

Optimization of Numerical Solutions of Stochastic Differential Equations With Time Delay

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Abstract

The research focuses on the optimization of numerical solutions of neutral stochastic differential equations with time delay. Analyzing approaches such as Euler-Maruyama, backward Euler and θ -Euler-Maruyama methods, the goal is to investigate the characteristics of approximate solutions, especially stability and boundedness. This study contributes to the understanding of the complexity of stochastic processes, offering a perspective for further mathematical modeling and optimization. The study of the characteristics of approximate solutions includes a detailed analysis of their stability and limitations, providing insight into the system's behavior in dynamic conditions. This analysis lays the foundations for the improvement of numerical methods and more precise modeling of stochastic processes with a time delay. The aforementioned approaches, such as the Euler-Maruyama, backward Euler and θ -Euler-Maruyama methods, provide tools for understanding and solving complex mathematical challenges. Through an interdisciplinary approach, this study sheds light on the field of optimization of numerical solutions, encouraging further development of theoretical and practical aspects of stochastic differential equations.

Keywords: Stochastic Process, Stochastic Differential Equations, Numerical Methods, Optimization of Solutions, Time Delay.

1. Introduction

The concept of stochastic processes was introduced at the beginning of the last century when the natural sciences of engineering and physics were engaged in the study of phenomena that change over time, while the theory of probability still had a developed methodology for training such phenomena. Thus, the above was an entry card for the development of stochastic processes within which random variables that are time-dependent would be considered. As for the mathematicians who dealt with this problem, we highlight Sluckii, Wiener, Kolmogorov, Carmer, Doob and many others. The theory of stochastic processes contributed to the development of many other mathematical theories that are important for other sciences such as economics, mechanics, electrical engineering, engineering, medicine. Each of the scientists introduced the concept of stochastic process on different examples and studied it on different examples. We can single out that Kolomogorov, in the work [27] and [28], established the terms of conditional probability for the development of Markov-type stochastic processes with an infinite parameter set, while Cramer elaborated the theory on the Gaussian process [1].

2. Stochastic Processes

Stochastic processes represent a mathematical concept that describes the evolution of some random variable over time. This process includes elements of uncertainty or randomness, which distinguishes it from deterministic processes that are completely

predictable. In the context of stochastic processes, "stochastic" means that the future behavior of the system is conditioned by random or random variables. Examples of stochastic processes include stock price movements, temperature fluctuations, or changes in financial markets.

Formally, a stochastic process can be defined as a set of random variables, usually indexed by time, that evolve according to certain probabilistic regularities. The analysis of stochastic processes plays a key role in many fields, including statistics, finance, engineering, control theory, and other scientific disciplines.

Definition 1: The family $\{x(t), t \in T\}$ of random measurable functions $x(\omega, t) : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}^d, \mathcal{B})$ is called a stochastic process with phase space $(\mathbb{R}^d, \mathcal{B})$ and parameter set T , where \mathcal{B} is the Borel σ -algebra over \mathbb{R}^d .

On the basis of the previous definition, it can be concluded that for each fixed $t \in T$, a random variable, or \mathcal{F}_t -measurable function $x(\omega, t) : (\Omega, \mathcal{F}_t) \rightarrow (\mathbb{R}^d, \mathcal{B})$ is obtained. For any fixed $\omega \in \Omega$, $x(\omega, t)$ represents a function of the real argument $t \in T$, called the trajectory or realization corresponding to the outcome $\omega \in \Omega$. If $T = \mathbb{N}$, i.e. if the time interval is discrete, then it is a stochastic sequence $\{x(\omega, n), n \in \mathbb{N}\}$.

A stochastic process determines a family of finite-dimensional

distribution functions

$F_{t_1, \dots, t_n}(x_1, \dots, x_n) = P\{x(t_1) < x_1, \dots, x(t_n) < x_n\}$,
where $x_i \in \mathbb{R}, t_i \in T, i=1, 2, \dots, n, n \in \mathbb{N}$.

It is required that this family meets the following conditions:

(symmetry properties)

$F_{t_1, \dots, t_n}(x_1, \dots, x_n) = F_{t_{i_1}, \dots, t_{i_n}}(x_{i_1}, \dots, x_{i_n})$, where (i_1, \dots, i_n) is an arbitrary permutation of numbers $(1, \dots, n)$, $n \in \mathbb{N}$,

(agreement property)

$F_{t_1, \dots, t_j, t_{j+1}, \dots, t_n}(x_1, \dots, x_j, \infty, \dots, \infty) = F_{t_1, \dots, t_j}(x_1, \dots, x_j)$, for $j < n$. [2]

Theorem 1: (Kolmogorov's Theorem) For every family of finite-dimensional distribution functions that satisfies conditions (1) and (2) there is a probability space (Ω, \mathcal{F}, P) and a stochastic process $x(t)$, $t \in T$ defined on it corresponds to that family of finite-dimensional distribution functions.

Definition 2: Stochastic processes $x(t)$, $t \in T$ and $\tilde{x}(t)$, $t \in T$, defined on the same probability space and with the same set of states, are stochastically equivalent if for arbitrary $t \in T$

$P\{\omega \in \Omega : x(\omega, t) = \tilde{x}(\omega, t)\} = 1$.

In that case, the process $\{\tilde{x}(t), t \in T\}$ is said to be a stochastic modification (version) of the process $\{x(t), t \in T\}$ and vice versa.

In the definition of a stochastic process, it is not emphasized what the set T is, and the problem arises when the behavior of a stochastic process needs to be observed on an uncountable set of parameters t . In order to eliminate this difficulty, the concept of separability is introduced.

Definition 3: The stochastic process $\{x(t), t \in T\}$ is separable if there exists a countable set $G \subset T$ and a fixed event $\Lambda \subset \Omega$ of probability zero, so that for an arbitrary closed set $K \subset \mathbb{R}^d$ and an arbitrary open interval $I \subset T$, sets

$\{\omega : x(\omega, t) \in K, t \in I\}$ and $\{\omega : x(\omega, t) \in K, t \in I \cap G\}$

differ on a subset of Λ . The set G is called a separant.

Theorem 2. (Doob) For every stochastically continuous stochastic process $\{x(t), t \in T\}$ there exists a stochastically equivalent, separable and measurable stochastic process $\{\tilde{x}(t), t \in T\}$, defined on the same probability space and with the same set of values.

The stochastic process $\{\tilde{x}(t), t \in T\}$ from the previous theorem is called a separable and measurable modification of the stochastic process $\{x(t), t \in T\}$.

The stochastic process $\{x(t), t \in T\}$ is continuous in the middle order p , i.e. L_p -continuous, at the point $t \in T$, if $E|x(t)|^p < \infty$, for each $t \in T$ and $E|x(t+h) - x(t)|^p \rightarrow 0, h \rightarrow 0$. The stochastic process is L_p -continuous on the set $S \subseteq T$ if (1.1.2) holds for

$$x(t) = x_0 + \int_{t_0}^t f(x(s), s) ds + \int_{t_0}^t g(x(s), s) dw(s), x(t_0) = x_0, t \in [t_0, t]$$

every $t \in S$.

$P\{\omega \in \Omega : x(\omega, t) \text{ has a break at } [a, b]\} = 0$.

The examination of almost certain continuity is often performed by applying the Kolmogorov criterion, which is expressed by the following theorem.

Theorem 3: (Kolmogorov Criterion) Let p, q and k be positive constants such that for each $T > 0$ and $0 \leq t, s \leq T$ holds $E|x(t) - x(s)|^p \leq k|t - s|^{1+q}$ [3].

Then the stochastic process $\{x(t), t \in T\}$ almost certainly has continuous modification.

Definition 4: The stochastic process $\{x(t), t \in T\}$ is of the second order (L_2 -process) if $E|x(t)|^2 < \infty$, for every $t \in T$.

Definition 5. The stochastic process $\{x(t), t \in T\}$ is a Markov process if for arbitrary $0 \leq t_1 < \dots < t_k$ and $B \in \mathcal{B}^d$, $P\{x(t_k) \in B | \text{holds } x(t_1), \dots, x(t_{k-1})\} = P\{x(t_k) \in B | x(t_{k-1})\}$, almost certain.

Interpreting the random variables $x(t_1), \dots, x(t_{n-2})$ as the past, $x(t_{n-1})$ as the present and $x(t_n)$ as the future, Markov processes can be described as processes in which the prediction of the future depends only on of the present, and not from the past. Markov processes are used to describe phenomena in the theory of mass catering, for example stochastic characteristics of waiting lines, or in the economics of share prices. An important class in the theory of stochastic processes, introduced strictly mathematically by Doob, are martingales [4].

Definition 6: The stochastic process $\{x(t), Ft, t \geq 0\}$ for which $E|x(t)| < \infty$, for each $t \geq 0$ is:

martingale, if $E(x(t) | \mathcal{F}_s) = x(s)$ s.i. for every $0 \leq s \leq t$;

submartingale, if $E(x(t) | \mathcal{F}_s) \geq x(s)$ s.i. for every $0 \leq s \leq t$;

supermartingale, if $E(x(t) | \mathcal{F}_s) \leq x(s)$ s.i. for every $0 \leq s \leq t$.

Submartingales and supermartingales together are called semimartingales. The definition of local martingale is based on the concept of stopping time [5,6].

3. Stochastic Differential Equations

The stochastic differential equation of the unknown n -dimensional stochastic process $\{x(t), t \in [t_0, T]\}$ is an equation of the form $dx(t) = f(x(t), t) dt + g(x(t), t) dw(t)$, $t \in [t_0, T]$, $x(t_0) = x_0$, where $w = \{w(t), t \geq 0\}$ is an m -dimensional Brownian motion, the initial condition x_0 is a d -dimensional random variable that is stochastically independent in relation to Brownian motion and the functions $f : \mathbb{R}^d * [t_0, T] \rightarrow \mathbb{R}^d$ and $g : \mathbb{R}^d * [t_0, T] \rightarrow \mathbb{R}^d * \mathbb{R}^m$ are non-random Borel-measurable functions with their domains. The equation in integral form reads:

In the following, it will be assumed that $F_t = \sigma\{x_0, \omega(s), 0 \leq s \leq t\}$

A measurable stochastic process $x(t)$, $t \in [t_0, T]$ is a strict solution of the equation if it satisfies the following conditions: $x(t)$ is F_t -measurable for every $t \in [t_0, T]$;

$$\int_{t_0}^t T |f(x(t), t)| ds < \infty \text{ s.i.}, \int_{t_0}^t T |fg(x(t), t)| ds < \infty \text{ s.i.},$$

$$x(t_0) = x_0 \text{ s.i.};$$

the equation in the integral sense is satisfied s.i. for each $t \in [t_0, T]$ [7].

Definition 7: The equation has a unique strict solution if $P\{x(t) = \tilde{x}(t), t \in [t_0, T]\} = 1$ holds for every two strict solutions $x(t)$ and $\tilde{x}(t)$.

One of the essential characteristics of differential equations is stability. The stability of the solution of stochastic differential equations is based on the examination of the state of the system in relation to small changes in the initial condition or some other parameters of the system. Stochastic stability is one of the highly researched areas of stochastic analysis and many mathematicians have contributed to its development. Considering that many stochastic differential equations are not explicitly solvable, it is necessary to apply numerical methods, and the stability of numerical solutions is also important [8].

Definition 8: A solution $\{x(t), t \geq 0\}$ of a stochastic differential equation is stochastically stable or probability stable if for every $\varepsilon \in (0, 1)$ and $r > 0$, there exists $\delta = \delta(\varepsilon, r, t_0) > 0$ so that, for $|x_0| < \delta$, $P\{|x(t; t_0, x_0)| < r, t \geq t_0\} \geq 1 - \varepsilon$.

Definition 9: The solution $\{x(t), t \geq 0\}$ of the stochastic differential equation is stochastically asymptotically stable if for every $\varepsilon \in (0, 1)$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that, for $|x_0| < \delta$, valid:

$$P\{\lim_{t \rightarrow \infty} x(t; t_0, x_0) = 0, t \geq t_0\} \geq 1 - \varepsilon.$$

Definition 10: The solution $\{x(t), t \geq 0\}$ of the stochastic differential equation is exponentially stable in the mean order p if for each $x_0 \in \mathbb{R}^d$, $E|x_0|^p < \infty$ holds:

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log E(|x(t; t_0, x_0)|^p) \leq 0.$$

Definition 11: The solution $\{x(t), t \geq 0\}$ of the stochastic differential equation is almost certainly exponentially stable if for each $x_0 \in \mathbb{R}^d$:

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log (|x(t; t_0, x_0)|^p) \leq 0. [9]$$

3.1 Numerical Methods For Solving Stochastic Differential Equations

With the increase in the complexity of real problems in different areas of life, mathematical models consisting of systems of equations are increasingly used to describe key phenomena or processes relevant to the challenges posed. While analytical solutions can sometimes be found for simpler models, most models are complex, requiring the application of numerical methods to obtain approximate solutions. Different approximation methods are used to solve different types of stochastic differential equations, with criteria for comparing their efficiency. It is important that the method be simple to implement and that the approximate solutions retain the key characteristics of the exact solution [10].

In this paper, we first analyze the conditions under which there is a unique solution for the class of neutral stochastic differential equations with time-dependent delay and Markov transitions, as well as for the class of neutral stochastic differential equations with time-dependent delay. Numerical methods are then applied to obtain approximate solutions of these equations, while investigating the conditions under which the approximate solution shares the same characteristics as the exact solution, including stability - an aspect that is considered in more detail in the paper [11].

We will consider and compare explicit (like Euler-Maruyama method) and implicit (like backward Euler method and θ -Euler-Maruyama method) numerical approaches [12].

3.2 Euler-Maruyama Method

The Euler-Maruyama method represents an explicit approach to the numerical solution of both ordinary and stochastic differential equations. This method generates explicit approximate solutions, where standard existence and uniqueness conditions, such as the

Lipschitz condition, the linear growth condition and the L2-boundedness of the initial condition, are often required to prove the mean-square convergence of the corresponding approximate solutions [13].

The modeled stochastic differential equation has the form $dx(t) = f(x(t)) dt + g(x(t)) dw(t)$, where $t \in [0, T]$, $x(t) \in \mathbb{R}^d$ for each t , $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$, $g: \mathbb{R}^d \rightarrow \mathbb{R}^d \times m$, and $w(t)$ represents the m -dimensional process of Brownian motion. Pristup of the Euler-Maruyama method includes the calculation of discrete approximate solutions $Y_k \approx x(t_k)$, where $t_k = k\Delta$ and Δ represents.

3.3 Backward Euler's Method

Backward Euler's method is an implicit method for numerically solving stochastic differential equations. It often appears in the literature as a semi-implicit Euler method. Backward Euler's method is implicit by the argument of the transfer coefficient.

More precisely, let the solution $\{x(t), t \in [0, T]\}$ of the equation be approximated on an arbitrary partition of the interval $[0, T]$, $0 = t_0 < t_1 < \dots < t_n = T$. For the chosen step size $\Delta \in (0, 1)$, where $t_k = k\Delta$, for $k \in 0, 1, 2, \dots, n-1$, this method is based on solving the following equation:

$$Y_{k+1} = Y_k + f(Y_{k+1})\Delta + g(Y_k)\Delta w_k \text{ gdje je } Y_0 = x_0 \text{ i } \Delta w_k = w(t_{k+1}) - w(t_k).$$

These methods are applied to Eq $x(t) = f(x(t)) dt + g(x(t)) dw(t)$, $x(0) = x_0$. [14]

3.4 θ -Euler-Maruyama Method

The θ -Euler-Maruyama method is also an implicit method for numerically solving stochastic differential equations and is a generalization of the previously mentioned methods. The parameter $\theta \in [0, 1]$ is often called a measure of the implicitness of the numerical method. Specially, if $\theta = 0$, the Euler-Maruyama method is obtained, while for $\theta = 1$, the backward Euler method is obtained [15].

The solution of the equation $x(t) = f(x(t)) dt + g(x(t)) dw(t)$, $x(0) = x_0$, is approximated by this method on an arbitrary partition of the interval $[0, \infty]$. the selected step size $\Delta \in (0, 1)$ where $t_k = k\Delta$. for $k \in 0, 1, 2, \dots$, and is based on solving the following equation:

$$Y_{k+1} = Y_k + \theta f(Y_{k+1})\Delta + (1 - \theta)f(Y_k)\Delta + g(Y_k)\Delta w_k, \text{ gdje je } Y_0 = x_0 \text{ i } \Delta w_k = w(t_{k+1}) - w(t_k).$$

In order to introduce the numerical scheme, an auxiliary function is defined

$F: \mathbb{R}^d \rightarrow \mathbb{R}^d$ as follows

$$F(x) = x - \theta f(x)\Delta t.$$

Assuming that the one-sided Lipschitz condition is valid, i.e. there is a constant $\mu > 0$, so that for each $a, b \in \mathbb{R}^d$:

$$\langle a - b, f(a) - f(b) \rangle \leq \mu(a - b)^2$$

there is an inverse function F^{-1} and the solution Y_{k+1} can be represented as follows:
 $Y_{k+1} = F^{-1}(Y_k + (1-\theta)f(Y_k)\Delta t + g(Y_k)\Delta w_k)$ where the random variable Y_k is measurable.

4. Optimization of The Solution of A Stochastic Differential Equation Using A Numerical Method Using A Specific Example

Consider the following example of a neutral stochastic differential equation:

$$d[x(t) - \frac{1}{50} \sin x(t - p(t))] = -\frac{1}{48} x(t) dt + \frac{1}{20\sqrt{6}} \frac{x(t-p(t))}{1+x^2(t-p(t))} \cos x(t) dw(t) \dots \dots \dots (1)$$

for $t \in [0, 500]$, with the initial condition $\varphi(t) = 1, t \in [-\tau, 0]$, so that $\tau = 0.5$ and $\varphi \in C_{[-\tau, 0]}(\mathbb{R})$. The transmission coefficient $f(x, y) = -1/48 x$ satisfies the linear growth condition A1 for $K = 1/(48 \cdot 48)$, where the function $u(x) = 1/50 \sin x, x \in \mathbb{R}$ satisfies the assumption A2 for $\beta = 1/500$. Let the delay function of the form $p(t) = 1/4 - 1/4 \sin t, t \in [0, 500]$ Then:

$$p'(t) = -\frac{1}{4} \cos t \leq \frac{1}{4} = \bar{p}$$

$$|p(t) - p(s)| \leq \frac{1}{4} |t - s|, t, s \in [0, 500]$$

Now let's see that:

$$\begin{aligned} 2(x - u(y))f(x, y) + |g(x, y)|^2 &= -\frac{1}{24}x^2 + \frac{1}{1200}x\sin y + \frac{1}{2400}\frac{y^2}{(1+y^2)^2}\cos^2 x \\ &\leq -\frac{1}{24}x^2 + -\frac{1}{2400}x^2 + \frac{1}{2400}y^2 + \frac{1}{2400}y^2 \leq -\frac{33}{800}x^2 + \frac{1}{1200}y^2 \end{aligned}$$

That is, it is valid that $\alpha_1 = \frac{33}{800}$ AND $\alpha_2 = \frac{1}{1200}$ In addition, it is valid that:

$$\frac{\alpha_2}{1-p} = \frac{1}{900} < \frac{33}{800} = \alpha_1$$

As the assumption applies that for each choice of positive constants μ_1 and μ_2 , let $\mu_1 = \mu_2 = 1/5$, so that $\theta(\mu_1 + \mu_2)\Delta + \beta < 1$ for every $\Delta \in (0, 1)$. Now it can be concluded that the corresponding Q Euler Maruyama approximation equations have unique solutions. Bearing in mind that for $\theta = 3/4$ it follows that $q_{k=\varphi(k\Delta)}, k = -n^*, -n^*+1, \dots, 0$ while for $k = 0, 1, 2, \dots$,

$$\begin{aligned} q_{k+1} &= q_k + \frac{3}{200} \sin q_{k+1-\lfloor \frac{p(k+1)\Delta}{\Delta} \rfloor} - \frac{1}{100} \sin q_{k-\lfloor \frac{p(k\Delta)}{\Delta} \rfloor} - \frac{1}{200} \sin q_{k-1-\lfloor \frac{p(k+1)\Delta}{\Delta} \rfloor} \\ &\quad - \frac{1}{64} q_{k+1}\Delta - \frac{1}{192} q_k\Delta + \frac{1}{20\sqrt{6}} \frac{q_{k-\lfloor \frac{p(k\Delta)}{\Delta} \rfloor}}{q_{k-\lfloor \frac{p(k\Delta)}{\Delta} \rfloor}^2} \cos q_k \Delta w_k \dots \dots (2) \end{aligned}$$

Given that $C = \frac{\alpha_1}{6}$, it follows that $\Delta^* = 1$. Therefore, for each $\Delta \in (0, \Delta^*)$, the stability conditions of equation (2) are now examined.

The figure shows several trajectories of the solution of equation (2) with a step of $\Delta = 0.02$

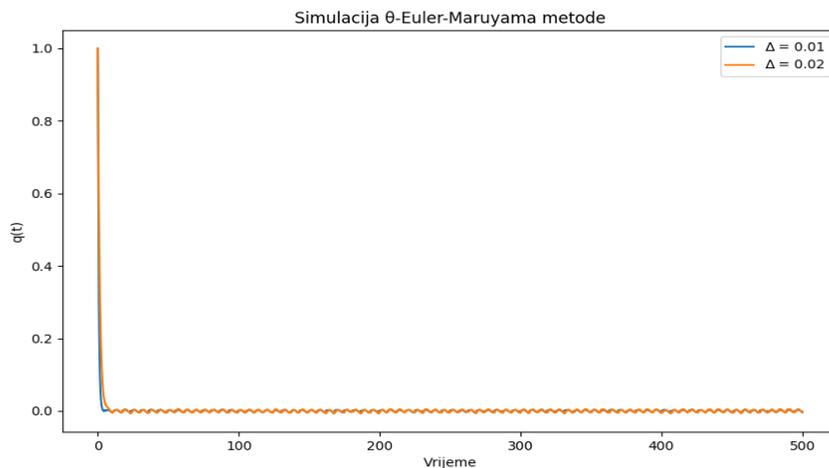


Figure 1: Simulation of the K Euler Marjjam Method for steps 0.01 and 0.02

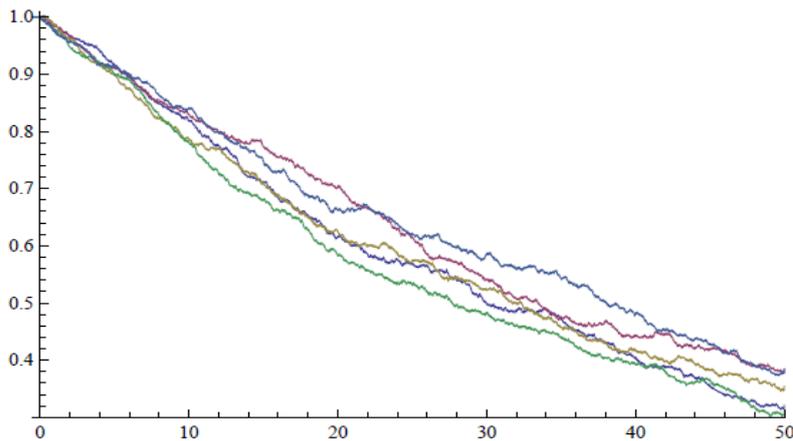


Figure 2: Solution trajectories for delta =0.01

How is it :

$$\frac{1}{4(9(1-\theta)^2+\theta^2+3)[(1-\eta)^{-1}+1]} \wedge 1 = \frac{1}{33}. \text{ Tada važi da je :}$$

$$\frac{33}{800} = \alpha_1 > \frac{12([(1-\eta)^{-1}]+1)}{1-4\beta^2(9(1-\theta)^2+\theta^2+3)[(1-\eta)^{-1}+1]} \times \left(\alpha_2 + K \left(\frac{1}{[(1-\eta)^{-1}]+1} + 1 \right) + 5(1-\theta)^2\beta^2 \right) = 0,03914.$$

The final one we have is

$$\alpha_1 \theta - \theta^2 K - (\alpha_2 \theta + \theta^2 K + 4\beta^2(1-\theta)^2) ([(1-\eta)^{-1}] + 1) = 0,0287$$

For some $\gamma \in \left(0, \frac{1}{\tau+1} \log[(1-\eta)^{-1}]+1\right) \wedge \log\left(\frac{3}{2} \wedge A\right) \wedge C$, it follows that:

$\lim_{k \rightarrow \infty} \frac{\log|q_k|}{k\Delta} \leq -\frac{\gamma}{2}$ where $A = \log \varepsilon$, while ε is the unique solution of our initial equation. By direct calculation we have that:

$$\frac{1}{\tau+1} \log[(1-\eta)^{-1}]+1 = 0,4621 \quad C = \frac{\alpha_1}{6} = \frac{11}{1600} \quad A = 34,1444$$

Now using the result we got with this method and the programming method, Figure 3 shows the graph of the solution with all three numerical methods and their approximations [16]:

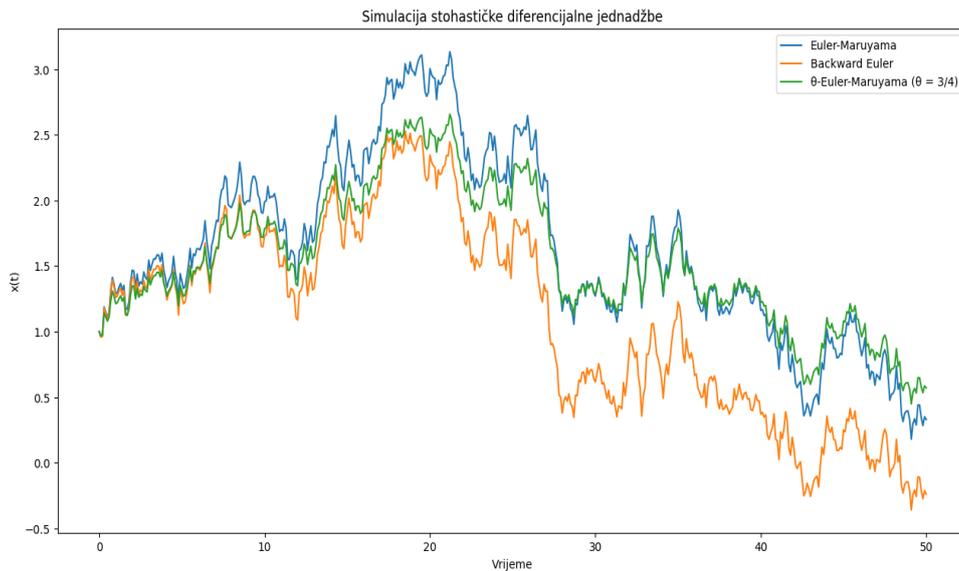


Figure 3: Approximation of The Solution of The Differential Equation Using All Three Numerical Methods

Optimizing the solution of the stochastic differential equation implies the following steps:

1. **Finer time discretization:** Increase the number of steps (N) to obtain a finer time discretization. This will result in a more accurate solution, but increases the computational load.
2. **Adaptive steps:** Consider using adaptive steps to automatically adjust the step size according to variations in the solution. This can improve accuracy and efficiency.

3. **Different solving methods:** Consider using other numerical methods, such as implicit methods (eg implicit Euler, implicit Runge-Kutta), to see if there is a difference in performance.

4. **Parameter Validation:** Check different parameters of the method (eg noise variance, time step) to see how they affect the results [17].

Now we have solved our method numerically using the Runge Kuta method using the implicit method as well, and we have obtained the following results, which we presented in Figure 4.

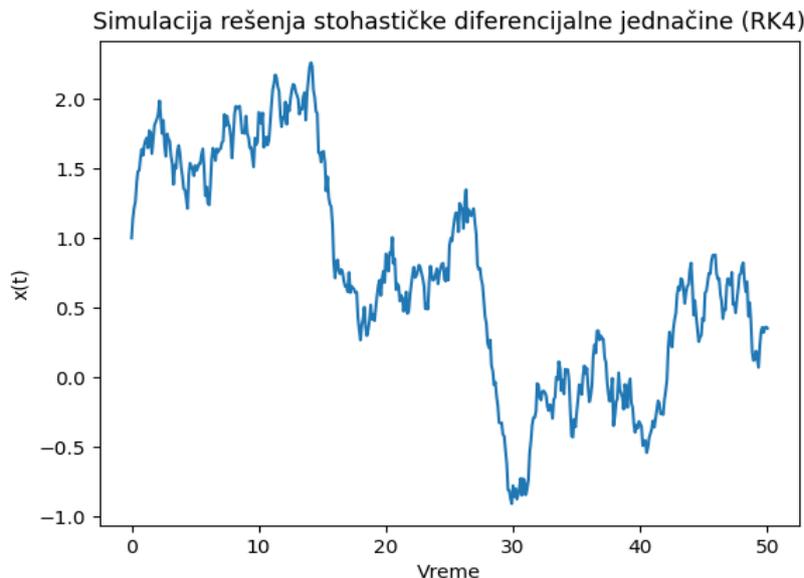


Figure 4: Simulation Stochastic Differential Equations

Based on the visualization of the graph and the difference in percentages, it can be noted that the fourth-order Runge-Kutta method (RK4) provided more accurate results compared to the Euler-Maruyama method. The difference between these two approaches is expressed in percentages and clearly indicates that RK4 is less prone to approximation errors compared to the

Euler-Maruyama method. This may indicate a higher accuracy of the RK4 method in solving stochastic differential equations, with a possible higher computational complexity. Optimization, in this context, refers to the choice of the method that gives more precise results, and in this case, it is the RK4 method [18].

The fourth-order Runge-Kutta method (RK4) is a numerical method used to solve differential equations. In this context, it was applied to the stochastic differential equation (SDE). RK4 is known for its high accuracy and stability in solving differential equation problems. The key advantage of RK4 compared to the Euler-Maruyama method, which was also used in the simulation, is reflected in the order of accuracy. RK4 has an order of accuracy of 4, which means that the approximation error decreases quadratically with the reduction of the time step, while the Euler-Maruyama method has an order of accuracy of 0.5. This difference in accuracy results in a more precise reproduction of the trajectories of the stochastic process [19].

When optimizing the RK4 method, it is crucial to choose the time step (dt) carefully. By optimizing the time step, a balance between precision and computational efficiency can be achieved. In this context, a proper choice of time steps can increase the accuracy of the simulation. In addition, it is important to note that stochastic processes include random noise, so the optimization of the parameter for generating random numbers is also important. Providing high-quality random numbers contributes to the reliability of the simulation [20]. In conclusion, RK4 excels in the accuracy of the results, and optimization is achieved by careful selection of time steps and random number generation, enabling accurate reproduction of stochastic processes [21].

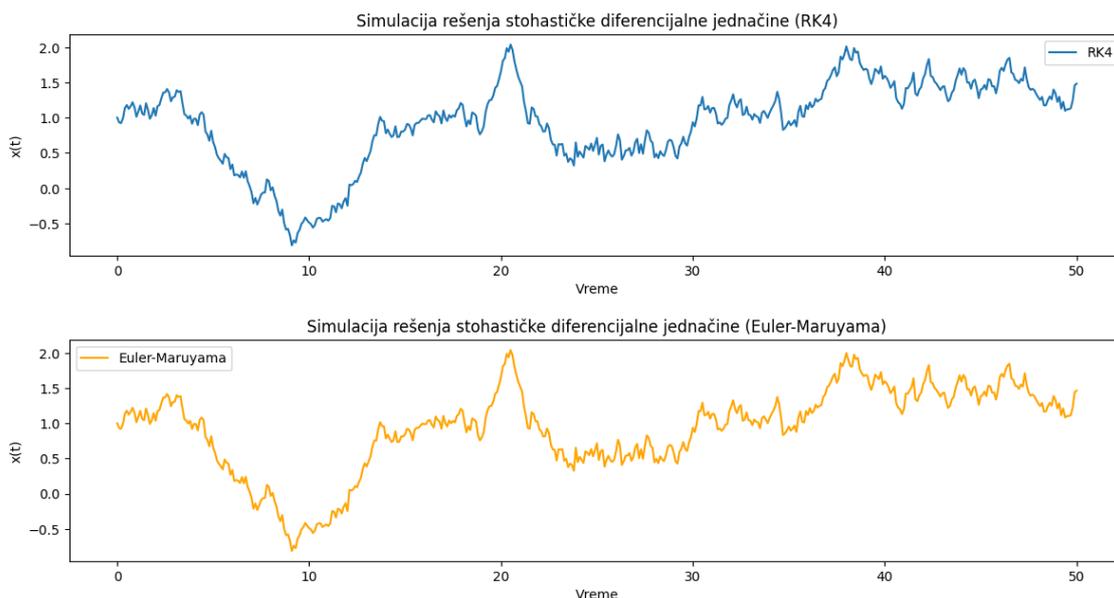


Figure 5: Comparison of Runge Kutta's And Euler Maruyama's Solution Optimization Methods

In the case of applying the fourth-order Runge-Kutta method (RK4) to a stochastic differential equation (SDE), the key parameters and functions include [22]:

1. Parameters:

- T (Time): Total simulation time.
- N (Number of steps): Number of time discretization steps [23].
- dt (Time Step): The time step between every two samples in the time discretization.
- ϕ (Initial condition): Initial value of the process [24].

2. Functions:

- $\delta(t)$: A delay function that defines the time relationship between the current moment and the delay moment.
- $f(x, y, t)$: Transfer coefficient used in SDE. In this case, the function describes a linear dependence on x , y and t .

3. Runge-Kutta Method:

- k_1, k_2, k_3, k_4 : Steps of the method used to evaluate the change of the variable x in each time step.
- dw : Stochastic term representing the random change at each step.

When we compare the Runge-Kutta method with the Euler-Maruyama method (which is the second method used in the simulation), the Runge-Kutta stands out by a higher order of

accuracy [25]. This order of accuracy makes RK4 more precise and reliable in reproducing stochastic process trajectories. (Figure 6).

Thus, the improvement comes from the mathematical nature of the RK4 method, which better approximates the changes in the stochastic process at each step. This mathematical approach allows obtaining more precise simulation results compared to simpler methods, such as the Euler-Maruyama method [26].

5. Conclusion Of Discussion

Investigating the stochastic differential equation provides a deeper understanding of the numerical methods applied to this type of problem. By analyzing the stability conditions, the optimal value of the step Δ was determined, which ensures the stability of the numerical solution. Through experiments with Q Euler-Maruyama and Runge-Kutta methods, we notice that Runge-Kutta, thanks to a higher order of accuracy, provides more precise solutions and reduces the error in the approximation of stochastic changes. This optimization results in a better approximation of process trajectories, which is crucial for the analysis of system behavior. The effort to achieve stability and precision in numerical solutions plays a key role in the reliability of approaches to stochastic differential equations. The discussion of the results indicates that the selection of the

appropriate numerical method depends on the specificity of the problem. The Runge-Kutta method, although it requires more computing resources, shows superior accuracy compared to the Q Euler-Maruyama method. Increasing accuracy in solving stochastic differential equations is important for the analysis of real systems, where accuracy can have a significant impact on decision-making [27-29].

In addition, the research points to the importance of the correct determination of parameters, such as the step Δ , in order to achieve stability and reliability of numerical solutions. This approach can be applied to various stochastic systems and contribute to the understanding of their behavior over time. In conclusion, the combination of stability analysis, selection of an appropriate numerical method and careful adjustment of parameters is key to successfully solving stochastic differential equations. This research provides a basis for further understanding of numerical aspects of stochastic processes and their application in real situations [30].

References

1. Arnold, L. (1974). Stochastic differential equations: theory and applications. (*No Title*).
2. Arnold, L., & Kliemann, W. (1983). Qualitative theory of stochastic systems. In *Probabilistic analysis and related topics* (pp. 1-79). Academic Press.
3. Basak, G. K., Bisi, A., & Ghosh, M. K. (1996). Stability of a random diffusion with linear drift. *Journal of Mathematical Analysis and Applications*, 202(2), 604-622.
4. Berger, M. A., & Mizel, V. J. (1980). Volterra equations with Itô integrals—I. *The Journal of Integral Equations*, 187-245.
5. Berger, M. A., & Mizel, V. J. (1980). Volterra equations with itô integrals—II. *The Journal of Integral Equations*, 319-337.
6. Burkholder, D. L., Davis, B. J., & Gundy, R. F. (1972, January). Integral inequalities for convex functions of operators on martingales. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (Vol. 2, pp. 223-240). Univ. California Press Berkeley, Calif.
7. Cramér, H. (1940). On the theory of stationary random processes. *Annals of Mathematics*, 215-230.
8. Dellacherie, C. (2019). *Capacités et processus stochastiques* (Vol. 67). Springer.
9. Dolean-Dade, C., & Meyer, P. (1970). Stochastic Integrals with Respect to Local Martingales. *Lecture Notes in Mathematics*, 124, 77-107.
10. Doob, J.L. (1953). Stochastic Processes. John Wiley.
11. Friedman, A. (1975). Stochastic differential equations and applications. In *Stochastic differential equations* (pp. 75-148). Berlin, Heidelberg: Springer Berlin Heidelberg.
12. Gihman, I. I. (1950). Certain differential equations with random functions. *Uskr. Math. Z.*, 2(3), 45-69.
13. Gikhman, I. I. (1950). On the theory of differential equations of random processes. Part 1. *Ukr. Mat. Zh.*, 4, 37-63.
14. Gikhman, I. I., & Skorokhod, A. V. (1969). Introduction to the theory of random processes. (*No Title*).
15. Gikhman, I. I., & Skorokhod, A. V. (1982). Stochastic Differential Equations and Their Applications. Naukova Dumka.
16. Has'minskii, R. Z. (1980). Stochastic Stability of Differential Equations. Sijthoff & Noordhoff.
17. Higham, D. J., Mao, X., & Stuart, A. M. (2002). Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM journal on numerical analysis*, 40(3), 1041-1063.
18. Higham, D. J., & Kloeden, P. E. (2005). Numerical methods for nonlinear stochastic differential equations with jumps. *Numerische Mathematik*, 101(1), 101-119.
19. Hu, L., Mao, X., & Shen, Y. (2013). Stability and boundedness of nonlinear hybrid stochastic differential delay equations. *Systems & Control Letters*, 62(2), 178-187.
20. Hu, J., & Xu, Z. (2014). Exponential stability of neutral stochastic functional differential equations with two-time-scale Markovian switching. *Mathematical Problems in Engineering*, 2014.
21. Galić, D., Stojanović, Z., & Čajić, E. (2024). Application of Neural Networks and Machine Learning in Image Recognition. *Tehnički vjesnik*, 31(1), 316-323.
22. Galić, R., & Čajić, E. (2023). Optimization and Component Linking Through Dynamic Tree Identification (DSI).
23. Čajić, E., Ibrišimović, I., Šehanović, A., Bajrić, D., & Šćekić, J. (2023, December). Fuzzy Logic and Neural Networks for Disease Detection and Simulation in Matlab. In *CS & IT Conference Proceedings* (Vol. 13, No. 23). CS & IT Conference Proceedings.
24. Shabani, E., Resic, S., Cajic, E., & Ramaj, V. (2024). Methods of solving partial differential equations and their application on one specific example. *J Math Techniques Comput Math*, 3(1), 01-16.
25. Ramaj, V., Elezaj, R., & Čajić, E. (2024). Fuzzy Numbers Unraveling the Intricacies of Neural Network Functionality.
26. Čajić, E., Stojanović, Z., & Galić, D. (2023, November). Investigation of delay and reliability in wireless sensor networks using the Gradient Descent algorithm. In *2023 31st Telecommunications Forum (TELFOR)* (pp. 1-4). IEEE.
27. Galić, R., Čajić, E., Stojanovic, Z., & Galić, D. (2023). Stochastic Methods in Artificial Intelligence.
28. Cajic, E. (2023). Optimization of nonlinear equations with constraints using genetic algorithm, 08 November 2023, PREPRINT available at Research Square.
29. Ibrišimović, I., Jasak, Z., Omerović, A., & Čajić, E. (2023). Practical Application of Out-of-Kilter Algorithm. *Chinese Business Review*, 22(2), 86-94.
30. Stojanovic, Z., Cajic, E. (2024). Application of Artificial Intelligence in Cultural Heritage / The Fourth International Scientific Conference The Importance of Media Interpretation for the Promotion of Cultural Heritage / [ed. Branislav Sančanin]. - ISBN 978-86-81866-04-7. - Vol. 4, p. 333-352.

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