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Postulations on the Behavior Exhibited By the Circumscribing Center of a Triangle, Alongside the Perpendicular Heights

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Abstract

The whole of the postulations made in this paper simply aim at describing the positioning and occurrence of the circumscribing center of a triangle so much so that given any specifications and orientation for a particular triangle, the position could be sketched to exact precision and accurate dimensions without a single construction. With these postulations we are able to to envision clearly and describe where the circumscribing center of a triangle will be located without a single construction detail, all stemming from the fact that by the postulations we are able to study the circumscribing center's behavior with respect to the angles in the triangle given a particular orientation. Contained also in this paper are the mathematical justifications for each postulation made. A rule analogous to the sine rule is also observed but here pertains to the three 'perpendicular heights' obtainable respectively from the three vertices in the triangle, wherein the other two maybe obtained when only one is given alongside all the angles in the triangle.

1. The Nine Postulations for Perfectly Locating the Circumscribing Center of a Triangle

For a clearer depiction let us consider choosing a particular orientation for diagrammatically depicting a triangle. This may otherwise be referred to as the 'conventions' which shall be used to illustrate and prove all the propositions made.

The conventions will therefore be to:

• Take the greatest angle in the triangle to be the 'vertical angle', or 'apex' or the 'top vertex' of the triangle, and

• To take the side opposite the greatest angle as the base of the triangle.

Sequel to the above chosen conventions we may want to describe a peculiar case of isosceles triangles where the greatest angle in the triangle occurs twice, that is, the two equal angles in the triangle are the greatest angles and the third angle happens to be the smallest we may want to in this discuss refer to such triangles as **irregular isosceles,** whilst we refer to the other isosceles whose greatest angle occur once and hence the two other equal angles in the triangle become the smallest angles as the **regular isosceles.**

Having put forth the aforementioned, we may then go ahead to consider the following postulations.

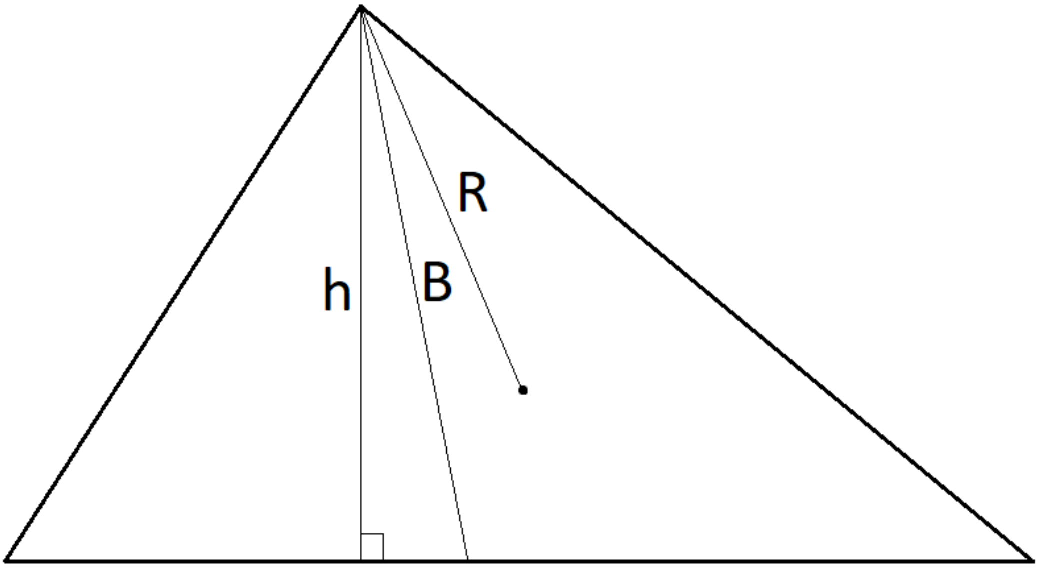
• **1st Postulation**

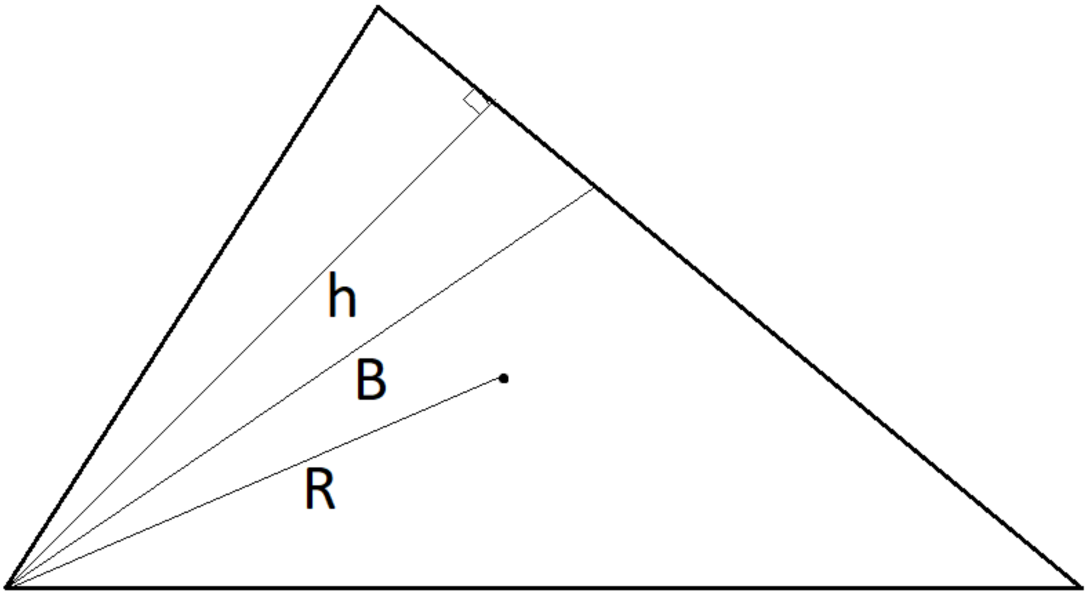
There exist for every vertex (angle) in a triangle, three projectable lines namely;

• A perpendicular height(h) relative to the side directly opposite it (or simply put a height perpendicular to the side directly opposite it)

• A line of bisection which divides it into 2 equal halves(otherwise called a bisectional height(B))

• The circumscribing radius of the triangle(R).


R
h
B


h
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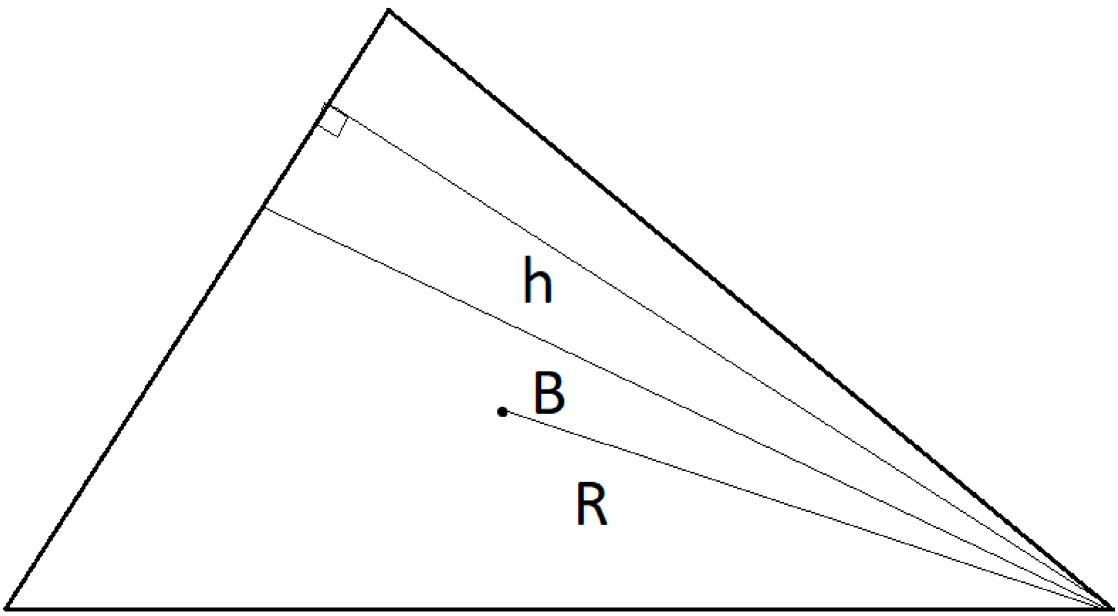

h
B
R

• 2nd Postulation

For each of the vertices of a triangle, the angle between the perpendicular height(h) and the circumscribing radius(R) is equal to the difference between the two other angles in the triangle.

• 3rd Postulation

For each of the vertices of a triangle, the Bisectional height(B) also perfectly bisects the angle between the perpendicular height(h) and the circumscribing radius(R). In other words the angle between the perpendicular height (h) and the bisectional height(B), is equal to the angle Between circumscribing radius(R) and the bisectional height.

• 4th Postulation

As a consequence of the 2nd and 3rd postulations, it may be observed that, for a vertex, the three projectable lines namely the circumscribing radius, the perpendicular height and the bisectional height may exist explicitly separate from one another provided the difference between the other two angles in the triangle is greater than zero(0). These three lines are however coincident one ontop another whenever the difference between the other two angles in the triangle is equal to 0

• 5th Postulation

For a vertex in a triangle, the circumscribing radius(R) is given by the following relationship:

R = h ÷ (cosk + cosko)

Where:

R = circumscribing radius

h = perpendicular height

K = the difference between the other two angles

k_o = the vertex(angle) being considered

In obedience to the above listed 5 statements a peculiar phenomenon is observed to occur with respect to the greatest angle present in the triangle, hence the following under listed postulations.

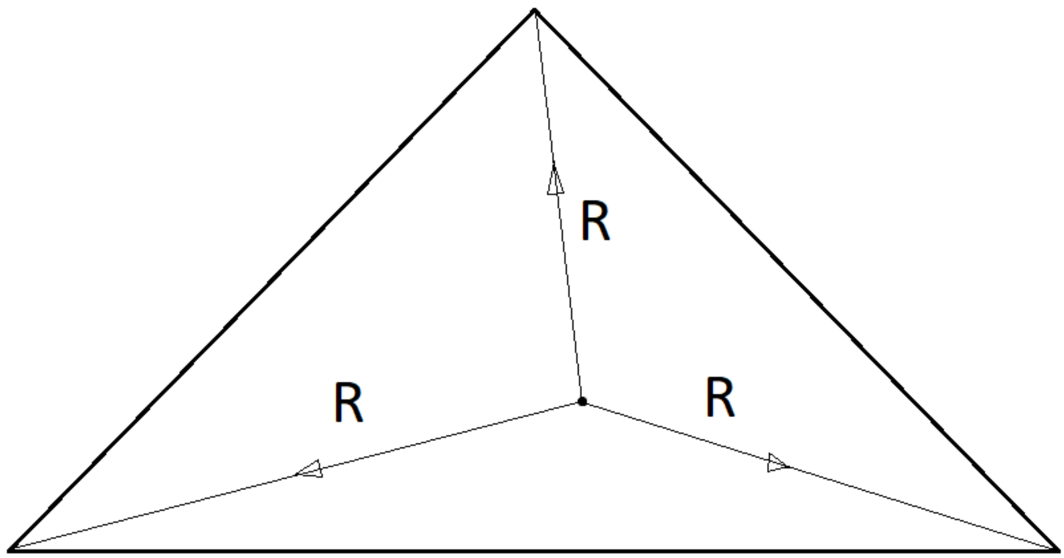
• 6th Postulation

In a triangle where there exists no angle greater than 90, the circumscribing center will always be resident inside of the triangle

The above statement can simply otherwise be stated thus:

2. When the Greatest Angle in the Triangle Is Less Than 90, the Circumscribing Center Exists Above The Base of the Triangle.

(Base here is as defined by the conventions suggested in the beginning)

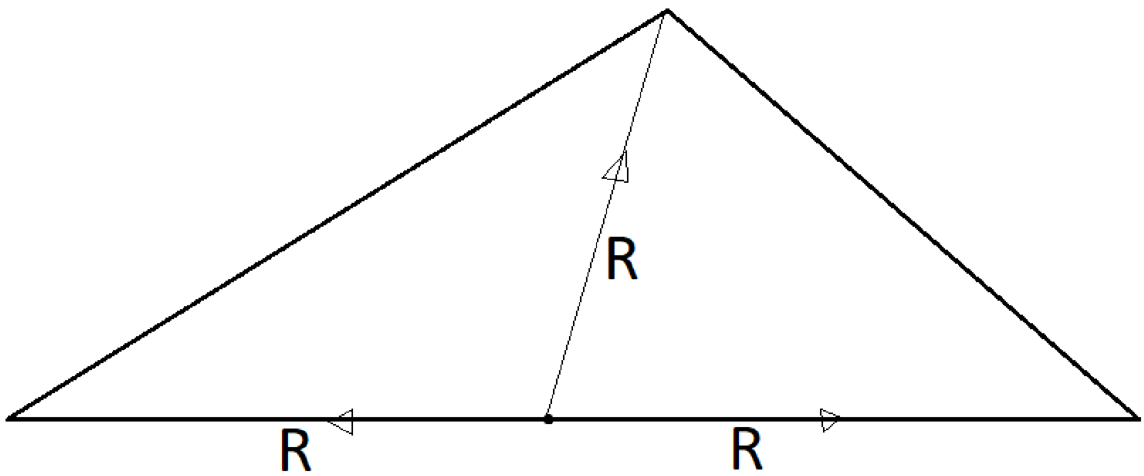


• 7th Postulation

When there exist the angle 90 inside of the triangle, the circumscribing center will occur exactly on the side opposite it.

The above statement can simply otherwise be stated thus:

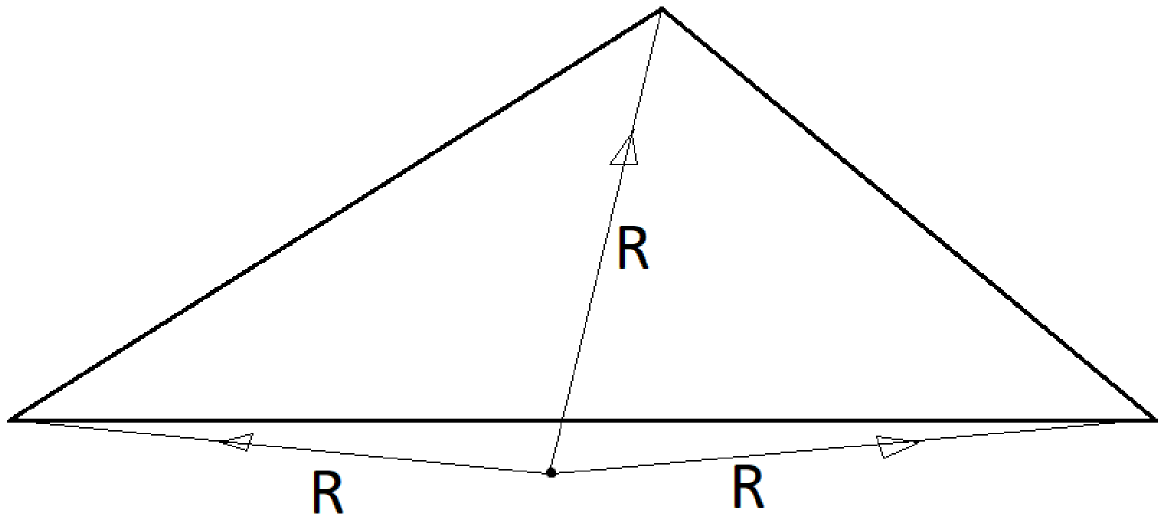
3. When the Greatest Angle in the Triangle Is Equal to 90, the Circumscribing Center Occurs Directly on the Base of the Triangle



• 8th Postulation

In a triangle where there exists an angle greater than 90, the circumscribing center will always occur outside of the triangle. The above statement can simply otherwise be stated thus:

4. When the Greatest Angle in the Triangle Is Greater Than 90, the Circumscribing Center Exists Below the Base of the Triangle



A final postulation exist so as to describe the positioning of the circumscribing center to the left or to the right of the triangle namely the:

• 9th Postulation

Whenever the three projectable lines are not coincident (that is k is not equal to 0), the circumscribing center for any particular triangular orientation always tends towards the side of the smaller of the two other angles in the triangle.

See proof of second and third postulations for images

5. Mathematical Proofs for Each of the Nine Postulations

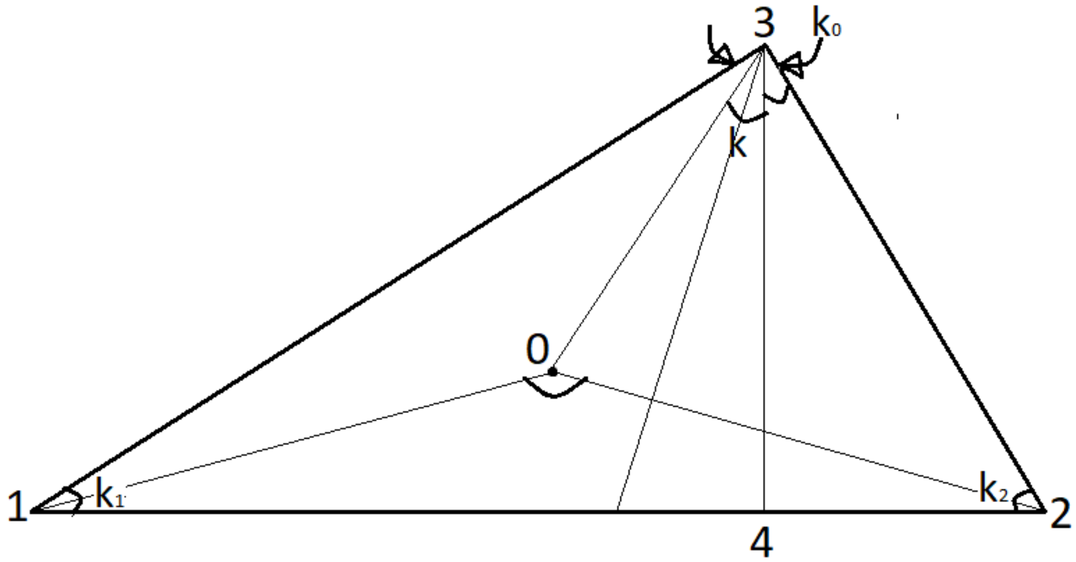
All proofs are made here Using the already stated conventions at the beginning of the discuss

• 1st Postulation

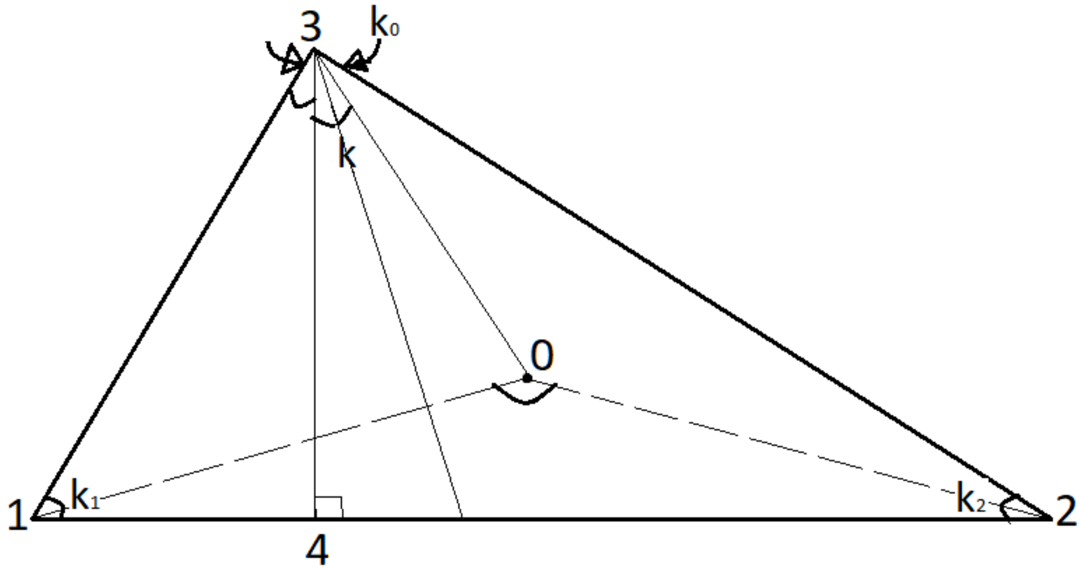
The first postulation really has no mathematical proof, however they can be easily verified by construction(see original statement of the first postulation for images)

• 2nd and 9th Postulations

Consider the triangle below



<302 = 2k_1 {<s at center and circumference}
<032 = <023 = (180 - <302)/2 {<s in an isosceles Δ}
=> <032 = (180 -2k_1)/2 = 90 - k_1 ---------------1
But <032 = <034 + <432,
But <432 = 90 - k_2 {<s in a right angled Δ}
=> <032 = k + 90 - k_2 ---------------2
Equating 1 and 2;
90 - k_1 = k + 90 - k_2
=> K = 90 - 90 - k_1 + k_2
=> K = k_2 - k_1 therefore $k_1 < k_2$, for all k element of +ve real numbers(try to use math symbols to rep these)
For a similar orientation where the circumscribing center is to the right of the perpendicular height rather than to the left of the perpendicular height as was the case of the above scenario, the formula would be given as:
K = k_1 - k_2 following the same mathematical arguments thus;

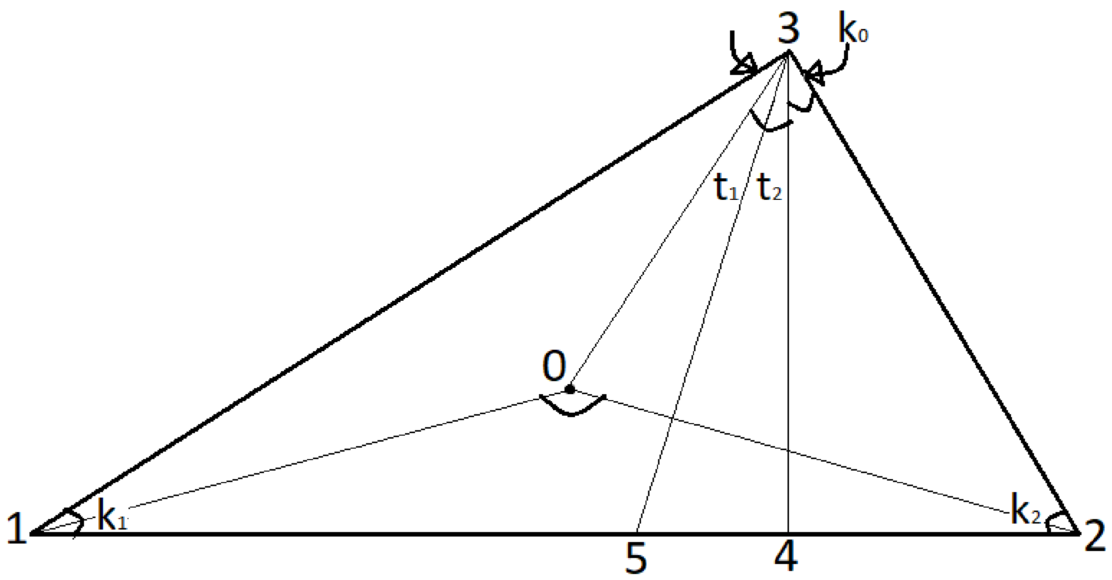


<301 = 2k_2 {<s at center and circumference}
<130 = <013 = (180 - <301)/2 {<s in an isosceles Δ}
=> <130 = (180 - 2k_2)/2 = 90 - k_2 ---------------1
But <130 = <134 + <430,
But <134 = 90 - k_1 {<s in a right angled Δ}
=> <130 = k + 90 - k_1 ---------------2
Equating 1 and 2;
90 - k_1 = k + 90 - k_2

$=> K = 90 - 90 - k_2 + k_1$
$=> K = k_1 - k_2$
This proof shows that k always being a positive value will always be given by the subtraction of the smaller value from the bigger value having being consistent in the use of signs and every other mathematical conventions during the proofs.

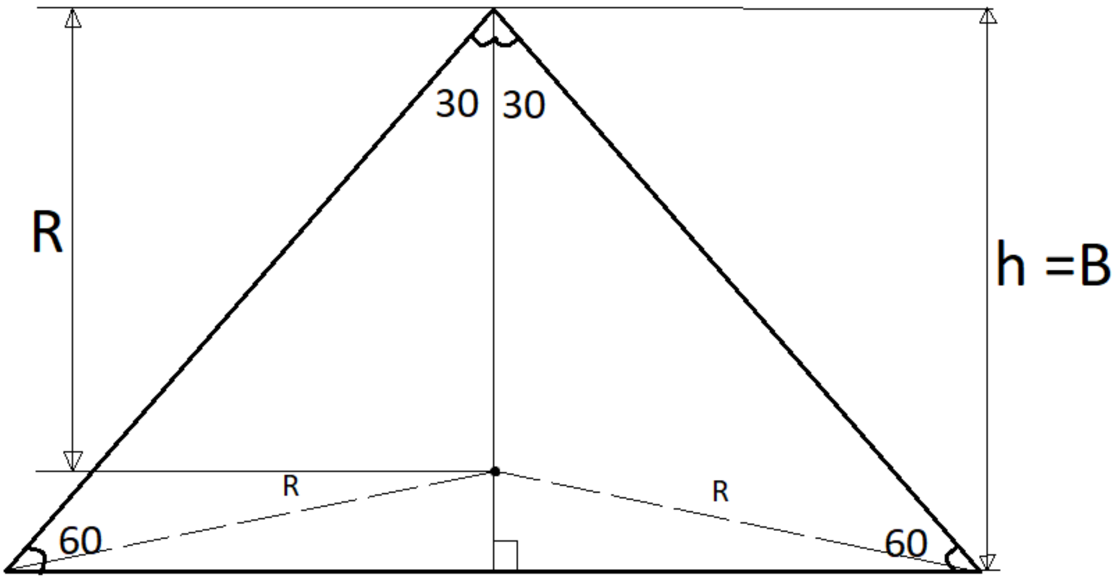
Thus proving that k_2 is the bigger value and hence the greater angle in the first triangular orientation while k_1 is the smaller value and hence the smaller angle. And that k_2 is the smaller value and hence the smaller angle in the second orientation, while k_1 is the greater of the two angles. Hence showing and thus proving the ninth postulation that states that the circumscribing center would always tend towards the smaller of the two other angles namely k_1 and k_2.
As one can clearly observe that for the first orientation where k_1 is smaller the circumscribing center is towards it and hence to the left of the perpendicular height. Similarly, in the second orientation, where k_2 is smaller the circumscribing center is towards it and hence to the right of the perpendicular height.

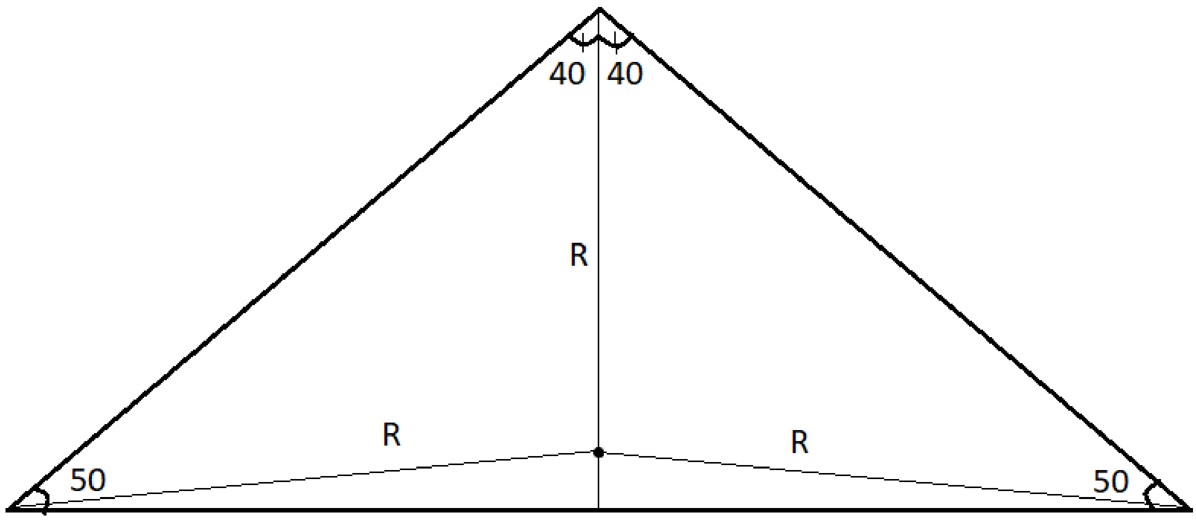
• 3rd Postulation
Let us assume any of the orientations above illustrated, say the first;

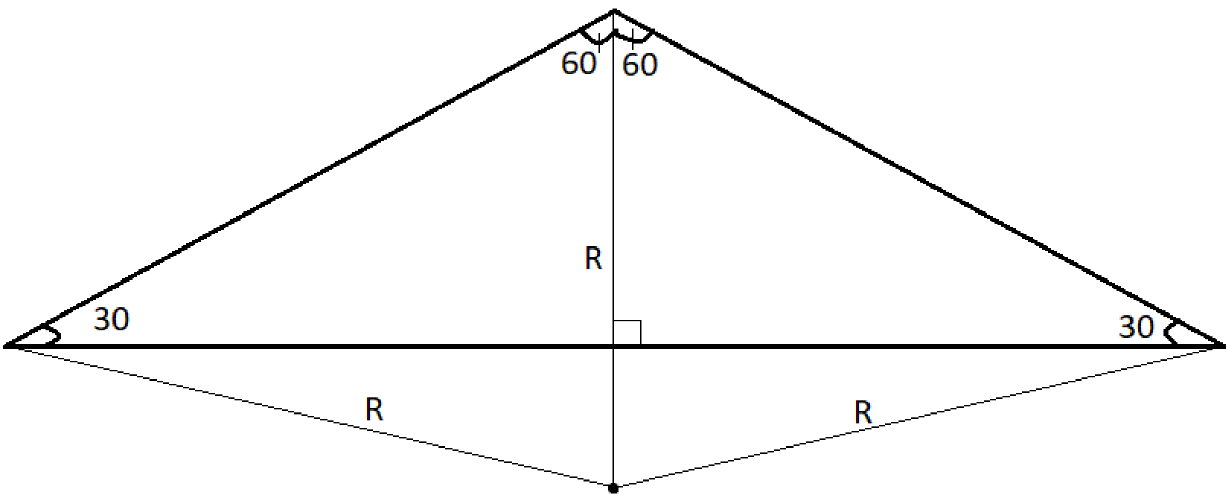


Where :
1. line 3-5 (35) represents the bisectional height such that <135 = <532
2. $t_1 + t_2 = k$
Thus:
<532 = <534 + <432
But <432 = $(90 - k_2)$ {<s in a right angled Δ}
=> <532 = $t_2 + (90 - k_2) = 90 + t_2 - k_2$
But <130 = < 013 = $(180 - 2k_2)$ {<s at center and circumference and <s in an isoceles Δ}
=> <130 = $90 - k_2$
=> <135 = $t_1 + 90 - k_2$
But <135 = <532
therefore: $t_1 + 90 - k_2 = 90 + t_2 - k_2$
=> $t_1 = t_2$

• 4th Postulation
The 4th postulation need not a mathematical proof save one by construction so as to verify the claim as one can easily see that when k = 0, there is no angle between R and h, and hence they are coincident. This is mostly the case for an isosceles and particularly always the case for an equilateral triangle where it may be observed thus.

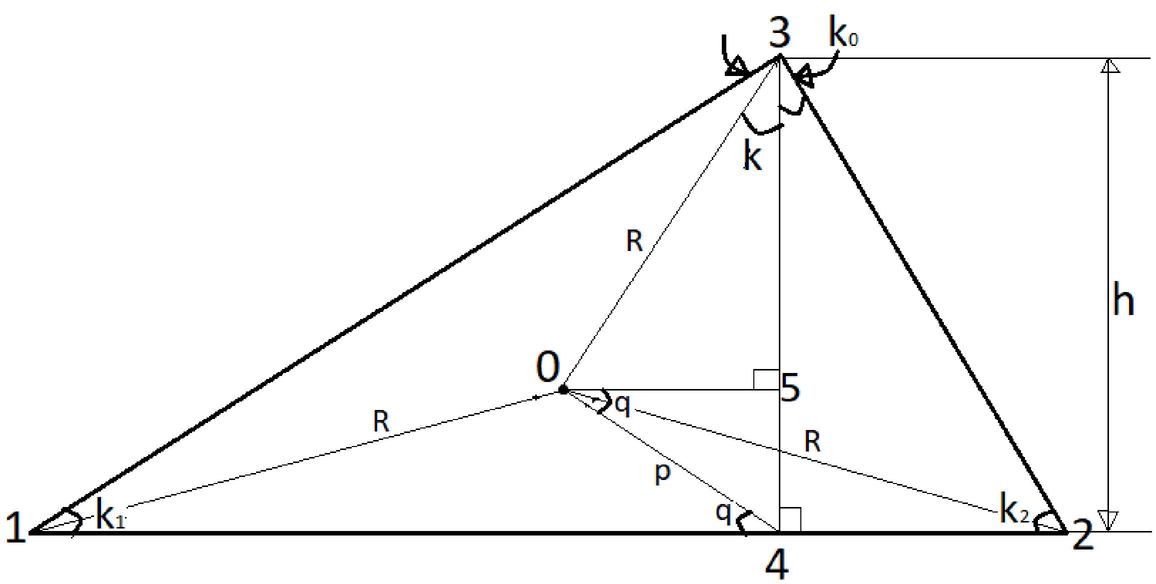






• 5th Postulation [R = h ÷ (cosk + cosko)]

Consider the following orientation



Where line 04 = P
<012 = <021 = (180 - 2k_o)/2 {<s at center and circumference and <s in an isosceles Δ}
Therefore: <012 = <021 = 90 - k_o
From Δ104, we may observe that
R/sin<041 = P/sin <012 {sine rule}
But <041 = q = < 405 {alt <s}
Therefore: R/sinq = P/sin(90 - k_o) ---------1
From Δ403:
R/sin<043 = P/sin k {sine rule}
But <043 = 90 - q {<s on a straight line}
Therefore: R/sin(90 - q) = P/sink ---------2
From 1; P = Rsin(90 - k_o)/sinq --------3
From 2; P = Rsink/sin(90 - q) --------4
Combining 3 and 4 ;
P = Rsin(90 - k_o)/sinq = Rsink/sin(90 - q)
Dividing through by R;
sin(90 - k_o)/sinq = sink/sin(90 - q)
But sin (90 - k_o) = cosko {trigonometric identities}
sin (90 - q) = cosq {trigonometric identities}
=> cosko / sinq = sink /cosq
=> tanq = cosko / sink -----------5
Reconsidering Δ403 again:
R/sin<043 = h/sin<403
But <403 = < 405 + <503,
<405 = q {alt <s} ,
<503 = 90 - k {<s in a right angled Δ}
Recall <043 = 90 - q
=> R/sin(90 - q) = h/sin(q + 90 - k)
But sin(q + 90 - k) = sinq•cos(90-k) + cosq•sin(90- k) {trigonometric identities}
=> R/cosq = h/[sinq•cos(90-k) + cosq•sin(90- k)]
But cos(90 - k) = sink, and
Sin(90 - k) = cosk
=> R/cosq = h/[sinq•sink + cosq•cosk]
Multipling through by cosq;
R = h•cosq/[sinq•sink + cosq•cosk] = h/[(sinq•sink)/cosq + (cosq•cosk)/cosq]
=> R = h/[tanq•sink + cosk]

But from 5: tanq = cosko / sink
=> R = h/[(cosk_o / sink)•sink + cosk] = h/[cosko + cosk]
Therefore; R = h ÷ (cosk + cosk_o) QED
From the above, one can quickly see how in an equilateral triangle, regular isosceles, and an irregular isosceles whose smallest angle is taken to be the vertical angle, the circumscribing radius R is just a fraction of the perpendicular height because in such a case the formula reduces to
R = h/(cosk + 1), since the two other two angles are equal and hence their difference is 0 thus;
K = 0 and cos0 = 1
Hence for an equilateral triangle, R is always given by;
R = h/(cos60 + 1) = h/((1/2) + 1) = h/(3/2) = 2h/3.

• 6th, 7th and 8th Postulations

From the already derived formula R = h ÷ (cosk + cosko), one can easily deduce the 6th, 7th and 9th postulations for an equilateral, regular and an irregular isosceles whose smallest angle is taken as the vertical angle, bearing in mind that for these classes of triangles, the aforementioned formula reduces to this:
R = h/(cosk + 1)
Hence we may observe the following
When k_o < 90,
R becomes ; R = h/(c + 1)
Where c is a positive value greater than one(i.e c > 1)such that when c + 1 divides h, the resulting value is definitely less than h and thus R becomes just a portion of h resident within the triangle. Hence the 6th postulation
(See original statement of 6th, 7th and 8th postulations for examples)

When k_o = 90,
R becomes: R = h/(cos90 + 1) = h/(0 + 1) = h
Therefore R = h, and h terminates at the base of the triangle, hence the circumscribing center is found to occur at the base.
(See original statement of 6th, 7th and 8th postulations for examples)

When k_o > 90,

R becomes ; R = h/((-c) + 1) = h/(-c + 1)
Where (-c) is a negative value less than one because for all k > 90, cosk = -cos(180 - k).
(-c + 1) then gives us a value less than one such that when it divides h, the resultant value becomes greater than h , hence R becomes greater than h and lies outside the triangle extending downwards beyond the base of the triangle, since h terminates at the base.

(See original statement of 6th, 7th and 8th postulations for examples)

So we observe easily that for triangles where k = 0, R exists as just greater fractions or lesser fractions of h. Depending on the value of ko, which in this case is the greatest angle, save for an irregular isosceles.
However it maybe well to note that for an irregular isosceles the smallest angle in the triangle suffices to show the 6th postulation very well because all angles in an irregular isosceles must be less than 90 and thus must always exhibit the 6th postulation no matter which vertex is used. However using the point of smallest angle makes it a lot easier to the describe it using the above analogy which is quite clear and simple.

We only easily see the 6th, 7th and 8th postulation materialize in this form for the above mentioned category of triangles(i.e the category for which k = 0), however the 6th, 7th and 8th postulation holds for all types of orientation and hence we may give a general proof which is suitable for all orientations and types of triangles.

6. The General Proof

Consider the following triangles exhibiting the 6th, 7th and 8th postulations, still using the initially stated conventions;

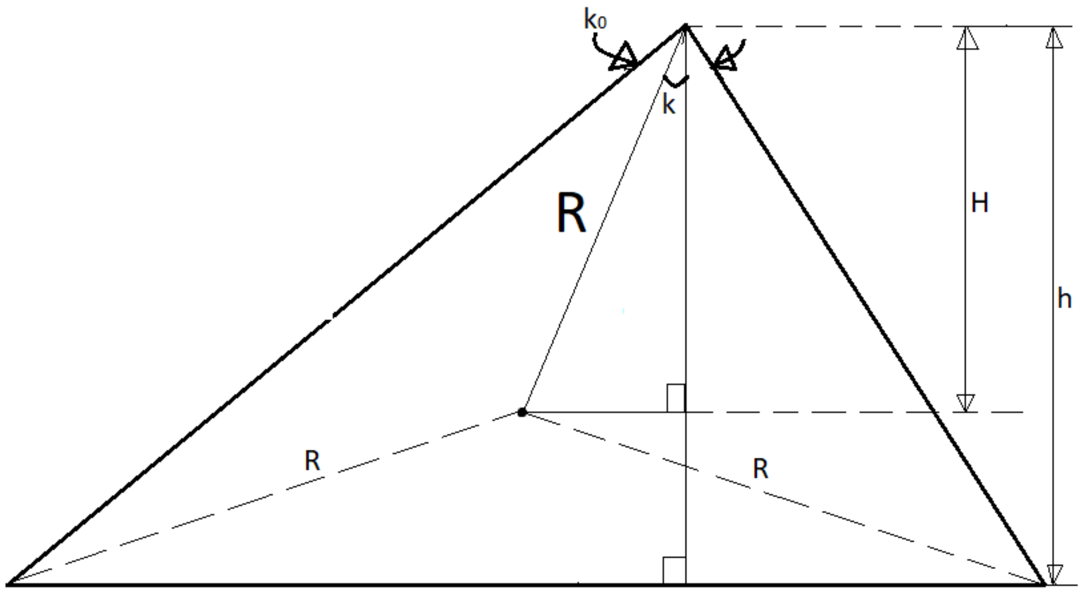


Figure 1. For $k_o < 90$ (6th postulation)

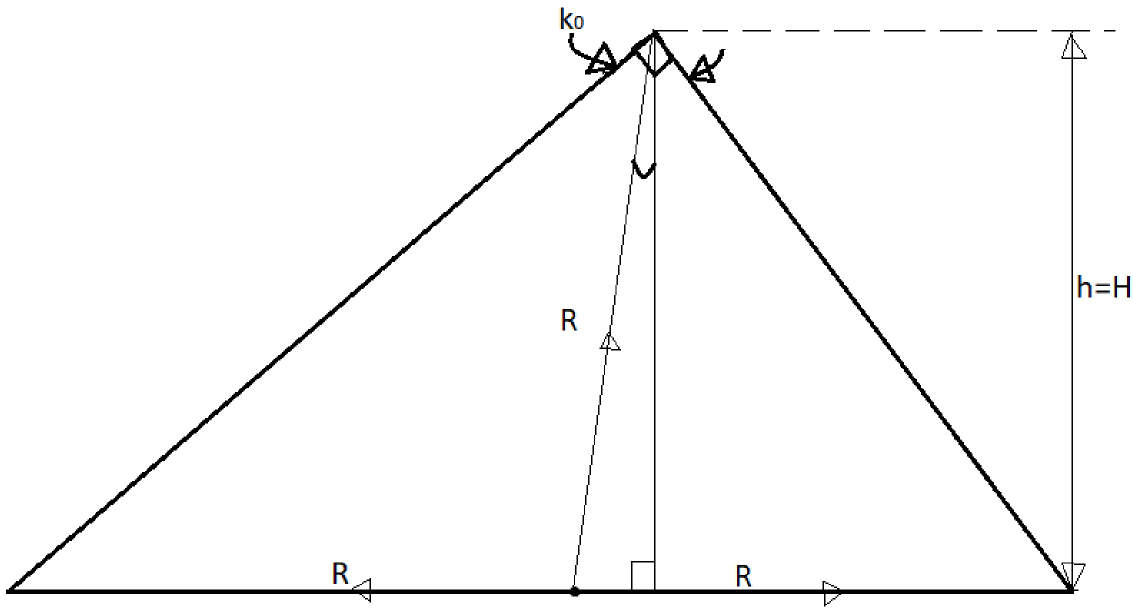


Figure 2. For $k_o = 90$ (7th postulation)

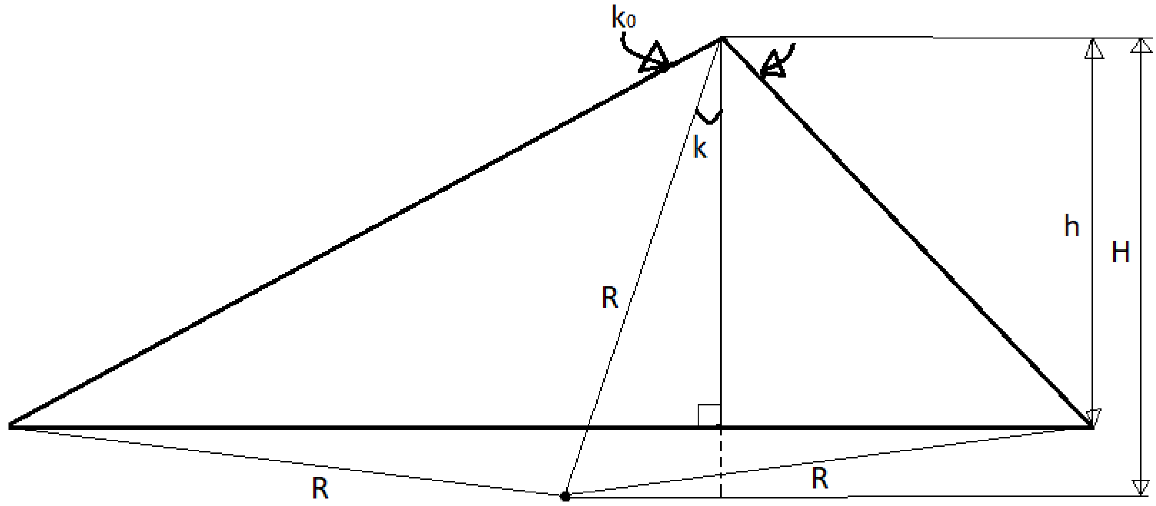


Figure 3. For $k_o > 90$ (8th postulation)

We define 'H' as the projection of R on h or the vertical component of R(the former is more appropriate)
From the above three illustrative triangles, one can clearly see by reason, by construction and by obvious observation that;
• When the circumscribing radius(R) of the triangle Is inside of the triangle, the projection of R on the perpendicular height(h), which is H, is less than h (I.e $H < h$) fig 1.
• When R is at the base, the projection of R on h becomes equal to h.(i.e $H = h$) fig 2.
• When R is outside of the triangle (i.e below the base), the projection of R on h becomes greater than h. (I.e $H > h$) fig 3.

Hence our proof reduces to showing that;
• When $H < h$, $k_o < 90$, (6th postulation)
• When $H = h$, $k_o = 90$ (7th postulation)
• When $H > h$, $k_o > 90$ (8th postulation)

Thus for the 6th postulation;
Given that:
$R = h/[\cos k_o + \cos k]$ {proven}
And $H = R\cos k$ {soh can toa}
=> $H = R\cos k = h\bullet\cos k/[\cos k_o + \cos k]$
Therefore if $H < h$
=> $h\cos k/[\cos k_o + \cos k] < h$
Dividing through by h
$\cos k/[\cos k_o + \cos k] < 1$
Dividing the numerator and denominator by cosk
$1/[1 + \cos ko/\cos k] < 1$
=> $1< 1\bullet(1 + \cos k_o/\cos k)$
=> $1< 1 + \cos k_o/\cos k$
=> $1 - 1 < 1 + \cos k_o/\cos k - 1$
=> $0 < \cos k_o/\cos k$
=> $0 < \cos k_o$
=> $\cos ko > 0$

But if; $\cos k_o > 0$, then, $0 < k_o < 90$ and $180 < ko < 270$ {interval for which cosine is positive }
But k_o cannot range from 180 to 270 {angles in a triangle}
Therefore ; $k_o < 90$
Hence When $H < h$, $k_o < 90$ QED

For the 7th postulation
By similar arguments,
$H = R\cos k = h\bullet\cos k/[\cos k_o + \cos k]$
If $H = h$,
=> $h\cos k/[\cos k_o + \cos k] = h$
Dividing through by h
$\cos k/[\cos k_o + \cos k] = 1$
Cross multiplying
$\cos k = \cos k_o + \cos k$
=> $\cos k - \cos k = \cos k_o$
=> $0 = \cos k_o$
=> $k_o = \cos\text{-}1\ 0 = 90$
=> $k_o = 90$
Therefore when $H = h$, $k_o = 90$.

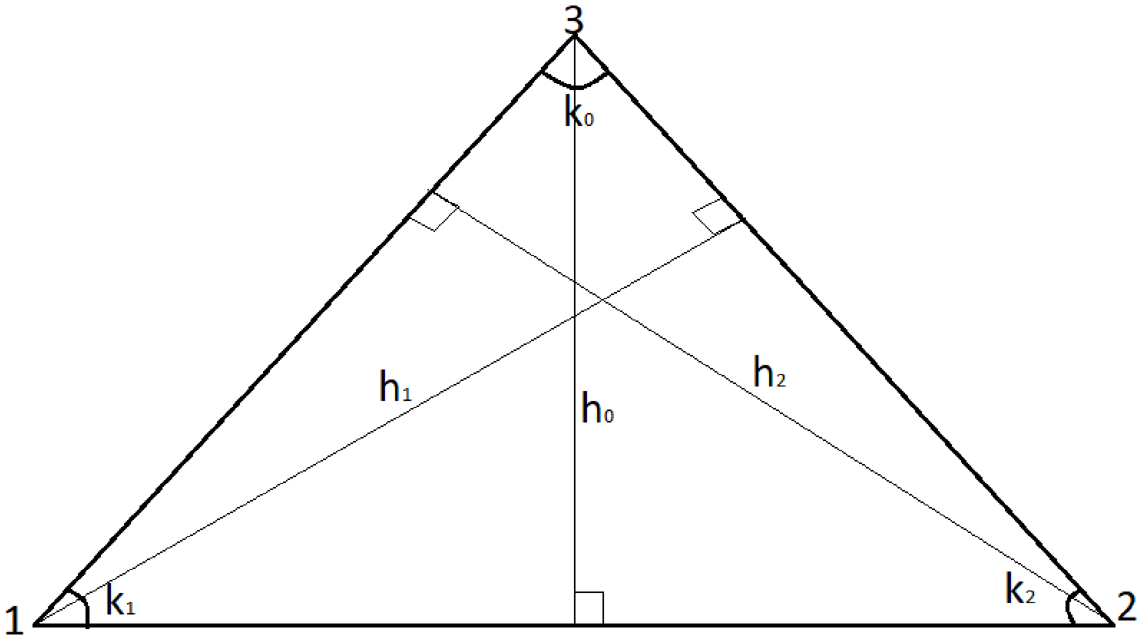
For the 8th postulation
Following the selfsame arguments ;
$R = h/[\cos k_o + \cos k]$ {proven}
And $H = R\cos k$ {soh can toa}
=> $H = R\cos k = h\bullet\cos k/[\cos k_o + \cos k]$
Therefore if $H > h$
=> $h\cos k/[\cos k_o + \cos k] > h$
Dividing through by h
$\cos k/[\cos k_o + \cos k] > 1$
Dividing the numerator and denominator by cosk
$1/[1 + \cos ko/\cos k] > 1$
=> $1> 1\bullet(1 + \cos k_o/\cos k)$
=> $1> 1 + \cos k_o/\cos k$
=> $1 - 1 > 1 + \cos k_o/\cos k - 1$
=> $0 > \cos k_o/\cos k$
=> $0 > \cos ko$
=> $\cos ko < 0$
But if ; $\cos k_o < 0$, then, $90 < k_o < 180$ and $270 < k_o < 360$ {interval for which cosine is negative }
But ko cannot range from 270 to 360 {angles in a triangle}

Therefore; $90 < k_o < 180$
Hence When $H > h$, $k_o > 90$ QED

Having achieved the primary purpose of this paper we may then go on to show forth an extra discuss which by it's very nature is but a consequence of the above illustrated postulations, particularly the first and fifth postulations.

There exists a rule very much analogous to the sine rule, but unlike the sine rule which relates the sides in a triangle to the angles opposite them, this rule relates the perpendicular heights projected from the three vertices in a triangle to each other, and to the angles in the triangle. We may also choose to call it a tenth postulation which can be stated thus.

• **10th Postulation**
The product of the perpendicular height projected from a vertex and the sine of the vertex angle is a constant for all the vertices in a particular triangle.



Where h_o is the perpendincular height from vertex 3,
h_2 is the perpendicular height from vertex 2 and
h_1 is the perpendicular height from vertex 1
Hence
$\mathbf{h_o sink_o = h_1 sink_1 = h_2 sink_2 = constant}$

7. Proof

We recall the already derived relation for the circumscribing radius of a triangle with respect to a given angle(vertex) as:
$R = h/[cosk_o + cosk]$
But ko = $180 - (k_1 + k_2)$ {angles in a triangle}
=> $cosk_o = cos180•cos(k_1 + k_2) + sin180•sin(k_1 + k_2)$ {trigonometric identities}
But $cos180 = -1$ and $sin180 = 0$
=> $cosk_o = -1•cos(k_1 + k_2) + 0•sin(k_1 + k_2)$
=> $cosk_o = -cos(k_1 + k_2)$
But $-cos(k_1 + k_2) = -(cosk_1•cosk_2 - sink_1•sink_2) = -cosk_1•cosk_2 + sink_1•sink_2$
Recall also that $k = k_1 - k_2$
=> $Cosk = cos(k_1 - k_2) = cosk_1•cosk_2 + sink_1•sink_2$
Therefore:
$R = h/[-cosk_1•cosk_2 + sink_1•sink_2 + cosk_1•cosk_2 + sink_1•sink_2]$
=> $\mathbf{R = h/(2sink_1 sink_2)}$
Therefore this maybe further expressed for all the vertices(angles) in the triangle as;
$R_o = h_o/(2sink_1 sink_2)$
$R_1 = h_1/(2sink_o sink_2)$
$R_2 = h_2/(2sink_o sink_1)$
But for a particular triangles $R_o = R_1 = R_2$
=> $h_o/(2sink_1 sink_2) = h_1/(2sink_o sink_2) = h_2/(2sink_o sink_1)$
Multiplying through by 2
$h_o/sink_1 sink_2 = h_1/sink_o sink_2 = h_2/sink_o sink_1$
Considering: $h_o/sink_1 sink_2 = h_1/sink_o sink_2$
=> $h_o sink_o sink_2 = h_1 sink_1 sink_2$
Dividing through by $sink_2$
$h_o sink_o = h_1 sink_1$
In similitude;
$h_o/sink_1 sink_2 = h_2/sink_o sink_1$
=> $h_o sink_o sink_1 = h_2 sink_1 sink_2$
Dividing through by sink1
$h_o sink_o = h_2 sink_2$
Therefore
$h_1 sink_1 = h_o sink_o = h_2 sink_2$
Hence;
$\mathbf{h_o sink_o = h_1 sink_1 = h_2 sink_2 = constant}$