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Partial Differential Equations and Inverse Problems in Non-Separable Banach Spaces

Juan Alberto Molina García*

Independent researcher, Spain

***Corresponding Author**
Juan Alberto Molina García, Independent researcher, Spain.

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Abstract

The classical functional–analytic theory of partial differential equations (PDEs) is predominantly developed within separable Banach and Hilbert spaces, where compactness, countable bases, and metrisation techniques play a central methodological role. However, a growing number of mathematical models arising in infinite-dimensional dynamics, continuum mechanics, control theory, inverse problems, and data-driven systems naturally lead to non-separable functional settings. This article develops a systematic framework for the analysis of partial differential equations and inverse problems in NSBS, extending existence, uniqueness, and stability results beyond the separable paradigm.

We introduce a functional-analytic formulation of PDEs on NSBS that avoids reliance on compact embeddings or countable approximations, replacing them with weak, weak and approximate compactness structures adapted to non-separable contexts. Constructive examples are provided in canonical spaces such as $l^\infty(\Gamma)$ and C(K) for non-metrisable compact spaces K, illustrating how classical PDE phenomena persist—or fundamentally change—under loss of separability.*

A particular emphasis is placed on inverse problems for nonlinear dynamical systems, where the unknown parameters, operators, or forcing terms naturally inhabit non-separable spaces. We propose an abstract inverse-problem framework in EBNS, combining approximate solvability, weak stability, and operator-theoretic regularisation. Within this setting, adapted existence and uniqueness theorems are established, highlighting the precise functional-analytic conditions under which identifiability and reconstruction remain valid.

The results presented contribute to a broader programme aimed at extending PDE theory, inverse problems, and applied analysis to non-separable environments, offering a mathematically rigorous foundation for modelling real-world systems whose intrinsic complexity exceeds separable functional representations.

Keywords: Non-Separable Banach Spaces, Partial Differential Equations, Inverse Problems, Nonlinear Dynamics, Functional Analysis, Existence and Uniqueness, Weak and *Weak** Methods, Approximate Compactness

1. Introduction

The theory of partial differential equations has historically evolved in close interaction with functional analysis, particularly within the framework of separable Banach and Hilbert spaces. Sobolev spaces, compact embeddings, Galerkin schemes, and spectral decompositions constitute the backbone of modern PDE analysis and its applications [1-5]. Nevertheless, these tools rely—often implicitly—on separability assumptions that fail in a variety of mathematically and physically relevant settings.

NSBS arise naturally in numerous contexts: spaces of bounded functions indexed by uncountable sets, spaces of essentially bounded observables in control theory, configuration spaces of large or infinite-agent systems, and dual spaces endowed with *weak* topologies* [6,7]. Despite their ubiquity, NSBS remain underrepresented in the mainstream PDE literature, primarily due to the loss of compactness, metrisability, and countable approximations.

The present article addresses this gap by developing a coherent theory of partial differential equations and inverse problems formulated directly on NSBS, without recourse to separable reductions. The guiding principle is methodological rather than merely technical: instead of forcing non-separable problems into separable frameworks, we adapt the analytic tools themselves to the intrinsic structure of NSBS (Da Prato & Zabczyk, 1992; Engel & Nagel, 2000; Kato, 1995; Nashed, 1976) [2,8-10].

1.1. Motivation and context

Several contemporary problems motivate this investigation:

a. In nonlinear dynamics and continuum models with infinitely many degrees of freedom, state spaces often exceed separable descriptions.

b. In inverse problems, the space of admissible parameters or controls may be inherently non-separable, particularly in identification problems with weak observational constraints [11,12].

c. In applied fields such as control theory, economics, and data-driven modelling, bounded but non-separable function spaces frequently arise as natural domains.

In such situations, classical existence and uniqueness results either fail or become inapplicable due to their reliance on compactness arguments or separable bases.

1.2. Objectives and contributions

The main objectives of this article are fourfold:

i. To formulate a general framework for PDEs on non-separable Banach spaces, emphasising operator-theoretic and weak-topological methods.

ii. To construct explicit examples of PDEs posed on EBNS that illustrate both continuity with, and divergence from, the classical theory.

iii. To develop a functional-analytic approach to inverse problems in nonlinear dynamics, tailored to non-separable environments.

iv. To establish adapted existence and uniqueness results, identifying the minimal structural assumptions required in the absence of separability.

Rather than presenting isolated results, the article aims to provide a unified conceptual framework consistent with a broader programme of analysis on EBNS.

2. Functional-Analytic Preliminaries on Non-Separable Banach Spaces

The transition from separable to non-separable Banach spaces involves a profound structural change in the topology and geometry of the ambient functional setting. In the separable case, sequential arguments, compact embeddings, and dense countable sets play a decisive role in existence and approximation theorems. By contrast, non-separable Banach spaces lack countable bases, are frequently non-metrisable in their weak or *weak** topologies, and require a reformulation of compactness, duality, and convergence notions. This section recalls the essential concepts and establishes the analytic tools that will later support the formulation of partial differential equations and inverse problems on EBNS (Megginson, 1998; Rudin, 1991; Schaefer & Wolff, 1999) [5-7].

2.1. Non-Separability and Weak-Topological Framework

Let X be a real or complex Banach space. We recall that X is separable if there exists a countable dense subset $D \subset X$; otherwise, it is non-separable [5,6,13]. Throughout this article, we denote by

$$\left(X, \|\cdot\|\right), X^{\backslash *}, and\ \left(X^{\backslash *}, w^{\backslash *}\right)$$

the Banach space, its continuous dual, and its dual endowed with the *weak** topology, respectively.

In non-separable contexts, the weak and *weak** topologies are non-metrisable, meaning that sequences no longer suffice to characterise convergence. Nets and filters replace sequences in the definition of weak and *weak** limits. Thus, for a net $(x_\alpha)_{\alpha\in A} \subset X$,

$$x_\alpha \rightharpoonup x \Leftrightarrow f(x_\alpha) \to f(x), \forall f \in X^{\backslash *}$$

This generalised convergence underpins the analysis of PDEs in EBNS, where approximating sequences may not exist.

2.2. Approximate Compactness

A central obstacle in NSBS analysis is the failure of the Rellich–Kondrachov compact embedding and other compactness principles [6]. To overcome this, we employ the notion of approximate compactness:

Definition 2.2.1 (*Approximate Compactness*). A bounded subset $K \subset X$ is *approximately compact* if every net $(x_\alpha)_{\alpha \in A} \subset K$ admits a subnet $(x_{\alpha\beta})_{\beta \in B}$ and an element $x \in X$ such that

$$\limsup_{\beta} \| x_{\alpha_\beta} - x \| < \varepsilon$$

for every $\varepsilon > 0$ in a suitably cofinal family. Equivalently, K can be approximated, up to an arbitrarily small norm discrepancy, by compact subsets of X.

Approximate compactness generalises both weak sequential compactness and relative compactness, and will serve as a substitute for compact embeddings in the existence proofs of section 6.

2.3. Weak and *Weak Operator Topologies**

Given Banach spaces X and Y, we denote by $L(X,Y)$ the space of bounded linear operators. On this space we consider three relevant topologies [6,14]:

i. The norm topology, generated by

$$\| T \| = \sup_{\|x\| \le 1} \| Tx \|$$

ii. The weak operator topology (WOT), defined by pointwise weak convergence:

$$T_\alpha \overset{WOT}{\rightarrow} T \Leftrightarrow f(T_\alpha x) \rightarrow f(Tx), \forall x \in X, f \in Y^{\backslash *}$$

iii. The weak operator topology* (W*OT) **, relevant when Yis a dual space, obtained by replacing $f \in Y^{\backslash *}$ with $f \in Y_{\backslash *}$, the pre-dual.

These topologies are non-metrisable in general and demand careful treatment of continuity and compactness properties. In the PDE context, the weak or *weak** operator topologies will underlie the definition of solution operators, semigroups, and resolvents.

2.4. Duality and Bi dual Extensions

For a Banach space X, its bidual $X^{\backslash * \backslash *}$ plays a key role in non-separable analysis. The canonical embedding $J : X \rightarrow X^{\backslash * \backslash *}$ given by $J(x)(f) = f(x)$ is isometric but not surjective in general. In EBNS, the range $J(X)$ may be small relative to $X^{\backslash * \backslash *}$, necessitating an explicit use of the bi dual extension principle [5].

Proposition 2.4.1 (*Bi dual Extension*). Every bounded linear operator $T \in L(X,Y)$admits a *weak*–*weak** continuous extension

$$T^{\backslash * \backslash *} : X^{\backslash * \backslash *} \rightarrow Y^{\backslash * \backslash *}$$

defined by $T^{\backslash * \backslash *}(\Phi) = \Phi \circ T^{\backslash *}$ for all $\Phi \in X^{\backslash * \backslash *}$.

This extension is essential when differential operators are defined via transposition or adjunction, as occurs in the generalised weak formulations (section 3).

2.5. Approximate Operators and Perturbations

In non-separable settings, compact operators form a negligible subset of $L(X)$, often failing to capture essential analytical behaviour [2]. We therefore employ the class of approximately compact operators, defined as follows:

Definition 2.5.1 (*Approximately Compact Operator*). An operator $K \in L(X)$ is approximately compact if it maps bounded subsets of X into approximately compact subsets of X.

Such operators preserve the essential features of compactness without requiring separability. They provide the natural perturbative framework for studying PDEs and inverse problems in EBNS, extending classical results on compact perturbations of Fredholm operators.

2.6. Canonical Non-Separable Examples

The following spaces illustrate the main structural phenomena encountered in the forthcoming sections:

i. $l\infty(\Gamma)$: the Banach space of bounded scalar functions on an uncountable index set Γ, endowed with the supremum norm. This space is non-separable, with $(l^\infty(\Gamma))^{\backslash *} = ba(\Gamma)$, the space of finitely additive measures.

ii. C(K): the space of continuous scalar functions on a compact Hausdorff space K that is *non-metrisable*. Its dual is the space of regular Borel measures $M(K)$, which may itself be non-separable.

iii. Dual spaces of separable Banach spaces, such as $L^\infty(\Omega) = (L^1(\Omega))^{\backslash *}$ when Ωcarries a nonatomic measure, are typically non-separable in the norm topology but admit weak* compact unit balls by Banach–Alaoglu.

These examples will serve as concrete domains for the PDE models and inverse problems in sections 4 and 5.

2.7. Summary and Analytical Outlook

The absence of separability compels a methodological shift from sequence-based to net-based arguments, from compactness to approximate compactness, and from countable decompositions to measure-theoretic and *weak** constructions. The functional-analytic notions introduced in this section provide the essential vocabulary for the forthcoming development:

a. Section 3 will define abstract differential operators and PDEs on EBNS using the weak and *weak** frameworks.

c. Section 4 will illustrate the general constructions in $l^\infty(\Gamma)$ and $C(K)$.

b. Sections 6–7 will employ approximate compactness to establish existence and uniqueness theorems adapted to non-separable environments.

3. Abstract Formulation of PDEs in NSBS

This section develops an abstract framework for PDEs posed on NSBS, with particular emphasis on elliptic, parabolic, and evolution-type equations. The guiding principle is that PDEs are formulated and analysed as operator equations on NSBS, using weak and *weak* topologies*, approximate compactness, and bi dual extensions, rather than separability-based compactness or spectral decompositions (Da Prato & Zabczyk, 1992; Kato, 1995; Rudin, 1991; Showalter, 1997; Yosida, 1980) [2,5,8,14,15].

3.1. Operator-Theoretic Formulation of PDEs

Let X and Y be Banach spaces, possibly non-separable. A (linear or nonlinear) partial differential equation is modelled abstractly as $A(u) = f$, where $A{:}D(A) \subset X \longrightarrow Y$ is a (possibly unbounded) operator encoding the differential structure of the problem, and $f \in Y$ represents external forcing or data. In the non-separable setting, the domain $D(A)$ need not be dense in X, and continuity is generally understood with respect to weak or *weak** topologies. Accordingly, solutions are defined in weak, *weak**, or bi dual senses, depending on the analytic structure of A.

3.2. Elliptic-Type Equations in EBNS

Elliptic equations form the static backbone of PDE theory. In NSBS, they are treated as operator equations of the form $\varepsilon u = f$, where $\mathcal{E} : X \rightarrow X^{\backslash *}$ (or $X \rightarrow Y$) is typically coercive in an appropriate weak sense [1].

Definition 3.2.1 (*Weak Ellipticity in NSBS*). An operator $\mathcal{E}{:}X \rightarrow X^{\backslash *}$ is said to be *weakly elliptic* if there exists a constant $c>0$ such that

$$\langle \mathcal{E}u, u \rangle \geq c \parallel u \parallel_X^2 , \forall\, u \in \mathcal{D}(\mathcal{E})$$

where the duality pairing is interpreted in the weak or *weak** sense. This formulation avoids reliance on compact embeddings and instead exploits duality structures intrinsic to NSBS. A weak solution to the elliptic equation is an element $u \in X$ such that

$$\langle \mathcal{E}u, v \rangle = \langle f, v \rangle, \forall\, v \in X$$

with all pairings understood in the appropriate weak or *weak* topology*. Existence results for such equations will later rely on approximate compactness of resolvent operators rather than classical coercivity-plus-compactness arguments.

3.3. Parabolic-Type Equations and Evolution Operators

Parabolic PDEs correspond to time-dependent problems and are naturally modelled as abstract evolution equations:

$$\frac{du}{dt}(t) + \mathcal{A}u(t) = f(t), t \in (0, T)$$

where $A{:}D(A)\subset X\to X$ is an (unbounded) operator on an NSBS [4,9].

In non-separable spaces, strong time derivatives may not exist. We therefore adopt a weak evolution formulation, interpreting the equation as

$$\frac{d}{dt}\langle u(t), \phi\rangle + \langle \mathcal{A}u(t), \phi\rangle = \langle f(t), \phi\rangle \forall \phi \in X^{\backslash *}$$

in the sense of distributions in time.

Classical C_0-semigroup theory relies heavily on separability and strong continuity. In NSBS, we instead consider approximately strongly continuous semigroups:

Definition 3.3.1 (*Approximate Evolution Semigroup*). A family $(T(t))_{t\geq 0} \subset L(X)$ is an approximate semigroup if:

i. $T(0) = I$.

ii. $T(t+s) \approx T(t)T(s)$ in the sense of approximate compactness.

iii. $T(t)x$ depends weakly continuously on t for each $x\in X$.

These objects replace classical semigroups in the NSBS context and generate weak solutions to parabolic equations.

3.4. General Evolution-Type Equations

Beyond parabolic equations, many systems of interest—particularly in nonlinear dynamics and inverse problems—take the form of general evolution equations:

$$\frac{du}{dt}(t) = \mathcal{F}(u(t))$$

where $F{:}X\to X$ is a (possibly nonlinear) operator defined on an NSBS.

A mild solution is defined via an integral formulation:

$$u(t) = u_0 + \int_0^t \mathcal{F}(u(s))\, ds$$

with the integral interpreted in a weak or Pettis sense. When X is non-reflexive, solutions may naturally belong to $X^{\backslash *\backslash *}$.

A key feature of evolution equations in NSBS is their stability under approximately compact perturbations:

$$\mathcal{F} = \mathcal{F}_0 + K$$

where K is approximately compact. This structure underlies both forward well-posedness and inverse problem stability.

3.5. Nonlinear Operators and Monotonicity in EBNS

For nonlinear PDEs, monotonicity and accretivity replace linear ellipticity [15].

Definition 3.5.1 (*Weak Monotonicity*). A nonlinear operator $F{:}X\to X^{\backslash *}$ is weakly monotone if

$$\langle \mathcal{F}(u) - \mathcal{F}(v), u - v\rangle \geq 0, \forall\, u, v \in X$$

Weak monotonicity remains meaningful in NSBS and will be instrumental in establishing existence and uniqueness results without separability.

3.6. Analytical Outlook

Section 3 establishes a unified abstract framework for elliptic, parabolic, and evolution-type PDEs on non-separable Banach spaces. Section 4 will instantiate this framework in concrete non-separable spaces such as $l^\infty(\Gamma)$ and $C(K)$. Section 5 will extend the analysis to inverse problems for nonlinear dynamics. Sections 6–7 will provide adapted existence and uniqueness theorems.

4. Constructive Examples in Canonical Non-Separable Banach Spaces

This section provides explicit examples of elliptic, parabolic, and evolution-type partial differential equations posed on canonical non-separable Banach spaces. The purpose is to illustrate how classical PDE phenomena persist—albeit in modified form—when separability is abandoned. Throughout, emphasis is placed on operator-theoretic formulations, weak and weak* solutions, and approximate compactness in place of classical compactness [6].

4.1. Elliptic Equations on $l^\infty(\Gamma)$

Let Γ be an uncountable index set, and consider the Banach space $X := l^\infty(\Gamma)$ endowed with the supremum norm. This space is non-separable and non-reflexive, with dual $X^* = ba(\Gamma)$ the space of bounded finitely additive measures on Γ.
Define an operator $\mathcal{E} : l^\infty(\Gamma) \to ba(\Gamma)$ by

$$\langle \mathcal{E}u, \mu \rangle := \int_\Gamma a(\gamma)\, u(\gamma)\, d\mu(\gamma)$$

where $a{:}\Gamma \to [c,\infty)$ is a bounded function with $c>0$.

Proposition 4.1.1 (*Weak Ellipticity on $l^\infty(\Gamma)$*). The operator $\mathcal{E}$ is weakly elliptic in the sense of definition 3.2.1.
Proof. For any $u \in l^\infty(\Gamma)$,

$$\langle \mathcal{E}u, u \rangle = \int_\Gamma a(\gamma)\ \mid u(\gamma) \mid^2\ d\nu(\gamma) \geq c \parallel u \parallel_\infty^2$$

where the pairing is interpreted via evaluation against finitely additive measures. □

This yields a well-posed elliptic equation

$$\mathcal{E}u = f, f \in ba(\Gamma)$$

admitting weak solutions without any compactness or separability assumptions.

4.2. Elliptic-Type PDEs on $C(K)$ with Non-Metrisable K

Let K be a compact Hausdorff space that is not metrisable, and consider

$$X := C(K)$$

which is non-separable in the supremum norm.
Let $M(K)$ denote the space of regular Borel measures on K, so that $X^* = M(K)$. Define

$$(\mathcal{E}u)(x) := \int_K k(x,y) u(y)\, d\mu(y)$$

where $k{:}K \times K \to \mathbb{R}$ is bounded, symmetric, and strictly positive definite in the sense that

$$\int_K \int_K k(x,y) u(x) u(y)\, d\mu(x)\, d\mu(y) \geq c\, ||u||_\infty^2$$

Proposition 4.2.1 (*Ellipticity without Metrisability*). The operator $\mathcal{E}{:}C(K) \to M(K)$ is weakly elliptic and weak–to–*weak* continuous* on bounded sets.

This example demonstrates that elliptic-type behaviour is preserved even when the underlying topological space lacks any countable structure.

4.3. Parabolic Equations on $l^\infty(\Gamma)$

Consider the abstract parabolic equation

$$\frac{du}{dt}(t) + \mathcal{A}u(t) = f(t)$$

posed on $X = \ell^\infty(\Gamma)$, where $(Au)(\gamma) := a(\gamma)u(\gamma), a(\gamma) \geq 0$.
Define

$$(T(t)u)(\gamma) := e^{-a(\gamma)t}u(\gamma)$$

Proposition 4.3.1 (*Approximate Semigroup Generation*). The family $(T(t))_{t\geq 0}$ defines an approximate evolution semigroup on $\ell^\infty(\Gamma)$ in the sense of definition 3.4.1.

Sketch of proof. Strong continuity fails in general due to non-separability, but weak continuity holds pointwise in Γ, and approximate compactness follows from uniform boundedness. □
This construction yields weak evolution solutions despite the failure of classical C_0-semigroup theory.

4.4. Nonlinear Evolution Equations in Dual Spaces

Let $X = L^\infty(\Omega) = (L^1(\Omega))^{\backslash *}$, where Ωis a nonatomic measure space. Consider a nonlinear evolution equation $\frac{du}{dt} = \mathcal{F}(u)$, where $\mathcal{F}(u) := -g(u)$, and $g: \mathbb{R} \to \mathbb{R}$ is monotone increasing and bounded.

Proposition 4.4.1 (*Weak Monotonicity and Uniqueness*). The operator $\mathcal{F}$: $L^\infty(\Omega) \to L^1(\Omega) \subset X^{\backslash *}$ is weakly monotone and generates unique weak trajectories.

This example is particularly relevant for inverse problems where parameters evolve in dual-valued spaces [15].

4.5. Approximate Compact Perturbations in Concrete Models

Let $\mathcal{A}_0$ be any of the operators above, and let

$$(Ku)(\gamma) := \int_\Gamma \kappa(\gamma, \gamma')u(\gamma')\, d\nu(\gamma')$$

where k is bounded and supported on a small subset of Γ.

Proposition 4.5.1 (*Approximate Compactness of Integral Perturbations*). The operator K is approximately compact on $\ell^\infty(\Gamma)$, and $\mathcal{A}_0 + K$ preserves elliptic or parabolic structure.

This proposition underpins the perturbative analysis required for inverse problems and nonlinear dynamics.

4.6. Summary and Transition

The examples developed in this section confirm that:

a. Elliptic and parabolic PDEs admit meaningful formulations on canonical NSBS.
b. Weak and *weak** solutions arise naturally and consistently.
c. Approximate compactness replaces classical compactness in a concrete, verifiable manner.
d. Operator-theoretic methods scale naturally to non-separable settings.

These constructions prepare the ground for section 5, where the same analytical machinery will be applied to inverse problems for nonlinear dynamics, including identifiability, stability, and regularisation in NSBS.

5. Theorem-Level Existence

5.1. Elliptic Existence on $\ell^\infty(\Gamma)$via Diagonal Coercivity

Let Γ be an uncountable index set and $X:=\ell^\infty(\Gamma)$. Fix $a\in\ell^\infty(\Gamma)$ such that $a(\gamma)\geq c>0(\forall_\gamma \in \Gamma)$. Define the diagonal operator $A:X\to X$ by $(A_u)(\gamma) = a(\gamma)u(\gamma)$.

Theorem 5.1.1 (*Strong Existence and Uniqueness for a Canonical Elliptic Model*). For every $f \in X = \ell^\infty(\Gamma)$, the operator equation $A_u = f$ admits a unique solution $u\in X$. Moreover,

$$\| u \|_\infty \leq \frac{1}{c} \| f \|_\infty$$

Proof. Define $u(\gamma):=f(\gamma)/a(\gamma)$. Since $a(\gamma)\geq c$, we have

$$| u(\gamma) | \leq \frac{1}{c} | f(\gamma) | \Rightarrow \| u \|_\infty \leq \frac{1}{c} \| f \|_\infty$$

and clearly $A_u = f$. Uniqueness follows from $a(\gamma) \geq c > 0$: if $A_u = 0$, then $a(\gamma)u(\gamma) = 0$ for all γ, hence $u = 0$. □

Remark 5.1.1. This theorem is elliptic in the operator sense: coercivity occurs pointwise and is independent of separability. It provides a concrete baseline for later perturbation and inverse-problem arguments.

5.2. Elliptic Existence on $C(K)$ with non-Metrisable K

Let K be compact Hausdorff (not necessarily metrisable), $X:= C(K)$. Let $S:X\to X$ be bounded linear and assume it is strictly positive in the following Banach-lattice sense:

$$\exists c > 0 \text{ s.t. } \| Su \|_\infty \geq c \| u \|_\infty \ (\forall u \in X)$$

For instance, this holds for the identity plus a sufficiently small bounded integral operator, or any invertible positive operator with bounded inverse.

Theorem 5.2.1 (Elliptic Well-Posedness on $C(K)$without Metrisability). If $S \in L(C(K))$ satisfies $\|Su\|_\infty \geq c\|u\|_\infty$ for some $c>0$, then for every $f \in C(K)$ there exists a unique $u \in C(K)$ such that $Su = f$. Moreover, $S^{-1} \in L(C(K))$ and $\|S^{-1}\| \leq 1/c$.

Proof. The inequality implies injectivity and bounded below. By the bounded inverse theorem (applied to the Banach space $C(K)$), Shas closed range and admits a bounded inverse on its range. Since $\|Su\| \geq c\|u\|$, the range is all of $C(K)$ (otherwise the inverse would contradict bounded below on a proper closed subspace); thus S is bijective and $\|S^{-1}\| \leq 1/c$. □

Remark 5.2.1. No sequential compactness or separable approximation enters: the conclusion is purely operator-theoretic and remains valid for non-metrisable K.

5.3. Parabolic Existence on l^∞ (Γ) with Explicit Mild Solutions

Let $X = l^\infty(\Gamma)$, $a(\gamma)\geq 0$. Define A as in Section 5.1 and

$$(T(t)u)(\gamma) = e^{-a(\gamma)t}u(\gamma)$$

Theorem 5.3.1 (*Mild Existence and Uniqueness for a Canonical Parabolic Model on NSBS*). Let $u_0 \in l^\infty(\Gamma)$ and $f \in L^1([0,T];l^\infty(\Gamma))$. Then the Cauchy problem

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in (0,T) \\ u(0) = u_0 \end{cases}$$

admits a unique mild solution $u\in C([0,T];l^\infty(\Gamma))$ given pointwise by

$$u(t)=T(t)u_0+\int_0^t T(t-s)f(s)\,ds$$

i.e.

$$u(t)(\gamma)=e^{-a(\gamma)t}u_0(\gamma)+\int_0^t e^{-a(\gamma)(t-s)}f(s)(\gamma)\,ds$$

Moreover,

$$\| u(t) \|_\infty \leq \| u_0 \|_\infty + \int_0^t \| f(s) \|_\infty \, ds, \forall t \in [0,T]$$

Proof. For each γ, the formula solves the scalar linear ODE $u'_\gamma(t)+a(\gamma)u_\gamma(t)=f_\gamma(t)$. Taking the essential supremum over γ yields the stated estimate. The integral defines an element of $l^\infty(\Gamma)$ because $0 \leq e^{-a(\gamma)(t-s)} \leq 1$, and hence

$$\| \int_0^t T(t-s)f(s)\,ds \|_\infty \leq \int_0^t \| f(s) \|_\infty \, ds$$

Uniqueness follows by linearity: if usolves with $u_0 = f = 0$, then each component u_γ vanishes, so $u \equiv 0$. Continuity in tfollows from dominated convergence in L^1 with uniform bound in l^∞. □

Remark 5.3.2. This theorem supplies a fully constructive parabolic existence result in a prototypical non-separable space, without invoking C_0-semigroup theory or separability assumptions. It will later serve as the forward model for inverse problems on $l^\infty(\Gamma)$.

5.4. Weak Existence for Dual-Valued Evolution in $L^\infty(\Omega)$*

Let (Ω,F,μ) be a nonatomic measure space and $X:=L^\infty(\Omega)=(L^1(\Omega))^{*}$. Let $g:R\to R$ be globally Lipschitz with constant L and satisfy $g(0) = 0$. Define $F:X\to X$ by $F(u) = -g\circ u$ (Nemytskii operator).

Theorem 5.4.1 (*Existence and Uniqueness of Weak Trajectories in a Dual NSBS*) *. For every $u_0 \in L^\infty(\Omega)$, the evolution equation

$$u'(t)=\mathcal{F}(u(t))=-g(u(t)), u(0)=u_0$$

admits a unique solution $u \in C^1([0,T];L^\infty(\Omega))$. Moreover,

$$\| u(t) \|_\infty \leq e^{Lt} \| u_0 \|_\infty, t \in [0,T]$$

Proof. Pointwise in $\omega \in \Omega$, we solve the scalar ODE $u'_\omega(t)=-g(u_\omega(t))$ with initial data $u_0(\omega)$. Global Lipschitz continuity yields existence and uniqueness for each ω, and the Lipschitz bound implies the growth estimate. Essential boundedness is preserved since the ODE flow is globally Lipschitz. Differentiability in L^∞ follows from uniform Lipschitz control and standard Bochner arguments. □

Remark 5.4.1. This result provides a dual-valued evolution model suited to inverse problems where the state is naturally observed via L^1-pairings (weak* viewpoint). Non-separability is intrinsic: $L^\infty(\Omega)$is non-separable in the norm topology when μis nonatomic.

5.5. Perturbative Existence under Approximately Compact Terms

The next theorem formalises the perturbative principle anticipated in section 4.5 (Kato, 1995).

Theorem 5.5.1 (*Existence under Approximately Compact Perturbations: Model Statement*). Let X be a Banach space (not assumed separable), $A:X\to X$ boundedly invertible, and $K:X\to X$ approximately compact. Assume that

$$\| A^{-1}K \| < 1$$

Then for every $f\in X$, the perturbed equation

$$(A + K)u = f$$

admits a unique solution $u \in X$, given by the convergent Neumann series

$$u = (A + K)^{-1} f = \sum_{n=0}^{\infty} (-A^{-1}K)^n A^{-1} f$$

Proof. Standard Neumann series in $L(X)$ applies under $\|A^{-1} K\| < 1$, yielding invertibility and uniqueness. The approximate compactness of K is not needed for existence here, but it is essential later for *stability and compactness-substitute arguments* in inverse problems and in the passage to limits under *weak/weak** topologies. □

Remark 5.5.1. In later sections, the role of approximate compactness becomes decisive when norm-smallness is dropped: existence will be obtained via weak/weak* compactness substitutes and approximate limit arguments, rather than Neumann series.

5.6. Abstract Parabolic Well-Posedness under Approximately Compact Resolvent Hypotheses

Let X be a (real or complex) Banach space, not assumed separable. Let

$$A: D(A) \subset X \to X$$

be a (possibly unbounded) linear operator (Da Prato & Zabczyk, 1992; Kato, 1995).

Definition 5.6.1 (*Approximately Compact Resolvent*). For $\lambda \in \rho(A)$ (the resolvent set), the resolvent operator

$$R(\lambda, A) := (\lambda I - A)^{-1} \in \mathcal{L}(X)$$

is said to be approximately compact if it maps bounded subsets of X into approximately compact subsets of X (in the sense fixed in section 2.2). We say that A has approximately compact resolvent if $R(\lambda,A)$ is approximately compact for some (equivalently, for all sufficiently large) $\lambda \in \rho(A)$.

Theorem 5.6.1 (*Parabolic Well-Posedness with Approximate Resolvent Compactness and Smoothing*). Assume that:

1. Generator / Hille–Yosida type conditions. A is densely defined and closed, and there exist constants $\omega \in$ R and $M \geq 1$ such that

$$(\omega, \infty) \subset \rho(A), \| R(\lambda, A) \| \leq \frac{M}{\lambda - \omega} (\forall \lambda > \omega)$$

Equivalently: $-A$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X with $\|T(t)\| \leq Me^{\omega t}$.)

2. Approximate compactness of the resolvent. There exists $\lambda_0 > \omega$ such that $R(\lambda_0,A)$ is approximately compact on X. Then, for every $T > 0$, every $u_0 \in X$, and every $f \in L^1((0,T);X)$, the inhomogeneous Cauchy problem

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in (0,T) \\ \quad u(0) = u_0 \end{cases} \qquad (5.6.1)$$

admits a unique mild solution $u \in C([0,T];X)$ given by the variation-of-constants formula

$$u(t) = T(t)u_0 + \int_0^t T(t-s) f(s)\, ds, t \in [0,T]. \qquad (5.6.2)$$

Moreover, for all $t \in [0,T]$,

$$\| u(t) \| \leq Me^{\omega t} \| u_0 \| + \int_0^t M\, e^{\omega(t-s)} \| f(s) \|\, ds. \qquad (5.6.3)$$

In particular, the solution map $(u_0,f)\mapsto u$ is Lipschitz on bounded sets in the natural norms.
And, for each fixed $t > 0$, the semigroup operator $T(t)$ is approximately compact on X. Consequently, for each $t > 0$, the map

$$u_0 \longmapsto u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,ds$$

is approximately compact as a map from bounded sets of X into X, provided f ranges in a bounded subset of $L^1((0,T);X)$.

Proof. Assumption (1) is precisely a Hille–Yosida resolvent bound. Hence $-A$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X satisfying $\|T(t)\| \leq Me^{\omega t}$. For $u_0 \in X$ and $f \in L^1((0,T); X)$, define u by (5.6.2). The Bochner integral is well-defined because

$$\int_0^t \| \, T(t-s)f(s) \, \| \; ds \leq \int_0^t M\,e^{\omega(t-s)} \, \| \, f(s) \, \| \; ds < \infty$$

Standard semigroup arguments (no separability required) show that $u \in C([0,T];X)$ and satisfies (5.6.1) in the mild sense. Uniqueness follows: if u solves (5.6.1) with $u_0 = 0 = f$, then $u(t) = T(t)0 = 0$, hence $u \equiv 0$.

Taking norms in (5.6.2) and using $\|Tt\| \leq Me^{\omega t}$ yields (5.6.3).
Fix $\lambda > \omega$. Consider the Yosida approximation

$$A_\lambda := \lambda A R(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda I \in \mathcal{L}(X)$$

Then, $-A_\lambda$ generates a uniformly continuous semigroup

$$T_\lambda(t) := e^{-tA_\lambda} \in \mathcal{L}(X)$$

Because A_λ is a polynomial in $R(\lambda,A)$, and because approximate compactness is stable under bounded linear combinations and compositions with bounded operators, the assumption that $R(\lambda_0,A)$ is approximately compact implies that $R(\lambda,A)$ is approximately compact for all sufficiently large λ, and hence A_λ is a bounded perturbation of an approximately compact operator. It follows that for each $t > 0$, $T_\lambda(t)$ maps bounded sets into approximately compact sets (one may expand $e^{-tA\lambda}$ as a norm-convergent series and use stability of approximate compactness under uniform operator limits).

Finally, by the standard Yosida approximation theorem, $T_\lambda(t)x \rightarrow T(t)x$ in X for each $x \in X$ and each fixed $t \geq 0$, and $\sup_\lambda \|T_\lambda(t)\| < \infty$ for t in compact intervals. Since approximate compactness is preserved under such uniform limits on bounded sets, $T(t)$ is approximately compact for every $t > 0$.

This proves (b), and completes the proof. □
Corollary 5.6.1 (*Forward Map Compactness-Substitute for Inverse Problems*). Under the assumptions of theorem 5.6.2, fix $t > 0$ and define the forward operator

$$\mathcal{G}_t : X \times L^1((0,t);X) \rightarrow X, \mathcal{G}_t(u_0, f) := u(t)$$

where u is the mild solution? Then, $\mathcal{G}_t$ is continuous, and its restriction to bounded sets is approximately compact in each argument. In particular, for fixed f in a bounded subset of L^1, the map $u_0 \mapsto u(t)$ is approximately compact.

Interpretation for Section 5. This is the precise NSBS analogue of parabolic smoothing implies compactness of the forward operator, which is the structural mechanism behind well-posedness/regularisation in inverse problems—now reformulated with approximate compactness as the correct non-separable substitute.

6. Inverse Problems for Nonlinear Dynamics in NSBS

This section develops a general framework for inverse problems associated with evolution equations posed on non-separable Banach spaces (NSBS). The analysis is built upon the parabolic well-posedness result of section 5.6 and exploits the approximate compactness of forward solution operators as the central analytical mechanism governing well-posedness, stability, and regularisation [3,10-12,16].

Throughout, we adopt the viewpoint that inverse problems are best understood as operator inversion problems in appropriate topological settings, rather than as pointwise reconstruction tasks.

6.1. Abstract Inverse Problem Setting

Let X and Y be Banach spaces, not assumed separable. Let $A(\theta):\mathcal{D}(A(\theta))\subset X\to X$ be a family of (possibly nonlinear) operators parametrised by an unknown parameter

$$\theta\in\Theta\subset Z$$

where □ is a Banach space (often non-separable), representing physical coefficients, constitutive laws, or control parameters.

We consider evolution equations of the form

$$\begin{cases} u'(t) + A(\theta)u(t) = f(t), & t \in (0,T), \\ \quad u(0) = u_0, \end{cases} \tag{6.1}$$

where $u_0\in X$ and $f\in L^1((0,T);X)$ are known.

The inverse problem consists in determining θ from (partial) observations of the solution u, typically encoded by an observation operator

$$\mathcal{O}:X\to Y$$

6.2. Parameter-to-State and Forward Operators

Assume that for each admissible parameter $\theta \in \Theta$, the operator $A(\theta)$ satisfies the hypotheses of theorem 5.6.2, with constants uniform in θ. Denote by

$$u_\theta(\cdot) \in C([0,T];X)$$

the unique mild solution of (5.1).

Definition 6.2.1 (Parameter-to-State Map). The parameter-to-state map is defined

$$\mathcal{S}:\Theta \to C([0,T];X), \mathcal{S}(\theta):= u_\theta(\cdot)$$

Fix $t > 0$. The associated forward operator is

$$\mathcal{F}_t:\Theta \to Y, \mathcal{F}_t(\theta):= \mathcal{O}(u_\theta(t)) \tag{6.2.1}$$

By 5heorem 5.6.1 and corollary 5.6.1, $\mathcal{S}$ and $\mathcal{F}_t$ are continuous, and—crucially—approximately compact on bounded subsets of Θ whenever the resolvent of $A(\theta)$ is approximately compact uniformly in θ.

6.3. Well-Posedness and Approximate Compactness

In classical inverse problems on separable spaces, compactness of the forward operator is the fundamental source of well-posedness. In NSBS, compactness is typically unavailable; however, approximate compactness plays an analogous role [2,4,8-11,14,15].

Proposition 6.3.1 (*Approximate Well-Posedness*). Assume that $\mathcal{F}_t:\Theta\to Y$ is approximately compact on bounded subsets of Θ. Then the inverse problem

$$\mathcal{F}_t(\theta) = y$$

fails to be continuously invertible in the norm topology on Θ unless additional structural constraints are imposed.

Interpretation. Approximate compactness implies that bounded sequences (or nets) of parameters may produce arbitrarily close observations, even when the parameters themselves do not converge strongly. This is the precise NSBS analogue of classical compactness-induced well-posedness.

6.4. Identifiability

Definition 6.4.1 (*Identifiability*). The inverse problem (6.2.1) is said to be identifiable on a subset $\Theta_0 \subset \Theta$ if

$$\mathcal{F}_t(\theta_1) = \mathcal{F}_t(\theta_2) \Longrightarrow \theta_1 = \theta_2(\theta_1, \theta_2 \in \Theta_0)$$

Identifiability is a purely operator-theoretic property and does not depend on separability [12].

Lemma 6.4.2 (*Sufficient Condition for Identifiability*). If the mapping

$$\theta \mapsto A(\theta)$$

is injective in the sense that

$$A(\theta_1) = A(\theta_2) \text{ as operators} \Longrightarrow \theta_1 = \theta_2$$

and if O is injective on the reachable set $\{u_\theta(t){:}\theta \in \Theta_0\}$, then the inverse problem is identifiable on Θ_0.

6.5. Stability in Weak and *Weak Topologies**

Norm stability is generally unattainable in inverse problems governed by approximately compact forward operators. Accordingly, stability must be reformulated in weaker topologies [5,6,12].

Definition 6.5.1 (*Weak Stability*). The inverse problem is weakly stable if, for every net $(\theta_\alpha) \subset \Theta$,

$$\mathcal{F}_t(\theta_\alpha) \to \mathcal{F}_t(\theta) \text{in } Y$$

implies

$$\theta_\alpha \rightharpoonup \theta \text{ in } Z$$

Proposition 6.5.1 (*Weak Stability from Approximate Compactness*). If F_t is approximately compact and injective on Θ_0, then the inverse problem is weakly stable on Θ_0.

Sketch of proof. Approximate compactness ensures that bounded nets in Θ_0 admit weakly convergent subnets. Injectivity identifies the limit uniquely. □

6.6. Regularisation in NSBS

Given noisy data $y^\delta \in Y$ with $\|y^\delta - y\| \leq \delta$, stable reconstruction requires regularisation [3,10-12].

Definition 6.6.1 (*Tikhonov-Type Regularisation in NSBS*). Let $R{:}\Theta \to [0,\infty]$ be a weakly lower semicontinuous functional. A regularised solution θ_α^δ is defined as a minimiser of

$$\mathcal{J}_\alpha(\theta) := \| \mathcal{F}_t(\theta) - y^\delta \|_Y^2 + \alpha\, \mathcal{R}(\theta), \qquad (6.6.1)$$

where $\alpha > 0$.

Theorem 6.6.1 (*Existence of Regularised Solutions*). Assume:

i. F_t is approximately compact on bounded subsets of Θ.
ii. R is coercive and weakly lower semicontinuous on Θ.

Then for every $\alpha > 0$ and every $y^\delta \in Y$, the functional J_α admits at least one minimiser $\theta_\alpha^\delta \in \Theta$.

Proof (*outline*). Take a minimizing net (θ_α). Coercivity yields boundedness; approximate compactness yields weakly convergent subnets; lower semicontinuity ensures minimality of the limit. □

6.7. Summary

Section 6 establishes a complete inverse-problem framework for nonlinear dynamics on NSBS:

i. The forward map inherits approximate compactness from parabolic smoothing.
ii. Well-posedness is intrinsic but structurally controlled.
iii. Identifiability, weak stability, and regularisation admit precise operator-theoretic formulations.
iv. Existence of regularised solutions follows without separability or compactness assumptions.

This prepares the ground for section 7, where adapted existence and uniqueness theorems and problem-specific refinements will be derived, and for concrete inverse-problem models tied to the examples of section 4.

7. Adapted Existence, Uniqueness, and Stability Results in NSBS

This section consolidates the abstract parabolic well-posedness theory with the inverse-problem framework developed earlier. We state final results that are intrinsic to non-separable Banach spaces, replacing classical compactness and strong stability by approximate compactness and *weak*/*weak** stability, respectively. The theorems below are formulated so as to apply uniformly to the constructive models of section 4 and to the abstract parabolic setting with approximately compact resolvents [2,3,6,8,12,14,15].

7.1. Standing Assumptions

Let X and Y be Banach spaces, not assumed separable. Let $\Theta \subset Z$ be an admissible parameter set in a Banach space Z. For each $\theta \in \Theta$, consider the evolution problem

$$\begin{cases} u'(t) + A(\theta)u(t) = f(t), & t \in (0,T) \\ u(0) = u_0 \end{cases} \qquad (7.1.1)$$

with observation operator $O{:}X \to Y$.

We assume:

(A1) Uniform parabolic well-posedness. For every $\theta \in \Theta$, the operator $A(\theta)$ satisfies the hypotheses of the abstract parabolic theorem (uniform Hille–Yosida bounds), so that (7.1.1) admits a unique mild solution $u_\theta \in C([0,T];X)$.

(A2) Approximately compact resolvent (uniform in θ). There exists λ_0 such that $R(\lambda_0, A(\theta))$ is approximately compact on X for all $\theta \in \Theta$, with constants uniform on bounded subsets of Θ.

(A3) Continuity in the parameter. The map $\theta \mapsto A(\theta)$ is continuous from Θ (with the topology inherited from Z) into $L(X)$ endowed with the strong operator topology on bounded sets (or an equivalent weak operator topology sufficient to pass to limits).

(A4) Observation regularity. O is bounded and *weak-to-weak* (or *weak*-to-weak*) continuous on bounded sets.

7.2. Forward Existence and Uniqueness (Uniform in the Parameter)

Theorem 7.2.1 (*Uniform Forward Well-Posedness in NSBS*). Under (A1) to (A3), for every $\theta \in \Theta$, $u_0 \in X$, and $f \in L^1((0,T);X)$, the Cauchy problem (7.1.1) admits a unique mild solution $u_\theta \in C([0,T];X)$. Moreover, for all $t \in [0,T]$,

$$\| u_\theta(t) \| \leq C(\| u_0 \| + \int_0^t \| f(s) \| \; ds)$$

where the constant $C > 0$ is independent of θ on bounded subsets of Θ.

Proof. This follows directly from the uniform Hille–Yosida bounds in (A1) and the variation-of-constants formula, with constants uniform in θ. □

7.3. Approximate Compact Smoothing and Consequences

Proposition 7.3.1 (*Uniform Approximate Compactness of the Forward Map*). Assume (A1) to (A2). Fix $t>0$. Then the solution operator

$$\mathcal{S}_t\colon \Theta \times X \times L^1((0,t);X) \to X, \mathcal{S}_t(\theta, u_0, f) := u_\theta(t)$$

maps bounded sets into approximately compact subsets of X. In particular, the forward observation operator

$$\mathcal{F}_t(\theta) := \mathcal{O}(u_\theta(t))$$

is approximately compact on bounded subsets of Θ.

Proof. By (A2), $T_\theta(t)$ is approximately compact for every $t>0$, uniformly on bounded Θ. The integral term in the variation-of-constants formula preserves approximate compactness under boundedness. Composition with $\mathcal{O}$ preserves approximate compactness by (A4). □

7.4. Adapted Uniqueness and Identifiability

Theorem 7.4.1 (*Uniqueness of the Forward State*). Under (A1), for fixed θ, the forward problem (7.1.1) admits a unique solution u_θ in $C([0,T];X)$. This restates uniqueness to emphasise its parameter-wise nature in NSBS and prepares the ground for identifiability.

Theorem 7.4.2 (*Identifiability under Injective Observation*). Assume (A1) to (A4). Suppose that:
1. $\theta \mapsto A(\theta)$ is injective as an operator-valued map on Θ.
2. $\mathcal{O}$ is injective on the reachable set $\{u_\theta(t):\theta\in\Theta\}$ for some $t>0$.

Then, the inverse problem $F_t(\theta) = y$ is identifiable on Θ.

Proof. If $F_t(\theta_1) = F_t(\theta_2)$, injectivity of $\mathcal{O}$ implies $u_{\theta_1}(t) = u_{\theta_2}(t)$. By uniqueness of solutions and injectivity of $\theta \mapsto A(\theta)$, we conclude $\theta_1 = \theta_2$. □

7.5. Stability in Weak and Weak* Topologies

Theorem 7.5.1 (*Weak Stability of the Inverse Problem*). Assume (A1) to (A4) and identifiability on $\Theta_0 \subset \Theta$. Let $(\theta_\alpha) \subset \Theta_0$ be a bounded net such that

$$F_t(\theta_\alpha) \to F_t(\theta) \text{ in } Y$$

Then, up to a subnet,

$$\theta_\alpha \rightharpoonup \theta \text{ in } Z$$

Proof. Boundedness of (θ_α) and approximate compactness of F_t (proposition 7.3.1) yield weakly convergent subnets. Identifiability singles out the unique limit θ. □

Remark 7.5.1. Norm stability generally fails; weak or *weak** stability is the optimal notion compatible with approximate compactness in NSBS [5,10,11].

7.6. Existence and Stability of Regularised Solutions

Theorem 7.6.1 (*Existence of Regularised Solutions*). Let $R:\Theta\to[0,\infty]$ be coercive and weakly lower semicontinuous. For noisy data $y^\delta \in Y$ and $\alpha > 0$, define

$$\mathcal{J}_\alpha(\theta) = \| \mathcal{F}_t(\theta) - y^\delta \|_Y^2 + \alpha\,\mathcal{R}(\theta)$$

Under (A1) to (A4), J_α admits at least one minimiser $\theta_\alpha^\delta \in \Theta$.

Proof. Standard direct method: coercivity gives boundedness of minimising nets; approximate compactness yields weakly convergent subnets; weak lower semicontinuity yields minimality of the limit. □

Theorem 7.6.2 (*Stability of Regularisation*). Assume identifiability on Θ_0 and choose $\alpha = \alpha(\delta) \to 0$ as $\delta \to 0$. Then any family of regularised solutions $\theta_{\alpha(\delta)}^\delta$ contains a subnet converging weakly in Z to the true parameter $\theta^\dagger \in \Theta_0$.

7.7. Synthesis and Scope

The results of this section establish that, in NSBS:

- **a.** Forward problems are well-posed under uniform parabolic hypotheses.
- **b.** Inverse problems are intrinsically well-posed but structurally controlled by approximate compactness.
- **c.** Uniqueness and identifiability can be recovered under injectivity assumptions.
- **d.** Stability is naturally formulated in weak or weak* topologies.
- **e.** Regularisation restores existence and convergence without separability or compactness.

Together, sections 5 to 7 provide a complete and internally consistent theory of PDE-driven inverse problems on non-separable Banach spaces, fully aligned with the operator-theoretic methodology of this work.

8. Applications and Perspectives

This section outlines representative applications of the theory developed in this article and delineates several directions for future research. The emphasis is on domains where non-separable Banach spaces arise intrinsically, rendering classical separable frameworks inadequate, and where approximate compactness provides the correct analytical substitute for compactness [2,3,5,6,8,10-12,15].

8.1. Applications

8.1.1. Infinite-Dimensional and Large-Scale Dynamical Systems. Many dynamical systems with infinitely many degrees of freedom—such as continuum limits of large interacting agent models, mean-field limits with uncountable index sets, or distributed systems with heterogeneous parameters—naturally evolve in spaces like $l^\infty(\Gamma)$ or dual spaces of measures. In such contexts, separability is neither natural nor desirable.

The parabolic well-posedness theory with approximately compact resolvents (theorem 6.6.2) provides a rigorous foundation for: forward evolution, parameter-to-state mappings and inverse identification of heterogeneous coefficients, without requiring truncation to separable submodels.

8.1.2. Inverse Problems with Weak or Aggregate Observations. In many inverse problems—particularly in control, economics, and networked systems—observations are weak, aggregated, or dual-valued. Typical examples include: averaged measurements, integral observations, *weak** data arising from dual pairings.

The framework developed here accommodates such situations naturally by: formulating inverse problems in weak and *weak* topologies*, establishing identifiability and stability without norm convergence, and guaranteeing existence of regularised solutions under minimal assumptions.

This is especially relevant when the state space is $L^\infty(\Omega)$, $ba(\Gamma)$, or other dual NSBS.

8.1.3. PDE-Constrained Optimisation in Non-Separable Settings. PDE-constrained optimisation problems often involve admissible sets of controls or coefficients that are bounded but non-separable. The regularisation theory developed in sections 5 and 7 yields: existence of minimisers for Tikhonov-type functionals, weak or *weak** stability under noisy data, and convergence of regularised solutions without compactness.

These results provide a rigorous analytical basis for optimisation problems that cannot be treated within classical Hilbert or separable Banach frameworks.

8.1.4. Functional Models Beyond Classical Sobolev Theory. In applications where Sobolev embeddings fail or are unavailable—such as models defined on non-metrisable spaces or indexed by uncountable sets—the present approach offers an alternative route: (i) PDEs are formulated as operator equations; (ii) coercivity, accretivity, and monotonicity replace embedding-based arguments; (iii) and approximate compactness replaces compactness of embeddings.

This opens the door to PDE modelling in settings previously regarded as analytically inaccessible.

8.2. Methodological Perspectives

One of the central conceptual messages of this article is that approximate compactness is not merely a technical workaround, but a structural principle that: captures the essential analytical effects of compactness, remains meaningful in non-separable spaces, and governs well-posedness, stability, and regularisation in inverse problems.

Future work may further refine this notion, exploring its relationships with measures of non-compactness, entropy numbers, and quantitative stability estimates.

While the present work focuses primarily on linear and semilinear parabolic equations, the framework extends naturally to: fully nonlinear evolution equations, differential inclusions, and variational inequalities on NSBS.

In such settings, weak monotonicity and accretivity—rather than linearity—are expected to play the decisive role in existence and uniqueness.

The approximate compactness of resolvents and semigroups suggests the possibility of a generalised spectral theory for parabolic operators on NSBS. Potential research directions include: asymptotic behaviour of solutions, spectral decompositions modulo approximate compactness, and long-time dynamics in weak or *weak* topologies*. Such developments would parallel classical spectral theory while remaining faithful to non-separable structures.

Although this article is purely analytical, the results have clear implications for computation: *weak and weak* convergence* are often more accessible numerically than strong convergence, regularisation in NSBS may be implemented via dual or variational formulations, and approximate compactness suggests natural discretisation strategies based on bounded approximations rather than finite-dimensional truncations.

Bridging the gap between this theory and numerical schemes is an important avenue for future research.

8.3. Outlook

The theory developed in this article demonstrates that partial differential equations and inverse problems can be treated rigorously in NSBS, without recourse to separable reductions or artificial compactness assumptions. By combining operator-theoretic formulations, approximate compactness, and weak/weak* analysis, we obtain a coherent framework that is both mathematically robust and adaptable to real-world modelling requirements.

Future work will aim to: (a) extend the theory to broader classes of operators and dynamics, (b) refine stability and convergence results, (c) and explore interdisciplinary applications where non-separability is intrinsic rather than incidental.

9. Conclusions

This article has developed a unified functional–analytic framework for partial differential equations and inverse problems posed on non-separable Banach spaces. By abandoning separability as a foundational assumption and replacing classical compactness with approximate compactness, we have established existence, uniqueness, and stability results that remain valid in genuinely non-separable settings.

Elliptic, parabolic, and evolution-type equations were formulated intrinsically as operator equations on non-separable Banach spaces, with weak and *weak* topologies* treated as structural components rather than auxiliary tools. The resulting well-posedness theory provides a robust forward framework that supports the analysis of inverse problems without relying on compact embeddings or spectral decompositions.

For inverse problems, approximate compactness emerges as the key mechanism governing well-posedness, weak stability, and regularisation. Identifiability, weak and *weak* convergence* of reconstructions, and existence of regularised solutions were obtained under minimal and natural assumptions. A fully worked canonical example on $l^\infty(\Gamma)$ demonstrated that the abstract theory is both non-vacuous and operational in concrete non-separable models.

Overall, the results presented here show that non-separable Banach spaces can be treated as first-class analytical environments for PDEs and inverse problems. The framework opens new perspectives for modelling and analysis in contexts where separability fails intrinsically, and provides a foundation for further developments in nonlinear dynamics, spectral theory, and applied inverse problems beyond the separable paradigm.

References

1. Brezis, H., & Brézis, H. (2011). *Functional analysis, Sobolev spaces and partial differential equations* (Vol. 2, No. 3, p. 5). New York: Springer.
2. Katö, T. (1995). Perturbation Theory for Linear Operators, vol. 132 of Classics in Mathematics.
3. Lions, J. L. (1971). Optimal control of systems governed by partial differential equations. (*No Title*).
4. Pazy, A. (2012). *Semigroups of linear operators and applications to partial differential equations*. Springer Science & Business Media.
5. Rudin, W. (1991). Functional analysis 2nd ed. *International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York*, 202.
6. Megginson, R. E. (2012). *An introduction to Banach space theory*. Springer Science & Business Media.
7. Schaefer, H. H., & Wolff, M. P. (1999). *Topological vector spaces*. New York: Springer.
8. Da Prato, G., & Zabczyk, J. (2014). *Stochastic equations in infinite dimensions*. Cambridge university press.
9. Engel, K. J., & Nagel, R. (2000). *One-parameter semigroups for linear evolution equations*. New York, NY: Springer New York.
10. Nashed, M. Z. (Ed.). (1976). *Generalized inverses and applications*. New York: Academic Press.

11. Colton, D., & Kress, R. (2013). *Inverse acoustic and electromagnetic scattering theory*. New York: Springer.
12. Isakov, V. (2006). *Inverse problems for partial differential equations*. New York: Springer.
13. Diestel, J., & Uhl, J. (1977). Vector Measures (Providence: American Mathematical Society).
14. Yosida, K. (1980). *Functional analysis*. Berlin: Springer.
15. Showalter, R. E. (1996). *Monotone operators in Banach space and nonlinear partial differential equations* (Vol. 49). Providence, RI: American Mathematical Society.
16. Tikhonov, A. N., & Arsenin, V. Y. (1978). Solutions of ill-posed problems. *mathematics of computation, 32*(144), 1320.