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Primacohedron, Riemann Hypothesis, and abc Conjecture

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Abstract

The Primacohedron provides a unifying adelic framework in which numbertheoretic, spectral, and geometric structures arise from prime-indexed resonance modes. Building on its interpretation of the non-trivial zeros of the Riemann zeta function as the spectrum of a Hilbert–Pólya–type operator, this work extends the construction to Diophantine geometry and the abc conjecture. We show that radicals and height functions naturally correspond to spectral-energy sums of prime resonances, while the abc inequality emerges as a curvature-stability condition on an underlying adelic manifold. Within this spectral–Diophantine duality, violations of RH or abc manifest as curvature singularities of a unified spectral–height geometry. We further introduce an adelic operator pair (H_{spec},H_{ht}) encoding L-function zeros and arithmetic heights simultaneously, propose a curvature-anomaly correspondence linking analytic and Diophantine pathologies, and outline a programme suggesting how a completed Primacohedron—extended to motivic L-functions and Vojta theory—could imply both RH and abc through a single geometric regularity principle. This synthesis positions the Primacohedron as a candidate framework for an arithmetic spacetime whose curvature governs both the analytic behavior of zeta and L-functions and the Diophantine behaviour of rational points, offering a geometric route toward long-standing conjectures in number theory [1-5].

Keywords: p-adic String Theory, Dedekind Zeta Function, Riemann Hypothesis, Random Matrix Theory, Hilbert–Pólya Conjecture, Emergent Geometry, Black-Hole Entropy, Holography, Complexity–Action Duality, Cosmology, Arithmetic Quantum Chaos, abc Conjecture

1. Spectral–Diophantine Duality: Primacohedron, RH, and the abc Conjecture

The Primacohedron has so far been developed as a spectral framework in which spacetime emerges from prime-indexed resonances, and the non-trivial zeros of the Riemann zeta function arise as the spectrum of an adelic Hilbert–Pólya-type operator H_ζ, in the spirit of the Hilbert–Pólya paradigm and its modern reformulations [6,7]. On the analytic side this connects to the classical explicit formula and the extensive literature on the Riemann zeta function and its zeros [1,2,8]. In this section we extend the picture to Diophantine geometry and articulate a conjectural duality between:

• *Spectral coherence*, encoded by the distribution of zeros of zeta and related L-functions (Riemann Hypothesis and its generalizations), and
• *Diophantine coherence*, encoded by height bounds and radical inequalities (the *abc* conjecture and Vojta-type statements).

We will interpret the radical rad(*abc*) as a spectral-energy sum over prime resonances, relate *abc* to curvature constraints in the adelic manifold, and describe a roadmap by which an eventual proof of RH inside the Primacohedron could, in an extended motivic setting, also imply *abc*.

1.1. The abc Conjecture as A Prime-Energy Constraint

Let $a, b, c \in \mathbb{Z}$ be non-zero, pairwise coprime integers satisfying $a + b = c$. The abc conjecture asserts that for every $\epsilon > 0$ there exists a constant $K(\epsilon)$ such that

$$|c| \;\leq\; K(\epsilon)\,\mathrm{rad}(abc)^{1+\epsilon}, \qquad \mathrm{rad}(n) := \prod_{p|n} p. \tag{1.1}$$

This conjecture, independently formulated by Masser and Oesterlé and related closely to Vojta's conjectures, has far-reaching consequences for Diophantine equations and Diophantine geometry [3,4, 9-11].

The quantity rad(*abc*) keeps track of which primes divide *a, b, c* but ignores their multiplicities. In our framework, each prime *p* defines a local resonance with frequency

$$\omega_p = \ln p, \tag{1.2}$$

so that

$$\ln(\mathrm{rad}(abc)) = \sum_{p|abc} \ln p = \sum_{p|abc} \omega_p =: E_{\mathrm{prim}}(abc), \tag{1.3}$$

is naturally interpreted as the *total prime-resonance energy* of the triple (*a, b, c*).
Equation (1.1) can therefore be rewritten as

$$\ln|c| \;\le\; (1+\epsilon)\, E_{\mathrm{prim}}(abc) + O_\epsilon(1), \tag{1.4}$$

which states that the additive amplitude ln |*c*| cannot grow faster than the prime-resonance energy budget $E_{prim}(abc)$ up to a factor (1 + ϵ) and a bounded error. In the Primacohedron, this becomes a *curvature stability* condition: no Diophantine configuration is allowed to inject more "geometric amplitude" into spacetime than is supported by the
activated prime modes.

1.2. Spectral Side: Explicit Formula and RH Revisited

On the spectral side, the Primacohedron encodes primes in the oscillatory part of the spectral density of H_ζ. The explicit formula takes the schematic form

$$\rho_{\mathrm{osc}}(t) = -\frac{1}{\pi}\sum_p \sum_{m\ge 1} \frac{\ln p}{p^{m/2}} \cos(t\, m \ln p), \tag{1.5}$$

so that primes appear as periodic orbits with action $S_{p^m} \sim m \ln p$, as in the classical explicit formulae of Riemann, Weil, and their modern developments [1,2,8]. Under the Hilbert–Pólya paradigm, *t* is an eigenvalue of H_ζ and (1.5) expresses the spectrum as an interference pattern of prime resonances, in line with the quantum-chaotic interpretations of Montgomery, Odlyzko, Berry, Keating, and Katz–Sarnak [5,12-14].

RH in this language asserts that all non-trivial zeros lie on the critical line, $\mathrm{Re}(s) = \frac{1}{2}$, which in the Primacohedron corresponds to the requirement that the spectral manifold has no curvature anomalies: the local curvature proxies extracted from spacing statistics remain finite and compatible with GUE universality [5,12,13]. Deviations from the critical line would appear as *spectral curvature singularities*, forbidden by the adelic consistency conditions that glue the local p-adic sectors into a smooth global spacetime.

Thus:
RH $\Longleftrightarrow$ *no curvature singularity in the spectral manifold associated with* H_ζ.

1.3. Diophantine Side: Heights, Radicals, And Curvature

On the Diophantine side, one typically studies *heights* rather than raw integers. For a rational point *P* on an algebraic curve, the (logarithmic) height *h*(*P*) measures arithmetic complexity, aggregating contributions from all places *v* of $\mathbb{Q}$,

$$h(P) = \sum_v \lambda_v(P), \tag{1.6}$$

where λ_v is a local height at v [15,16].

In the Primacohedron, each place v is already present as either the Archimedean sector or a *p*-adic sector. The *adelic* sum of local resonance energies

$$E_{\mathrm{adelic}}(P) = \sum_v E_v(P) \tag{1.7}$$

is then a natural arithmetic analogue of the total curvature of the spectral manifold. The radical rad(abc) is a particularly simple height-like quantity: it records precisely which primes contribute to the local energies, in line with the standard height interpretations of the abc conjecture and its relation to Vojta theory [3,4,11].

The *abc* inequality (1.4) can therefore be read as

$$\text{additive height of } (a, b, c) \lesssim \text{adelic prime-resonance energy}, \tag{1.8}$$

with a small exponent overhead. A violation of *abc* would require an additive configuration whose emergent amplitude $|c|$ is "too large" for the available prime energy—in the emergent-geometry picture, this is a *Diophantine curvature anomaly*.

1.4. Spectral–Diophantine duality diagram

The duality can be summarized qualitatively as follows. Imagine adelic Primacohedron sits at the center, encoding the operator H_ζ, zeta zeros, GUE statistics, and emergent curvature of the spectral side on the left, and radical and height data for triples (a, b, c) and more general rational points, with inequalities such as abc and Vojta's conjecture of the Diophantine side on the right [1,4,5,9-13].

Adelic coherence forbids anomalies in both directions. Spectral anomalies correspond to off-critical zeros; Diophantine anomalies correspond to height/radical configurations violating abc. The Primacohedron suggests that both kinds of anomalies are different facets of the same geometric obstruction in the adelic spectral manifold.

1.5. Towards A Joint Operator Framework for RH and *abc*

The most ambitious step is to embed both phenomena into a single adelic operator. On the spectral side, we have the Hilbert–Pólya-type operator H_ζ and its generalizations to Dedekind and automorphic L-functions [8,17]. On the Diophantine side, heights and radicals are encoded by local contributions of primes to archimedean and non-archimedean metrics [15,16].

Definition 1.1 (Spectral–height operator). *A spectral–height operator* for an arithmetic object (e.g. a curve, variety, or motive) is a pair (H_{spec}, H_{ht}) acting on a common adelic Hilbert space, where:

i. H_{spec} has spectrum related to zeros of the relevant L-function(s).
ii. H_{ht} encodes logarithmic heights and radical-like quantities as expectation values or eigenvalues.

The Primacohedron suggests identifying H_{spec} with a suitable extension of H_ζ and constructing H_{ht} as an operator whose spectral measure is supported on the prime-resonance energies $\omega_p = \ln\ _p$, with multiplicities determined by Diophantine data.

Conjecture 1.2 (Curvature anomaly correspondence). *Within the Primacohedron, offcritical zeros of L-functions and violations of abc correspond to curvature singularities of a unified spectral–height manifold. In particular, if the manifold admits a smooth adelic metric with bounded curvature, then both RH (for the relevant L-functions) and abc (for the corresponding Diophantine data) hold.*

This conjecture formalizes the idea that the Primacohedron simultaneously controls analytic and Diophantine pathologies via a single geometric regularity condition, in the spirit of Vojta's dictionary between value-distribution theory and Diophantine approximation [4,11].

1.6. A Toy Model: Radical Bounds from Spectral Constraints

To illustrate how spectral constraints might lead to radical bounds of *abc*-type, consider a simplified setting in which the prime-resonance energies ω_p obey a spectral density $\rho(\omega)$ derived from the eigenvalues of a finite-dimensional approximation $H_\zeta^{(N)}$, analogous to finite-rank random-matrix models [18,19]. Suppose that for a given triple (a, b, c), the primes dividing abc occupy a subset $\mathcal{P}(abc)$ of the spectrum with total energy

$$E_{\text{prim}}(abc) = \sum_{p \in \mathcal{P}(abc)} \omega_p.$$

Assume further that emergent geometry imposes a constraint of the form

$$\mathcal{A}(a,b,c) \;\le\; C\, E_{\mathrm{prim}}(abc), \tag{1.9}$$

where $\mathcal{A}$ is a geometric observable proportional to the effective "size" of the configuration induced by (*a, b, c*) (for example, a boundary area or a curvature-integrated measure). If we can relate $\mathcal{A}$ to the additive amplitude via

$$\mathcal{A}(a,b,c) \sim \ln|c| + O(1),$$

then (1.9) becomes a logarithmic *abc*-type inequality.

While this toy model suppresses many subtleties (heights on curves, dependence on number fields, etc.), it indicates a plausible mechanism: *geometric bounds on curvature and area, when translated into the language of prime- driven spectral energies, become Diophantine bounds on radicals and heights*, in the spirit of the height-inequality philosophy of Vojta [4,11].

1.7 Roadmap from Primacohedron to *abc*

We conclude this section with a concrete programme:

1. Complete RH for H_ζ and its generalizations. Establish the self-adjointness and spectral completeness of H_ζ and extended operators for Dedekind and automorphic L-functions, showing that all non-trivial zeros lie on their critical lines [5,8,17].

2. Construct an adelic height operator. Define H_{ht} whose local components encode logarithmic heights and radicals (e.g. via expectation values associated with local padic and archimedean metrics) [15,16].

3. Couple spectral and height operators via curvature. Introduce a unified information-geometry metric on the space of joint spectral–height distributions and derive curvature flow equations ensuring bounded curvature, inspired by ideas from information geometry and random-matrix theory [18,19].

4. Identify *abc* as a curvature bound. Show that violations of abc would force curvature singularities in the joint manifold, contradicting the existence of smooth solutions to the spectral–height flow. This would upgrade the toy inequality (1.9) into a rigorous Diophantine theorem, in the spirit of Vojta's conjectural framework [4,11].

5. Extend to Vojta's conjecture. Generalize the argument to global height inequalities on curves and higher-dimensional varieties, interpreting Vojta-type inequalities as global curvature-balance conditions on the adelic Primacohedron [4,11,15,16].

In summary, the Primacohedron suggests that RH and *abc* are not isolated conjectures but complementary projections of a single adelic regularity principle. The next section develops the motivic and Vojta-geometric aspects of this principle in more detail.

2. Motivic Extensions and Vojta Geometry

The Primacohedron has so far been developed primarily for the Riemann zeta function and its Dedekind generalizations. In order to fully capture the Diophantine complexity encoded by abc and Vojta's conjecture, we must extend the framework to *motivic* L-functions and their associated height theory. This section sketches such an extension, motivated by the Langlands programme and the theory of motives [17,20-22].

2.1. Motivic *L*-Functions in An Adelic Operator Setting

Let M be a pure motive over $\mathbb{Q}$ (or a suitable approximation, such as an algebraic variety endowed with a compatible cohomology theory). The associated motivic L-function $L(s,M)$ is expected to factor as an Euler product over primes,

$$L(s,M) = \prod_p L_p(s,M)^{-1},$$

where $L_p(s, M)$ encodes the Frobenius action on the local cohomology of M at p. The Langlands programme suggests that, in favourable cases, $L(s, M)$ should match an automorphic L-function $L(s, \pi)$ for a cuspidal automorphic representation π on a reductive group [17,21,23].

In the Primacohedron, the local Frobenius eigenvalues contribute additional primeindexed resonances atop the basic zeta-resonance $\omega_p = \ln p$. Thus, each motive M defines a refined adelic operator

$$H_M = \bigoplus_{p\le\infty} w_{p,M} H_{p,M},$$

whose spectrum is conjecturally related to the zeros of $L(s, M)$, paralleling the conjectural spectral interpretations of motivic L-functions [20,21].

2.2. Height Curvature and Vojta's Dictionary

Vojta's conjecture relates the distribution of rational points on varieties to height functions and discriminants, providing a far-reaching generalization of classical results such as the Mordell conjecture. Roughly, it asserts that certain *height inequalities*—involving canonical heights, discriminants, and local contributions—govern the structure of Diophantine sets [4,11].

In the Primacohedron, heights may be understood as *curvature* densities on the adelic manifold. For a rational point P on a variety X, we associate a spectral–height profile $\rho_P(H)$ whose moments encode:

- spectral data from H_M (zeros of $L(s, M)$), and
- height data from the local contributions of P [15,16].

The Fisher–Rao metric on the space of such profiles induces an information-geometric curvature tensor whose components correspond to second-order variations of both spectral and height quantities. Vojta-type inequalities can then be interpreted as conditions that prevent curvature from blowing up along Diophantine directions, in line with his dictionary between Nevanlinna theory and Diophantine approximation [4,11].

2.3. Towards A Motivic Primacohedron

We may summarize the desired structure as follows:

1. To each motive M (or variety X) we associate a motivic Primacohedron, an adelic spectral manifold encoding both the zeros of $L(s, M)$ and the height distribution of rational points on X [4,11,20,22].
2. The geometric data of the Primacohedron (curvature, entropy, complexity) controls both the analytic behaviour of $L(s, M)$ and the Diophantine behaviour of rational points.
3. Global regularity of the motivic Primacohedron (bounded curvature, absence of singularities) implies RH-type statements for $L(s, M)$ and Vojta-type inequalities for heights on X [3,4,11,16].

In this motivic setting, the abc conjecture appears as the simplest instance of a Vojtatype inequality for $X = \mathbb{P}^1$ minus three points, while RH appears as the simplest instance of a spectral regularity statement for the Riemann zeta function. The Primacohedron unifies these cases by viewing them as different shadows of the same adelic informationgeometry
object.

2.4. Outlook: From Number Fields to Arithmetic Spacetime

The extension to motives suggests a broader perspective: the Primacohedron should not be seen solely as a model for the physical spacetime of general relativity, but also as an *arithmetic spacetime* whose points correspond to motives and whose curvature encodes both analytic and Diophantine complexity. In this picture:

- RH and its generalizations enforce spectral regularity of arithmetic spacetime [1,5].
- abc and Vojta's conjecture enforce Diophantine regularity of the same spacetime [3,4,9-11].
- The absence of curvature anomalies in this arithmetic spacetime is the unifying principle behind both kinds of conjectures.

A complete theory of the motivic Primacohedron would thus constitute not only a spectral route to RH, but also a geometric route to abc and Vojta's conjecture, all embedded in a single adelic information-geometry framework.

A. Appendix: Radicals as Spectral–Energy Sums of Prime Resonances

In the Primacohedron, each prime p corresponds to a fundamental resonance mode with frequency

$$\omega_p = \ln p, \tag{A.1}$$

as introduced in Eq. (2) and consistent with the logarithmic weighting of primes in the explicit formula for $\zeta(s)$ [Edwards, 1974, Iwaniec and Kowalski, 2004]. For any integer n with prime factorization $n = \prod_p p^{v_p(n)}$, define its radical

$$\mathrm{rad}(n) = \prod_{p|n} p.$$

The logarithm of the radical is therefore

$$\ln(\text{rad}(n)) = \sum_{p|n} \ln p = \sum_{p|n} \omega_p, \tag{A.2}$$

which is naturally interpreted as the total spectral energy of the prime resonances activated by *n*. We formalize this by defining the *prime-resonance energy* of *n*,

$$E_{\text{prim}}(n) = \sum_{p|n} \omega_p. \tag{A.3}$$

Equation (A.2) then shows that

$$E_{\text{prim}}(n) = \ln(\text{rad}(n)). \tag{A.4}$$

For an *abc*-triple (*a, b, c*) with $a + b = c$ and $(a, b, c) = 1$, the *abc* conjecture may be written as

$$\ln |c| \; \leq \; (1+\epsilon)\, E_{\text{prim}}(abc) + O_\epsilon(1), \tag{A.5}$$

as in Eq. (1.4). In the emergent-geometry interpretation, the quantity $E_{prim}(abc)$ measures the total prime-resonance energy available to support the configuration (*a, b, c*), while ln |*c*| captures the effective geometric amplitude induced by the additive relation, in line with the height-theoretic viewpoint on abc [3,4,11,24].

A violation of *abc* would therefore correspond to a configuration whose geometric amplitude exceeds the admissible range allowed by the prime-resonance energies. In the Primacohedron, such a configuration would induce a curvature singularity in the adelic manifold. The abc conjecture can thus be viewed as the statement that *no such Diophantine curvature singularities exist*, mirroring the role of RH in forbidding spectral curvature singularities [1,5,12,13].

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