Review Article

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Quantum Physics in Phase Space: an Analysis of Simple Pendulum

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Abstract

In this work we present a brief review about quantum mechanics in phase space. The approach discussed is based in the notion of symplectic structure and star-operators. In this sense, unitary representations for the Galilei group are construct, and the Schrodinger equation in phase space is derived. The connection between phase space amplitudes and Wigner function is presented. As a new result we solved the Schrodinger equation in phase space for simple pendulum. PACS Numbers: 11.10.Nx, 11.30.Cp, 05.20.Dd

Introduction

The first formalism to quantum mechanics in phase space was introduced by Wigner in 1932 [1]. He was motived by the problem of finding a way to improve the quantum statistical mechanics, based on the density matrix, to treat the transport equations for superfluids. Then, the Wingner function has been applied in several areas such as quantum statistical mechanics, plasma physics, quantum computing and condensed matter. In the Wigner formalism [2], each operator, say A, defined in the Hilbert space, H, is associated with a function, say a_w (q, p), in phase space Γ. Then there is an application Ωw : A → a_w (q, p), such that, the associative algebra of operators defined in H turns out to be an associative (but not commutative) algebra is Γ, given by Ω : AB → a_w (q, p) ⋆ b_w (q, p), where the star-product (or Weyl product) ⋆ is defined by [3].

$$a_w(q,p)\star b_w(q,p) = a_w(q,p)\exp[\frac{i}{2}(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q})]b_w(q,p), \tag{1}$$

where the arrows indicate of the direction of action of the operators. Notice that Eq. (1) can be seen as an operator b Â acting on functions b_w, such that b Â(b_w) = $a_w \star b_w$. The star-product is interesting to study the irreducible unitary representations of kinematical groups considering operators of the type $a_w \star$. In this article, we present a brief review based in recents work of our research group [4, 5]. In this way, using the notion of symplectic structure and Weyl product of a non-commutative geometry, unitary representations of Galilei group were studied and the Schrödinger equation in phase space was obtained. This approach provides a new procedure to derive the Wigner function without the use of the Liouville-von Neumman equation. As a new result, we analyze the simple quantum pendulum in phase space, calculating the Wigner function relative to this system.

The presentation is organized in the following way. In section 2, we dene the Galilei algebra on Hilbert space $H(\Gamma)$ over a phase space with its natural symplectic structure and present the quantum mechanics in phase space.. $H(\Gamma)$ will turn out to be the space of representation of the Galileu group. In the section 3, we present a representation for the Schröinger equation in phase space is derived and the lagrangian density is written. In the section 4 results for the simple pendulum are presented. Finally, some closing comments are given in Section 5.

The Galilei Group In H (Γ) and Quantium Mechanics in Phase Space

In this section, we present the Galilei-Lie algebra based in staroperators dened in phase space. Using this algebra, we constuct a representation of Galilei group and realize the symplectic Schrödinger equation.

A. Hilbert Space H(Γ)

In this way, the Hilbert space H (Γ) associated to phase space Γ is introduced by considering the set of square integrable functions, $ø(q, p)$ in Γ, such that

$$\int dpdq\phi^*(q,p)\phi(q,p) < \infty. \tag{2}$$

Then we can write $\phi(q,p) = \langle q,p|\phi\rangle$, with

$$\int dpdq|q,p\rangle\langle q,p| = 1, \tag{3}$$

To be ⟨ø | the dual vector of |ø ⟩.

B. Symplectic Galilei-Lie Algebra

The Galilei Lie algebra in phase space can be constructed by using the Weyl operators given by $f\,\star$, where

$$f(q,p)\star g(q,p) = f(q,p)exp[\frac{i\hbar}{2}(\overleftarrow{\partial_q}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_q})]g(q,p).$$

The study of Galilei group in phase space is based in the following operators,

$$\widehat{Q} = q\star = q + \frac{i\hbar}{2}\partial_p, \quad (4)$$

$$\widehat{P} = p\star = p - \frac{i\hbar}{2}\partial_q, \quad (5)$$

$$\widehat{K} = k_i\star = mq_i\star - tp_i\star = m\widehat{Q_i} - t\widehat{P}_i, \quad (6)$$

$$\widehat{L_i} = \epsilon_{ijk}\widehat{Q_j}\widehat{P_k}, \quad (7)$$

$$\widehat{H} = \frac{\widehat{P}^2}{2m} = \frac{1}{2m}(\widehat{P_1}^2 + \widehat{P_2}^2 + \widehat{P_3}^2), \quad (8)$$

where t and m are parameters.
We can show that these operators satisfy the Galilei-Lie algebra with central extension, in which is given for the commutation relations,

$$[\widehat{L_i}, \widehat{L_j}] = i\hbar\epsilon_{ijk}\widehat{L_k}, \quad (9)$$

$$[\widehat{L_i}, \widehat{K_j}] = i\hbar\epsilon_{ijk}\widehat{K_k}, \quad (10)$$

$$[\widehat{L_i}, \widehat{P_j}] = i\hbar\epsilon_{ijk}\widehat{P_k}, \quad (11)$$

$$[\widehat{K_i}, \widehat{K_j}] = 0, \quad (12)$$

$$[\widehat{K_i}, \widehat{P_j}] = i\hbar m\delta_{ij}\mathbf{1}, \quad (13)$$

$$[\widehat{K_i}, \widehat{H}] = i\hbar\widehat{P_i}, \quad (14)$$

$$[\widehat{P_i}, \widehat{P_j}] = 0, \quad (15)$$

$$[\widehat{P_i}, \widehat{H}] = 0, \quad (16)$$

$$[\widehat{L_i}, \widehat{H}] = 0. \quad (17)$$

For the Galilean symmetry, $\widehat{P}$, $\widehat{K}$, $\widehat{L}$ and $\widehat{H}$ are generators of translations, boost, rotations and time translations. The physical content of this representation is derived by observing that $\widehat{Q}$ and $\widehat{P}$ are transformed by the boost according to:

$$exp(-i\mathbf{v}.\frac{\widehat{K}}{\hbar})\widehat{P}_j exp(i\mathbf{v}.\frac{\widehat{K}}{\hbar}) = \widehat{P}_j + mv_j\mathbf{1}, \quad (18)$$

$$exp(-i\mathbf{v}.\frac{\widehat{K}}{\hbar})\widehat{Q}_j exp(i\mathbf{v}.\frac{\widehat{K}}{\hbar}) = \widehat{Q}_j + v_j t\mathbf{1}. \quad (19)$$

Furthermore

$$[\widehat{Q}_j, \widehat{P}_n] = i\hbar\delta_{jn}\mathbf{1}, \quad (20)$$

Therefore, $\widehat{Q}$ and $\widehat{P}$ can be taken to be the physical observables of position and momentum, respectively, with Eq.(18) and Eq.(20) describing, consistently, the way $\widehat{Q}$ and $\widehat{P}$ transform under the Galilei boost. The invariants of the Galilei algebra in this representation are given by

$$I_1 = \widehat{H} - \frac{\widehat{P}^2}{2m} \quad and \quad I_2 = \widehat{L} - \frac{1}{m}\widehat{K}\times\widehat{P}. \quad (21)$$

The invariant I_1 describes the Hamiltonian of free particle, while I_2 is associated with the spin degrees of freedom. The parameters m and t are interpreted as mass and time. In this work, we will be limited with the scalar representations i.e. $I_2 = 0$.

The operators of position and momentum can be dened also by the following relations,

$$\widehat{P} = p\star = p\mathbf{1} - \frac{i\hbar}{2}\partial_q = p\mathbf{1} + \frac{1}{2}\widetilde{P},$$

and

$$\widehat{Q} = q\star = p\mathbf{1} + \frac{i\hbar}{2}\partial_p = q\mathbf{1} + \frac{1}{2}\widetilde{Q}.$$

If the c-*number* operators are dened as

$$\overline{P} = 2p\mathbf{1} \quad e \quad \overline{Q} = 2q\mathbf{1}, \quad (21)$$

the position and momentum operators can be written as

$$\widehat{P} = \frac{1}{2}(\overline{P} + \widetilde{P}) \quad and \quad \widehat{Q} = \frac{1}{2}(\overline{Q} + \widetilde{Q}). \quad (22)$$

In accord with the boost, $\overline{Q}$ and $\overline{P}$ transform as

$$exp(-iv\frac{\widehat{K}}{\hbar})2\overline{Q}exp(iv\frac{\widehat{K}}{\hbar}) = 2\overline{Q} + vt\mathbf{1}, \quad (23)$$

and

$$.exp(-iv\frac{\widehat{K}}{\hbar})2\overline{P}exp(iv\frac{\widehat{K}}{\hbar}) = 2\overline{P} + mv\mathbf{1}. \quad (24)$$

Consequently, we nd that $\overline{Q}$ and $\overline{P}$ transform as position and momentum. However, since $[\overline{Q}, \overline{P}] = 0$, $\overline{Q}$ and $\overline{P}$ cannot be interpreted as observables, althought they can be used to construct a phase space frame in the Hilbert space. So, a set of normalized eigenvectors, $|q, p\rangle$, are dened with $\{q\}$ and $\{p\}$, being a set of eigenvalues, that satisfy

$$\overline{Q}\psi(q,p) = q\psi(q,p),$$

and

$$\overline{P}\psi(q,p) = p\psi(q,p).$$

The operators $\overline{Q}$ and $\overline{P}$, with the eigenvalues $\{q, p\}$ are coordinates of a phase space Γ.

C. Schrödinger equation in Phase Space

Consider ψ(q, p, t) in $H(\Gamma)$ as a representative quantity describing the state of a quantum system, which is a wave function but not with the content of the usual quantum mechanics state, when q and p are the eigenvalues of the ancillary operators $\overline{Q}$ and $\overline{P}$.

The time evolution of the wave function is given by

$$\psi(q,p,t) = \exp\frac{-itH}{\hbar}\star\psi(q,p,0), \tag{25}$$

From these relations the following equation is derived(using the usual form for the Hamiltonian, $H = h + V(q) = \frac{p^2}{2m} + V(q)$),

$$i\hbar\partial_t\psi(q,p,t) = (\frac{p^2}{2m} - \frac{\hbar^2}{8m}\frac{\partial^2}{\partial q^2} - \frac{i\hbar p}{2m}\frac{\partial}{\partial q})\psi(q,p,t) + V(q + \frac{i\hbar}{2}\frac{\partial}{\partial p})\psi(q,p,t); \tag{26}$$

which is the Schrödinger equation in phase space.
The lagrangian that leads to equation above is given by

$$\begin{aligned}\mathcal{L} &= \frac{i\hbar}{2}(\psi^\dagger\partial_t\psi - \psi\partial_t\psi^\dagger) + \frac{i\hbar}{4m}p(\psi^\dagger\partial_q\psi - \psi\partial_q\psi^\dagger)\\ &- \frac{p^2}{2m}\psi\psi^\dagger + V(q)\star(\psi\psi^\dagger) - \frac{\hbar^2}{8m}\partial_q\psi\partial_q\psi^\dagger.\end{aligned}$$

In this sense the normalization condition is dened by

$$\int \psi^*(q,p,t)\star\psi(q,p,t)dqdp,$$

and the expected value of an observable is given by

$$\langle A\rangle = \int \psi^*(q,p,t)\star\widehat{A}\star\psi(q,p,t)dqdp. \tag{27}$$

The association with the Wigner function is given by

$$f_w(q,p) = \psi(q,p,t)\star\psi^\dagger(q,p,t). \tag{28}$$

It is easy to show that the function $\psi(q, p, t)\star\psi^\dagger(q, p, t)$ satisfy all properties of Wigner function, such as

- $\psi(q, p, t)\star\psi^\dagger(q, p, t)$ is real;
- $\psi(q, p, t)\star\psi^\dagger(q, p, t)$ is positive dened.

The eigenvalue equation to the wave functions $\psi(q, p)$ is given by

$$H\star\psi(q,p) = E\psi(q,p), \tag{29}$$

where H is the hamiltonian. In the next section we treat an example to 29.

III Simple Pendulum in Phase Space

The hamiltonian of pendulum is given by [6]

$$\widehat{H} = \frac{\widehat{L}^2}{8ml^2} + mgl(1-\cos(2\widehat{\theta}). \tag{30}$$

If we consider small oscillations, Eq.(30) becomes

$$\widehat{H} = \frac{\widehat{L}^2}{8ml^2} + 2mgl\widehat{\theta}^2. \tag{31}$$

Using the operators

$$\widehat{\theta} = \theta + \frac{i\hbar}{2}\frac{\partial}{\partial L}, \tag{32}$$

and

$$\widehat{L} = L - \frac{i\hbar}{2}\frac{\partial}{\partial\theta}, \tag{33}$$

that satises

$$[\widehat{\theta},\widehat{L}] = i\hbar,$$

The stationary Schroedinger equation

$$\widehat{H}\psi(L,\theta) = E\psi(L,\theta), \tag{34}$$

considering $m = l = g = \hbar = 1$, turns out

$$\frac{1}{8}\left(L^2 - iL\frac{\partial}{\partial\theta} - \frac{1}{4}\frac{\partial^2}{\partial\theta^2}\right)\psi(\theta,L) + 2\left(\theta^2 + i\theta\frac{\partial}{\partial L} - \frac{1}{4}\frac{\partial^2}{\partial L^2}\right)\psi(\theta,L) = E\psi(q,p). \tag{35}$$

Taking $z = \frac{L^2}{8} + 2\theta^2$, we obtain

$$(z-E)\psi(z) - \frac{1}{4}\frac{d\psi(z)}{dz} - \frac{1}{4}z\frac{d^2\psi(z)}{dz^2} = 0. \tag{36}$$

with ansatz $\psi(z) = e^{-1/2z}f(z)$, we get

$$z\frac{d^2f(z)}{dz^2} + (1-z)\frac{df(z)}{dz} + (E-\frac{1}{2}) = 0. \tag{37}$$

Eq.(37) is the Kummer dierential equation with solution given by

$$f(z) = M(1/2 - E, 1, z),$$

where $E - \frac{1}{2} = n$, n is the positive integer (n = 0, 1, 2, 3...). In this sense, the energy levels of simple pendulum are quantized,

$$E_n = \left(n + \frac{1}{2}\right)\sqrt{\frac{g}{l}}. \qquad (38)$$

The solutions of this systems are given by

$$\psi(\theta, L) = \sqrt{\frac{l}{g}} e^{\frac{L^2}{8ml^2}+2mgl\theta^2} L_n\left(\frac{L^2}{8ml^2} + 2mgl\theta^2\right), \qquad (39)$$

where L_n represents Laguerre's polynomials. Using Eq.(27) and Eq.(39), we can calculate the expected values to angular position Θ and kinetical energy K of pendulum, founding

$$\langle\Theta\rangle = 0,$$

and

$$\langle K\rangle = \frac{1}{2}mgl,$$

showing the consistence of formalism.

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IV. Concluding Remarks

In this work was presented a brief review of derivation of the Schrdinger equation in phase space from the representations of the Galilei group, using the star product. The analysis of the simple pendulum is performed. The central point is to obtain the Wigner function without use the Liouville-von Neumman equation.

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