum. Appendix B proves anomaly cancellation for the standard $\mathbf{\bar{5}} \oplus \mathbf{10}$ assignment while emphasizing that the assignment itself is not derived here.

## 6. The Selection Problem

Both pieces of Part IV's undetermined input — the singularity rank of Section 4.3 and the alignment sign of Theorem 5.13 — reduce, on inspection, to the same combinatorial fact: the anisotropy $c(k) = [k^3 + (5 - k)^3]/[25k(5 - k)]$ of the split $5 = k + (5 - k)$ is minimized at the balanced split $k = 3$ (equivalently $k = 2$), the split realizing $su(3) \oplus su(2) \oplus u(1)$. This is exactly the mechanism that selects the Standard Model over $SU(4) \times U(1)$ in Theorem 5.13 whenever $b > 0$.

**Definition 6.1** (Standard Model selection problem). *Let the UV structure group of the matter factor in Theorem 5.3 be SU(5) with edge data generated at ADE-degenerate nodes as in Section 4.3, and let the admissible coarse-graining kernels of Part III act on the matter factor. The selection problem is to determine whether, for generic UV graph ensembles in the geometric (percolating) phase of Part II, the attractor set of Conjecture 4.5 is* $\{H : D_\infty[H] = 0\} = \{S(U(3) \times U(2))\}$ *— equivalently, whether the relative-entropy RG flow dynamically prefers the balanced split k = 3 over k = 4, mirroring at the information-theoretic level the energetic preference c(3) < c(4) that drives Theorem 5.13 at the level of the vertex potential.*

We regard this coincidence of extremizing quantity — the same anisotropy ratio $c(k)$ governing both an energetic (Higgs-potential) and a candidate information-theoretic (relative-entropy) selection mechanism — as the sharpest question raised by combining Parts I–IV, and we leave it open.

## 7. Unified Status Ledger

| Component | Status | Reference |
|---|---|---|
| Single-link marginals of gauge-invariant measures are Haar | Proved | Lem. 2.1 |
| Unconditional RG monotonicity, lattice link variables | Proved | Thm. 2.5 |
| Dissipation identity, budget equation | Proved | Thm. 2.6, Cor. 2.7 |

| | | |
|---|---|---|
| Yang–Mills mass-gap program (persistence, strongcoupling closure, conditional gap theorem) | Established elsewhere; summarized, not repro-duced | Remark after Def. 2.8 |
| ADE singularity ⇒ $SU(N)$ gauge content (Katz–Vafa) | Imported | Result 3.4 |
| Metric/graviton emergence from enlarged connection | Established elsewhere; summarized, not repro-duced | Remark after Def. 3.1 |
| Vertex-Higgs symmetry breaking mechanism (general) | Established framework, reused | Post. 3.5, Result 3.6 |
| Percolation transition, area-law entropy (numerical) | Confirmed | Sec. 3.4 |
| Unconditional monotonicity, spin networks (transplant) | Proved | Thm. 4.3 |
| Attractor-classification conjecture | Open | Conj. 4.5 |
| Monotone decrease, quartic-stabilized numerics, crosschecked vs. arbitrary precision | Confirmed | Sec. 3.3.1 |
| Half-dissipation scale growth near percolation crossover | Suggestive trend, not proof | Sec. 3.3.2 |
| LQG sector exactly unchanged by enlargement | Proved | Thm. 5.3 |
| Rank-4 $A_4$ node ⇒ $SU(5)$ (specialization) | Imported | Sec. 4.3 |
| Borel–de Siebenthal classification, uniqueness of SM profile | Proved | Thm. 5.6, Prop. 5.7 |
| Exact adjoint branching | Proved | Thm. 5.8 |
| | | |
| Hypercharge normalization, $\sin^2\theta_W = 3/8$ | Proved | Lem. 5.9, Cor. 5.10 |
| Li vacuum alignment ($b > 0$ ⇒ SM; $b < 0$ ⇒ $SU(4) \times U(1)$) | Proved | Thm. 5.13 |
| $X/Y$ boson mass $M_X^2 = 25 g_5^2 V_0^2$ | Proved | Prop. 5.14 |
| Global structure $S(U(3) \times U(2))$, charge quantization | Proved | Thm. 5.15, Cor. 5.16 |
| Rank ($N = 5$) and sign ($b > 0$) dynamically selected | Open | Sec. 6 |
| Anomaly cancellation, $\bar{\mathbf{5}} \oplus \mathbf{10}$ | Proved (assignment not derived) | App. B |

| | | |
|---|---|---|
| Fermion content, chirality, generations, Yukawas | Not addressed | — |

**Table 2: Complete Status Ledger for the Unified Construction.**

## 8. Limitations

(1) The mass-gap-specific and metric/graviton-specific machinery of the two underlying programs (Parts I and II) is summarized, not reproduced; readers seeking those programs' full apparatus (Peierls arguments, Osterwalder–Schrader reconstruction, the toy holographic bulk, inflationary cosmology from the connection modulus) should note that only the pieces reused in Parts III–IV are proved in full here.

(2) Conjecture 4.5 is open; Section 3's numerics confirm only the unconditional monotonicity any admissible functional of this type must satisfy, plus a suggestive but inconclusive trend near the percolation crossover.

(3) The rank of the ADE singularity and the sign of the Higgs quartic coupling are the two inputs on which the entire Part IV output depends, and neither is derived from Parts I–III; Section 6 states the precise, currently open, selection problem.

(4) No claim is made about fermion content, chirality, generation number, Yukawa structure, coupling running, or proton-decay phenomenology.

(5) The gravitational (so(3,2)) and matter (su(5)) sectors are combined only as a direct sum; no unification of gravity and matter into a single simple structure group is attempted or claimed.

## 9. Conclusion

We have given a single, self-contained construction — combining a relative-entropy renormalizationgroup toolkit, a flux-gauge network model of ADE gauge-content generation and vertex-Higgs symmetry breaking, and their joint application to an enlarged Ashtekar connection — from which the Standard Model gauge group emerges as a chain of proved theorems: exact conservativity of the LQG kinematical sector, the unique Borel–de Siebenthal embedding, the exact adjoint branching with correct hypercharge normalization and $\sin^2 \theta W = 3/8$, the Li vacuum-alignment condition, the *X/Y* mass spectrum, and the global structure *S(U(3)×U(2))* with its automatic charge quantization. Two inputs carry the entire result and are not derived: the rank of the ADE singularity and the sign of the Higgs quartic anisotropy coupling, both of which reduce to the same combinatorial preference for the balanced split 5 = 3 + 2. We have formulated precisely, as the paper's central open problem, whether the relative-entropy attractor-classification conjecture of Part III supplies the missing dynamical selection mechanism for that preference. Fermion content, chirality, generations, Yukawas, and unification phenomenology remain entirely outside this paper's scope.

### A. Information-Theoretic Tools

For completeness we record, with proofs, the elementary information-theoretic facts used in the body of the paper; standard references for this material include [22].

**Lemma A.1** (Data-processing inequality). For any Markov kernel $T$, $D(\mu T \| \nu T) \leq D(\mu \| \nu)$.

*Proof.* Write $D$ as a supremum over finite measurable partitions of discrete divergences; pull back through $T$ and apply joint convexity of $(p,q)$ 7→ $p\log(p/q)$ (the log-sum inequality).

**Lemma A.2** (Chain rule). For $\pi,\pi'$ on $X \times Y$ with disintegrations $\pi(\cdot|y),\pi'(\cdot|y)$, $D(\pi\|\pi') = D(\pi_Y\|\pi'_Y) + \mathbb{E}_{y\sim\pi_Y}\left[D(\pi(\cdot|y)\|\pi'(\cdot|y))\right]$, symmetrically in $X,Y$.

*Proof.* By disintegration on standard Borel spaces, $\frac{d\pi}{d\pi'}(x,y) = \frac{d\pi_Y}{d\pi'_Y}(y) \cdot \frac{d\pi(\cdot|y)}{d\pi'(\cdot|y)}(x)$ $\pi$-a.e.; take logarithms and integrate against $\pi$, conditioning on y for the second term.

**Lemma A.3** (Pinsker's inequality). $D(\mu\|\nu) \geq 2\|\mu - \nu\|^2_{\mathrm{TV}}$.

**Lemma A.4** (Factorization / superadditivity). For $\nu = N_j \nu_j$ and any μ with marginals $\mu j$, $D(\mu\|\nu) = D(\mu\|N_j \mu_j) + P_j D(\mu_j\|\nu_j) \geq P_j D(\mu_j\|\nu_j)$.

### B. Anomaly Cancellation For The $\bar{5}$ ⊕ 10 Assignment

This appendix records, with proof, that the standard Georgi–Glashow fermion assignment is anomaly free; it does not derive that assignment from Parts I–IV.

**Proposition B.1.** *Let A(R) be the cubic anomaly coefficient,* $\mathrm{Tr}_R(T^A\{T^B,T^C\}) = \frac{1}{2}A(R)d^{ABC}$, $A(\mathbf{5}) = 1$. *Then* $A(\bar{\mathbf{5}})$

$= -1$, $A(\mathbf{10}) = +1$, $A(\bar{\mathbf{5}} \oplus \mathbf{10}) = 0$.

*Proof.* $A(\bar{R}) = -A(R)$ by conjugation ($T^A \mapsto -(T^A)^{\mathrm{T}}$, odd under the symmetrized trace). For $\mathbf{10} = \Lambda^2\mathbf{5}$, expanding pairwise weight sums with $\sum_i x_i = 0$: $\sum_{i<j}(x_i + x_j)^3 = (N-4)\sum_i x_i^3$ at general $N$, giving $A(\Lambda^2\mathbf{N}) = N - 4$; at $N = 5$, $A(\mathbf{10}) = 1$, and the sum vanishes.

## Statements and Declarations

- **Data Availability**
  This paper is entirely theoretical and mathematical, with the exception of the numerical tests reported in Sections 3.4 and 5.3.1–5.3.2 (self-contained Monte Carlo and exact-arithmetic simulations); source code and raw output are available from the author upon reasonable request. All group-theoretic constants were additionally verified by independent symbolic and numerical computation.
- **Competing Interests**
  The author declares no competing interests, financial or non-financial, relevant to the content of this manuscript. This research received no funding from any agency in the public, commercial, or not-for-profit sectors.

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