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Operator Algebras on Non-Separable Banach Spaces and Applications in Mathematical Physics

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Abstract
This paper develops a structural and functional framework for operator algebras acting on nonseparable Banach spaces (NSBS). While classical operator algebras—such as C^*- and W^*-algebras—are traditionally constructed on separable Hilbert spaces, many physical and mathematical contexts require non-separable or even transfinite structures: quantum field theories with infinitely many degrees of freedom, infinite tensor product systems, and algebras associated with non-measurable state spaces.

We extend the classical operator-algebraic formalism to NSBS by introducing approximate operator algebras, defined through directed nets of weakly compact projections and local separable subspaces. This approach restores the analytic machinery of functional calculus, spectra, and representations, while preserving topological and dual properties within locally separable components. The paper establishes several new results concerning approximate ideals, bicommutants, spectral continuity, and weak operator topologies in NSBS. Furthermore, we analyse the correspondence between approximate representations of C^*-algebras on NSBS and physical observables in quantum mechanics and field theory.

From a physical perspective, the proposed framework provides a rigorous mathematical description of systems with non-countable degrees of freedom, extending von Neumann's theory of operator algebras beyond separability. Applications include the representation of infinite spin systems, algebras of observables in non-separable Hilbert–Banach settings, and generalised state spaces in quantum statistical mechanics.

Keywords: Non-Separable Banach Spaces, Operator Algebras, Approximate Representations, Weak Compactness, Bicommutant Theorem, Quantum Mechanics, Quantum Field Theory, Functional Analysis

1. Introduction
The study of operator algebras has long played a central role in modern analysis and mathematical physics. Originating with the foundational work of von Neumann and Gelfand–Naimark, these structures were developed to formalise the algebraic and topological properties of bounded linear operators on Hilbert spaces, leading to the definition of C^* - and $W*$ -algebras. Such algebras provide the natural mathematical setting for the description of quantum observables, statistical ensembles, and the fundamental symmetries of physical systems [1]. Their theory has since evolved into a highly refined branch of functional analysis, intimately connected with spectral theory, noncommutative geometry, and quantum field theory [2-4].

In the classical framework, most developments in operator algebras assume separability of the underlying Hilbert or Banach space. Separability ensures the existence of countable dense subsets and enables the use of sequences to describe convergence in the strong, weak, or weak-* topologies. However, in many contexts of modern physics and analysis—particularly in infinite spin systems, continuous field models, quantum statistical mechanics, and operator-valued distributions—non-separable Banach or Hilbert spaces naturally arise. In these settings, the assumption of separability becomes not only restrictive but inadequate: sequential compactness fails, standard spectral decompositions are no longer valid, and the GNS representation may not extend globally [5,6].

The purpose of this paper is to develop a structural and functional framework for operator algebras acting on non-separable Banach spaces (NSBS), unifying algebraic, topological, and analytic aspects through the use of *approximate* constructions. Building on previous work on approximate compactness and spectral theory in NSBS, we introduce the concept of approximate operator algebras, defined via directed nets of weakly compact projections and local separable subspaces. This approach restores, in an approximate sense, the classical properties of closure under addition, multiplication, adjoint operations, and weak operator topologies, while preserving compatibility with local spectral and interpolation structures.

The proposed framework provides a consistent extension of the C^* - and W^* algebraic formalism to non-separable settings. It allows one to define approximate ideals, commutants, and bicommutants, as well as to formulate an Approximate Bicommutant Theorem for NSBS, establishing the stability of algebraic closure under weakly compact limits. Furthermore, the paper develops a theory of approximate representations and states, which generalises the Gelfand–Naimark–Segal (GNS) construction to locally separable subspaces, enabling the definition of approximate cyclic representations and weak*- continuous functionals.

From a physical perspective, these results provide the analytical infrastructure to model systems with non-countable degrees of freedom. They extend the mathematical formalism of operator algebras to non-separable spaces that appear in quantum field theory, infinite lattice models, and continuous ensembles in quantum statistical mechanics [4,7]. In this setting, observables correspond to elements of approximate operator algebras, and states to approximate positive functionals defined on these algebras.

The interplay between local separability and global non-separability captures the hierarchical structure of physical systems, from local interactions to global field configurations. The remainder of the paper is organised as follows.

Section 2 reviews the fundamental notions of operator algebras, topological duality, and weak operator topologies, emphasising the limitations of separability. Section 3 introduces approximate operator algebras on NSBS and establishes their basic algebraic and topological properties. Section 4 develops the theory of approximate representations and states, including an analogue of the GNS construction. Section 5 examines the structural properties of these algebras, including approximate bicommutant and closure results. Section 6 explores applications to mathematical physics, particularly in quantum field theory and statistical mechanics. Section 7 discusses the implications of the framework for non-separable functional analysis and operator theory, while Section 8 summarises the conclusions and outlines open research directions.

2. Preliminaries

The algebraic and topological study of bounded linear operators relies heavily on the structure of the ambient Banach space. Let $B(X)$ denote the space of all bounded linear operators on a Banach space X, endowed with the operator norm

$$\| T \| = \sup_{\|x\| \leq 1} \| Tx \|, T \in B(X)$$

Within $B(X)$, several topologies are of fundamental interest:
1. Norm topology, defined by the operator norm itself.
2. *Strong operator topology* (SOT), determined by pointwise convergence $T_\alpha x \to Tx$ for all $x \in X$.
3. Weak operator topology (WOT), defined by convergence of scalar functionals $\langle x^*, T_\alpha x\rangle \to \langle x^*, Tx\rangle$ for all $x \in X$, $x^* \in X^*$.
4. Weak*-topology, acting on dual spaces of operators, relevant to $B(X^*)$ and predual structures in the setting of W^*-algebras.

In the classical separable framework, these topologies can be characterised sequentially: a sequence (T_n) converges strongly (respectively, weakly) if and only if $T_n x \to Tx$ (respectively, $\langle x^*, T_n x\rangle \to \langle x^*, Tx\rangle$) for every x (and x^*).

This sequential description fails in non-separable Banach spaces (NSBS), where compactness and continuity cannot be captured by sequences alone; the appropriate generalisation requires directed nets and local compactness in separable subspaces.

2.1. Limitations of Separability

Separability ensures the existence of a countable dense subset $D \subset X$, which permits the use of sequences to approximate every element of X in norm. Many foundational results in operator theory—such as the Banach–Steinhaus theorem, compactness criteria, and spectral decompositions—rely on this property.

When separability fails, the topological structure of X becomes far more intricate:
a) *Weakly* convergent sequences may not capture the topology of weak convergence.

b) The unit ball B_X may be weakly compact without being sequentially compact.
c) Approximation by finite-rank operators may no longer be dense in $B(X)$ (Lindenstrauss & Tzafriri, 1979; Fabian et al., 2011).
d) Spectral measures and continuous functional calculi may lose their standard representations.

As a result, many of the algebraic structures associated with $B(X)$, such as C*- or W*-closures, fail to behave predictably under standard operator topologies. This motivates the development of approximate frameworks—analytic structures that reproduce separable behaviour locally and extend it globally via directed limits.

2.2. Directed Nets and Local Analysis

A net $(x_\alpha)_{\alpha\in A}$ in X is a function from a directed set A into X, denoted x_α, where for all $\alpha, \beta \in A$ there exists $\gamma \in A$ such that $\alpha, \beta \leq \gamma$.

Nets generalise sequences by capturing convergence in topologies that are not first-countable, as is the case for the weak and weak* topologies in NSBS. We say that $x_\alpha \to x$ weakly in X if $\langle x^*, x_\alpha\rangle \to \langle x^*, x\rangle$ for every $x^* \in X^*$. Similarly, a net of operators $(T_\alpha) \subset B(X)$ converges weakly to T if $\langle x^*, T_\alpha x\rangle \to \langle x^*, Tx\rangle$ for all $x \in X$, $x^* \in X^*$. This notion allows us to define weakly compact and weakly continuous operator limits even when no convergent subsequences exist.

In the non-separable setting, it is often possible to recover separable behaviour locally: every countable subset $C \subset X$ generates a separable subspace $X_C = \overline{span}(C)$. The collection of all such X_C, ordered by inclusion, forms a directed family of separable subspaces whose union is dense in X. This observation underlies the local–global principle that governs much of the approximate theory developed in subsequent sections.

2.3. Weakly Compact Projections

A projection $P \in B(X)$ is said to be weakly compact if $P(B_X)$ is relatively compact in the weak topology of X. Equivalently, for every bounded net $(x_\alpha) \subset X$, the image (Px_α) admits a weakly convergent subnet. Weakly compact projections play a fundamental role in the construction of approximate identities and locally separable decompositions of X.

For a non-separable Banach space X, consider a directed family $\{P_F\}_{F\in\mathcal{F}(I)} \subset B(X)$ of weakly compact projections indexed by the finite subsets F of some directed index set I, satisfying:

1. $P_F^2 = P_F$ for all $F \in \mathcal{F}(I)$;

2. $\| P_F \| \leq C < \infty$ uniformly;

3. $P_F \leq P_G$ whenever $F \subseteq G$;

4. For every $x \in X$, $P_F x \to x$ weakly as $F \uparrow I$.

The family $\{P_F\}$ is called a *weakly compact approximate identity* (WCAI). Such families generate the approximate local structures used to define operator algebras in NSBS. Each $P_F(X)$ is a separable subspace of X, and the net (P_F) approximates the identity operator in the weak topology. This structure enables the definition of approximate operator algebras $\mathcal{A}^{\text{approx}}$ as limits of operator families restricted to these separable components.

2.4. Approximate Topologies on $B(X)$

Given a WCAI $\{P_F\}$, we define *approximate strong and approximate weak* topologies on $B(X)$ by the following rules:
a) $T_\alpha \to T$ in the approximate strong topology if for every F, $P_F T_\alpha P_F x \to P_F T P_F x$ in norm for all $x \in X$;
b) $T_\alpha \to T$ in the approximate weak topology if for every F, $\langle x^*, P_F T_\alpha P_F x\rangle \to \langle x^*, P_F T P_F x\rangle$ for all $x \in X$, $x^* \in X^*$.

These topologies generalise the strong and weak operator topologies to non-separable spaces, preserving local separability and weak compactness. They are compatible with the operator algebraic operations (addition, multiplication, adjoint when applicable) and provide the foundation for defining *approximate operator algebras*.

2.5. Relation with Classical Operator Algebras

In the separable case, weakly compact approximate identities can be replaced by sequences of finite-rank projections, and the approximate topologies coincide with the classical ones. Thus, every separable Banach space admits a countable WCAI, and the approximate algebraic structures introduced here reduce to the standard framework of operator algebras.

The non-separable extension developed in this paper preserves these properties locally while extending them globally via directed limits.

3. Approximate Operator Algebras in Non-Separable Banach Spaces

The operator algebra $B(X)$ on a Banach space X possesses a rich algebraic and topological structure that underpins much of modern functional analysis and mathematical physics. However, when X is non-separable, several of the foundational mechanisms that make the separable theory work—particularly compactness, weak sequential convergence, and approximation by finite-rank operators—fail to hold globally.

In this section, we construct a generalised algebraic framework based on weakly compact approximate identities (WCAI) and directed nets of separable subspaces, allowing one to recover the essential features of operator algebras in the non-separable context.

3.1. Definition and Basic Structure

Let X be a non-separable Banach space, and let $\{P_F\}_{F\in\mathcal{F}(I)} \subset B(X)$ be a weakly compact approximate identity (as introduced in section 2.3). For each $F \in \mathcal{F}(I)$, define the local algebra

$$\mathcal{A}_F := P_F B(X) P_F = \{P_F T P_F : T \in B(X)\}$$

Each $\mathcal{A}_F$ is a Banach algebra acting on the separable subspace $X_F = P_F(X)$. The collection $\{\mathcal{A}_F\}$ forms a directed system under inclusion, with connecting morphisms

$$\rho_{FG}: \mathcal{A}_F \longrightarrow \mathcal{A}_G, \rho_{FG}(T) = P_G T P_G, \forall\, F \subseteq G$$

This allows us to define the approximate operator algebra on X as the direct limit:

$$\mathcal{A}^{approx} := \lim_{\substack{\rightarrow \\ F\in\mathcal{F}(I)}} \mathcal{A}_F$$

Definition 3.1.1 (Approximate Operator Algebra). The approximate operator algebra on a Banach space X is the inductive limit

$$\mathcal{A}^{approx} = \{(T_F)_{F\in\mathcal{F}(I)} : T_F \in \mathcal{A}_F, P_G T_F P_G = T_G \text{ whenever } F \subseteq G\}$$

endowed with the family of seminorms

$$\| T \|_F := \| T_F \|_{B(X_F)}, F \in \mathcal{F}(I)$$

The net of seminorms $\{\|\cdot\|_F\}$ induces a locally convex topology on $\mathcal{A}^{approx}$, called the approximate operator topology. Each element $T \in \mathcal{A}^{approx}$ corresponds to a consistent family of local operators (T_F) acting on the separable subspaces X_F . This structure generalises the standard topology of $B(X)$, reducing to it when X is separable and the net $\{P_F\}$ is sequential.

Proposition 3.1.1 (Algebraic Properties). The set $\mathcal{A}^{approx}$ is closed under addition, scalar multiplication, composition, and (when defined) adjoint operation. In particular:
1. $(S_F) + (T_F) = (S_F + T_F) \in \mathcal{A}^{approx}$;
2. $\lambda(T_F) = (\lambda T_F)$ for all scalars λ;
3. $(S_F)\,(T_F) = (S_F T_F) \in \mathcal{A}^{approx}$;
4. If $X$is a Hilbert space, $(T_F)^* = (T_F{}^*) \in \mathcal{A}^{approx}$.

Hence, $\mathcal{A}^{approx}$ is a locally *convex**-algebra.

Proof. Each $\mathcal{A}_F$ is a Banach*-algebra acting on a separable subspace. The compatibility conditions $P_G T_F P_G = T_G$ guarantee that the operations defined componentwise remain consistent across the directed system. Closure under the four operations follows immediately from the algebraic structure of $B(X_F)$.

3.2. Approximate Ideals and Commutants

Let $\mathcal{A}^{approx}$ be the approximate operator algebra on X . Given a subset $\mathcal{S} \subseteq \mathcal{A}^{approx}$, define:

$$\mathcal{S}' := \{T \in \mathcal{A}^{approx} : TS_F = S_F T,\ \forall S_F \in \mathcal{S}_F, F \in \mathcal{F}(I)\}$$

as the approximate commutant of $\mathcal{S}$, and

$$\mathcal{S}'' := (\mathcal{S}')'$$

as its approximate bicommutant.

Definition 3.2.1 (Approximate Ideal). A subset $\mathcal{J} \subseteq \mathcal{A}^{approx}$ is called a (two-sided) approximate ideal if:

$$\forall\ T \in \mathcal{J}, S \in \mathcal{A}^{approx};\ ST, TS \in \mathcal{J}$$

and if for each F , the local components T_F form an ideal in $\mathcal{A}_F$. The collection of approximate ideals forms a lattice under inclusion, generalising the classical ideal structure of $B(X)$.

Theorem 3.2.1 (Approximate Bicommutant Theorem). Let $\mathcal{S} \subseteq \mathcal{A}^{approx}$ be a *-subalgebra containing the identity. Then the approximate bicommutant $\mathcal{S}''$is the closure of $\mathcal{S}$ in the approximate weak topology. That is,

$$\mathcal{S}'' = \bar{\mathcal{S}}^{\,aWOT}$$

Proof. For each separable component X_F , the classical bicommutant theorem holds:

$$(\mathcal{S}_F)'' = \overline{\mathcal{S}_F}^{\,WOT} in\ B(X_F)$$

(Sakai, 1971; Kadison & Ringrose, 1983). The approximate bicommutant $\mathcal{S}''$ consists of nets (T_F) such that each $T_F \in (\mathcal{S}_F)''$. The directed limit preserves weak closure, yielding

$$\mathcal{S}'' = \lim_{F}\rightarrow\ (\mathcal{S}_F)'' = \lim_{F}\rightarrow\ \bar{\mathcal{S}_F}^{\,WOT} = \bar{\mathcal{S}}^{\,aWOT}$$

Hence, the bicommutant theorem extends naturally to the approximate context. □

3.3. Approximate C*-Algebras and Norm Compatibility

When X is a Banach space equipped with an additional inner product structure, the approximate algebra inherits properties analogous to those of a C*-algebra.

Definition 3.3.1 (Approximate C-Algebra)*. An approximate operator algebra $\mathcal{A}^{approx}$ is said to be an approximate C*-algebra if each local component $\mathcal{A}_F$ is a C*-algebra and the seminorms satisfy

$$\| T_F^* T_F \|_F = \| T_F \|_F^2, \forall\ F \in \mathcal{F}(I)$$

This ensures compatibility of the C*-identity across the directed system.

Proposition 3.3.1 (Norm Consistency). If X is a Hilbert space and $\{P_F\}$ is a WCAI of orthogonal projections, then for all $T \in \mathcal{A}^{approx}$,

$$\| T^* \|_F = \| T \|_F \text{ and } \| T^* T \|_F = \| T \|_F^2$$

Hence, $\mathcal{A}^{approx}$ satisfies the C*-property locally and globally.

3.4. Topological Properties

The approximate weak and strong operator topologies introduced in section 2.4 induce natural notions of convergence in $\mathcal{A}^{approx}$:

a) A net (T_α) converges to T in the approximate weak topology (aWOT) if $P_F T_\alpha P_F \to P_F T P_F$ weakly for all F.

b) Convergence in the approximate strong topology (aSOT) requires $P_F T_\alpha P_F x \to P_F T P_F x, \forall\ x \in X, F \in \mathcal{F}(I)$.

Proposition 3.4.1. If $\mathcal{A}^{approx}$ is bounded in the operator norm, then it is closed under both aSOT and aWOT limits.

Proof. Since each local algebra $\mathcal{A}_F$ is closed under WOT and SOT, the closure property is preserved under the inductive limit, as the projections P_F ensure consistency across separable components.

3.5. Relation to the Classical Theory

When X is separable, one may take $\{P_F\}$ as a countable sequence of finite-rank projections P_n with $P_n \uparrow I$. Then $\mathcal{A}^{approx} = B(X)$, and all approximate notions coincide with the classical ones. Thus, the framework of approximate operator algebras provides a genuine extension, not merely a reformulation, of the standard theory.

4. Representations and States

In the classical setting of separable Hilbert spaces, the representation theory of operator algebras—especially that of C*-algebras—relies fundamentally on the *Gelfand–Naimark–Segal (GNS) construction*. Given a positive linear functional φ on a C*-algebra $\mathcal{A}$, one obtains a representation $\pi_\varphi : \mathcal{A} \to B(\mathcal{H}_\varphi)$ and a cyclic vector $\xi_\varphi \in \mathcal{H}_\varphi$ such that $\varphi(T) = \langle \pi_\varphi (T) \ \ \xi_\varphi , \xi_\varphi \rangle$. However, this construction depends crucially on separability and on the existence of countable orthonormal bases.

In the non-separable case, where weak compactness replaces sequential compactness, such representations may fail to exist globally or to preserve continuity properties. To overcome these limitations, we introduce approximate representations and approximate states, which act on locally separable components and are consistent under directed weak limits. This framework restores many of the analytical and algebraic properties of the GNS theory in the context of non-separable Banach spaces.

4.1. Approximate Representations

Let $\mathcal{A}^{approx}$ be an approximate operator algebra on a non-separable Banach space X, constructed via a directed family of weakly compact projections $\{P_F\}$ $F \in \mathcal{F}(I)$. Each local algebra $\mathcal{A}_F = P_F B(X) P_F$ acts on the separable subspace $X_F = P_F(X)$. For each F , let $\pi_F : \mathcal{A}_F \to B(Y_F)$ be a continuous representation on a Banach space Y_F.

Definition 4.1.1 (Approximate Representation). An approximate representation of $\mathcal{A}^{approx}$ on a Banach space Y is a family

$$\pi^{approx} = (\pi_F)_{F \in \mathcal{F}(I)}$$

such that:
1. Each $\pi_F : \mathcal{A}_F \to B(Y_F)$ is a bounded representation;
2. The family $\{Y_F\}$ forms a directed system of separable Banach spaces with $Y = lim \to Y_F$;
3. For all $F \subseteq G$ and $T \in \mathcal{A}_F$, one has

$$\pi_G(P_G T P_G)\mid_{Y_F} = \pi_F(T)$$

The collection of all such families forms the category of approximate representations of $\mathcal{A}^{approx}$.

Proposition 4.1.1 (Continuity). If each π_F is continuous in the weak operator topology on $B(Y_F)$, then π^{approx} is continuous in the approximate weak topology on $B(Y)$.

Proof. For each F , continuity follows by hypothesis. Since the directed family $\{P_F\}$ defines consistent projections and each π_G restricts to π_F on YF, convergence $T_\alpha \to T$ in the approximate weak topology implies that

$$\langle y^*, \pi_F(T_\alpha) y \rangle \to \langle y^*, \pi_F(T) y \rangle$$

for all $y \in Y_F$, $y^* \in Y_F^*$. Hence, π^{approx} is weakly continuous.

4.2. Approximate States and Positive Functionals

A state on a C*-algebra $\mathcal{A}$ is a positive linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with $\varphi\ (\ I) = 1$. In non-separable settings, the absence of a global separable predual complicates the notion of positivity and weak*- continuity. We therefore adopt a local definition.

Definition 4.2.1 (Approximate Positive Functional). An approximate positive functional on $\mathcal{A}^{approx}$ is a family

$$\varphi^{approx} = (\varphi_F)_{F\in\mathcal{F}(I)}$$

where each $\varphi_F : \mathcal{A}_F \to \mathbb{C}$ is a positive linear functional satisfying:
1. $\varphi_G(P_G T P_G) = \varphi_F(T)$ whenever $F \subseteq G$;
2. $\sup_F |\varphi_F(T_F)| < \infty$ for all $T \in \mathcal{A}^{approx}$.
3. If, moreover, $\varphi_F(P_F) = 1, \forall F$, then φ^{approx} is an approximate state.

Proposition 4.2.1 (Local Positivity). If each φ_F is positive, then φ^{approx} is positive on $\mathcal{A}^{approx}$; that is,

$$\varphi^{approx}(T^*T) \geq 0,\ \forall T \in \mathcal{A}^{approx}$$

Proof. Each component satisfies $\varphi_F(T_F^* T_F) \geq 0$. Consistency ensures $\varphi_G(P_G T_F^* T_F P_G) = \varphi_F(T_F^* T_F), \forall\ F \subseteq G$. Hence, the net $(\varphi_F(T_F^* T_F))_F$ is non-negative and convergent, defining a positive functional globally.

4.3. Approximate GNS Construction

Given an approximate positive functional φ^{approx} , we can construct an approximate analogue of the GNS representation. For each F, define the pre-Hilbert space

$$\mathcal{H}_F = \{[T_F] : T_F \in \mathcal{A}_F\}, \langle [T_F], [S_F] \rangle = \varphi_F(S_F^* T_F)$$

Completing $\mathcal{H}_F$ yields a Hilbert space. The family $\{\mathcal{H}_F\}_{F\in\mathcal{F}(I)}$ forms a directed system, and the inductive limit

$$\mathcal{H}_{\varphi^{approx}} := \lim_{\substack{\to \\ F}}\ \mathcal{H}_F$$

is the approximate GNS Hilbert space.
Define the representation

$$\pi_{\varphi^{approx}}(T)[S_F] := [T_F S_F], T, S \in \mathcal{A}^{approx}$$

This is well defined, since consistency implies $P_G(T_F S_F)P_G = T_G S_G$ for $F \subseteq G$.

Theorem 4.3.1 (Approximate GNS Representation). Let φ^{approx} be an approximate positive functional on $\mathcal{A}^{approx}$. Then there exists a unique triple $(\pi_{\varphi^{approx}}, \mathcal{H}_{\varphi^{approx}}, \xi_{\varphi^{approx}})$ such that:
1. $\pi_{\varphi^{approx}}$ is a *-representation of $\mathcal{A}^{approx}$ on $\mathcal{H}_{\varphi^{approx}}$;

2. $\xi_{\varphi^{approx}}$ is cyclic;

3. $\varphi^{approx}(T) = \langle \pi_{\varphi^{approx}}(T)\xi_{\varphi^{approx}}, \xi_{\varphi^{approx}} \rangle$ for all $T \in \mathcal{A}^{approx}$.

Proof. Each local component φ_F defines a standard GNS triple $(\pi_F, \mathcal{H}_F, \xi_F)$. The directed limit construction ensures that these triples are compatible, i.e. $\pi_G(P_G T P_G) \mid_{\mathcal{H}_F} \pi_F(T)$ and ξ_G extends ξ_F . The inductive limit $\mathcal{H}_{\varphi^{\text{approx}}}$ is complete under the inner product induced by φ^{approx}, and $\pi_{\varphi^{\text{approx}}}$ is *-preserving. Cyclicity follows from the density of the span of $\pi_{\varphi^{\text{approx}}}(T)\, \xi_{\varphi^{\text{approx}}}, \forall\ T \in \mathcal{A}^{approx}$.

Corollary 4.3.1 (Local–Global Correspondence). For every separable component $X_F = P_F(X)$, the restriction of the approximate GNS representation $\pi_{\varphi^{\text{approx}}} : \mathcal{A}^{approx} \to B(\mathcal{H}_{\varphi^{\text{approx}}})$ to the local algebra $\mathcal{A}_F = P_F B(X) P_F$ coincides with the classical GNS representation associated with φ_F.

Proof. Recall that $\mathcal{A}^{approx} = \lim \to_{F\in\mathcal{F}(I)} \mathcal{A}_F$ is the inductive limit of the local algebras $\mathcal{A}_F$ with connecting maps

$$\rho_{FG}: \mathcal{A}_F \to \mathcal{A}_G, \rho_{FG}(T_F) = P_G T_F P_G (F \subseteq G)$$

Let $\varphi^{approx} = (\varphi_F)_{F\in\mathcal{F}(I)}$ be an approximate positive functional, i.e. $\varphi_G\big(\rho_{FG}(T_F)\big) = \varphi_F(T_F), \forall\ F \subseteq G$ and $T_F \in \mathcal{A}_F$.

For each F, let $(\pi_F, \mathcal{H}_F, \xi_F)$ be the classical GNS triple of $(\mathcal{A}_F, \varphi_F)$, constructed from the pre-Hilbert space

$$\mathcal{H}_F^{(0)} = \{[T_F]: T_F \in \mathcal{A}_F\}, \langle [T_F], [S_F]\rangle_F := \varphi_F(S_F^* T_F)$$

with completion $\mathcal{H}_F$, cyclic vector $\xi_F = [P_F]$, and representatio n$\pi_F(T_F)[S_F] = [T_F S_F]$.

For $F \subseteq G$, define the linear map:

$$J_{FG}: \mathcal{H}_F^{(0)} \longrightarrow \mathcal{H}_G^{(0)}, J_{FG}([T_F]) := [\rho_{FG}(T_F)] = [P_G T_F P_G]$$

Then, $\forall\ [T_F], [S_F] \in \mathcal{H}_F^{(0)}$,

$$\begin{aligned}\langle J_{FG}[T_F], J_{FG}[S_F]\rangle_G &= \varphi_G(\rho_{FG}(S_F)^*\, \rho_{FG}(T_F)) = \varphi_G(\rho_{FG}(S_F^* T_F)) = \varphi_F(S_F^* T_F)\\ &= \langle [T_F], [S_F]\rangle_F\end{aligned}$$

so J_{FG} is isometric and extends by continuity to an isometric embedding (still denoted J_{FG}) $\mathcal{H}_F \hookrightarrow \mathcal{H}_G$. The family $\{\mathcal{H}_F, J_{FG}\}$ is a directed system of Hilbert spaces.

By definition, the approximate GNS Hilbert space is the Hilbert inductive limit

$$\mathcal{H}_{\varphi^{approx}} := \lim_{\substack{\to\\ F}}\ \mathcal{H}_F$$

and we write $J_F: \mathcal{H}_F \to \mathcal{H}\varphi approx$ for the canonical isometric embeddings.

Define the canonical embedding of algebras

$$\begin{cases} \iota_F: \mathcal{A}_F \longrightarrow \mathcal{A}^{approx}\\ \iota_F(T_F) = (T_G)_{G\in\mathcal{F}(I)}\\ T_G := \rho_{FG}(T_F) = P_G T_F P_G\ (G \supseteq F)\end{cases}$$

and any choice consistent for $G \not\supseteq F$. By construction of $\pi_{\varphi^{approx}}$ on elementary classes,

$$\pi_{\varphi^{approx}}\ (\iota_F(T_F))\, J_F([S_F]) = J_F\ (\pi_F(T_F)[S_F]), \forall\, T_F, S_F \in \mathcal{A}_F$$

Indeed, both sides equal the class represented (at any level $G \supseteq F$) by $[\rho_{FG}(T_F)\, \rho_{FG}(S_F)]=[\rho_{FG}(T_F S_F)]$. Thus, for every F,

$$\pi_{\varphi^{approx}}\ |_{\iota_F(\mathcal{A}_F)}\ \ \textit{intertwines with } \pi_F \textit{ via the isometry } J_F: \mathcal{H}_F \to \mathcal{H}_{\varphi^{approx}}$$

Since J_F is an isometric embedding with dense range of the subspace it generates, the previous intertwining identity implies that the restriction of $\pi_{\varphi^{approx}}$ to the subalgebra $\iota_F(\mathcal{A}_F)$ is unitarily equivalent (via J_F) to the classical GNS representation π_F of $\mathcal{A}_F$. In particular, upon identifying $\mathcal{A}_F$ with its image $\iota_F(\mathcal{A}_F) \subset \mathcal{A}^{approx}$, we have

$$(\pi_{\varphi^{approx}})\ |_{\mathcal{A}_F} = J_F\ \pi_F(\cdot)\ J_F^{-1}\ on\ \ J_F(\mathcal{H}_F) \subset \mathcal{H}_{\varphi^{approx}}$$

Moreover, $J_F(\xi_F)$ is the cyclic vector for the embedded copy, and for all $T_F \in \mathcal{A}_F$,

$$\begin{cases} \langle \pi_{\varphi^{approx}}\, (\iota_F(T_F))J_F(\xi_F) \\ J_F(\xi_F)\rangle = \langle J_F(\pi_F(T_F)\xi_F) \\ J_F(\xi_F)\rangle = \langle \pi_F(T_F)\xi_F, \xi_F\rangle = \varphi_F(T_F) \end{cases}$$

showing that the matrix coefficients agree with those of the classical GNS functional at level F . Combining these considerations, the restriction of the approximate GNS representation to each local algebra $\mathcal{A}_F$ coincides (up to the canonical isometry J_F) with the classical GNS representation associated with φ_F . This proves the local–global correspondence.

Remark. The argument uses only (i) functoriality of the inductive system under the connecting maps □□□, and (ii) the consistency condition $\varphi_G \circ \rho_{FG} = \varphi_F$. Hence the result is robust under any choice of weakly compact approximate identity generating the directed system of separable components.

4.4. Weak-Continuity and Dual Structure*

Given that $\mathcal{A}^{approx}$ is a locally convex algebra with seminorms $\{\|\cdot\|_F\}$, its dual space can be identified as

$$(\mathcal{A}^{approx})^* = \lim_{F} \leftarrow\ (\mathcal{A}_F)^*$$

The *weak**-topology induced by this projective limit coincides with the topology of pointwise convergence on $\mathcal{A}^{approx}$.

*Proposition 4.4.1 (Weak- Continuity of Approximate States)**. Every approximate state φ^{approx} is continuous in the weak*-topology of $(\mathcal{A}^{approx})^*$. Moreover, the set of approximate states is convex and weak*-compact.

Proof. Each φ_F is weak*-continuous on $(\mathcal{A}_F)^*$ and satisfies $\|\varphi_F\| = 1$. Compactness follows from the Banach–Alaoglu theorem applied to each component and preserved under the projective limit.

4.5. Approximate Irreducibility

A representation π^{approx} is called approximately irreducible if the only closed subspaces $Y' \subseteq Y$ invariant under all $\pi^{approx}\,(T)$ are $Y' = \{0\}$ and $Y' = Y$. This property generalises irreducibility to the local–global framework and allows classification results for approximate C*-algebras.

Proposition 4.5.1. If each local representation π_F is irreducible, then π^{approx} is approximately irreducible.

Proof. Assume Y' is a closed invariant subspace under all $\pi^{approx}\,(T)$. Then $Y_F' = Y' \cap Y_F$ is invariant under $\pi_F(T_F)$ for each F. By local irreducibility, $Y_F' = \{0\}$ or $Y_F' = Y_F$. Directedness of the system implies $Y' = \{0\}$ or $Y' = Y$.

5. Structural Properties

The structural analysis of operator algebras on non-separable Banach spaces requires a reformulation of several classical notions, such as the spectrum, functional calculus, and duality. In separable spaces, these concepts are naturally compatible with sequential compactness and weak operator topologies. In non-separable Banach spaces, however, one must resort to local analysis on separable components and construct global properties through directed limits.

This section develops the approximate spectrum, approximate functional calculus, and approximate bicommutant structure, establishing the analytic and topological foundations of the approximate operator algebra framework.

5.1. Approximate Spectrum

Let T be an element of the approximate operator algebra A^{approx}, represented as a directed family (T_F) indexed by F in the directed set $\mathbb{F}(I)$, where each T_F belongs to the local algebra $A_F = P_F B(X)\, P_F$. For each F, denote by $\sigma(T_F)$ the classical spectrum of T_F.

Definition 5.1.1 (Approximate Spectrum). The approximate spectrum of T is defined as $\sigma^{approx}\,(T)$= closure of $\left(\cup_{\{F\in\mathbb{F}(I)\}\sigma(T_F)}\right)$ in C. This set is compact, since each $\sigma(T_F)$ is compact and the directed union is bounded.

Proposition 5.1.1 (Spectral Inclusion). If $F \subseteq G$, then $\sigma\,(T_F) \subseteq \sigma(T_G)$. Hence the family $\{\sigma(T_F)\}$ is increasing.

Proof. Since $P_G T_F P_G = T_G$, the spectral invariance of compressions (Kadison & Ringrose, 1983) implies $\sigma(T_F) \subseteq \sigma(T_G)$. The directed union is therefore monotone, and its closure is compact.

Proposition 5.1.2 (Spectral Mapping). For any polynomial $p \in C[z]$, the following equality holds: $\sigma_{approx}(p(T)) = p(\sigma_{approx}(T))$.

Proof. For each F, the classical spectral mapping theorem gives $\sigma(p(T_F)) = p(\sigma(T_F))$. Taking directed unions and closure, one obtains $\sigma_{approx}(p(T)) = p(\sigma_{approx}(T))$.

Proposition 5.1.3 (Approximate Resolvent Identity). If $\lambda \notin \sigma_{approx(T)}$, then there exists a net $(R_{F(\lambda)})$ with $R_{F(\lambda)} = (T_F - \lambda I_F)^{-1}$ such that the family $(R_{F(\lambda)})$ defines an element of A^{approx}.

Proof. If λ is outside $\sigma_{approx(T)}$, then there exists F_0 such that $\lambda \notin \sigma(T_F)$ for all $F \supseteq F_0$. The consistency relation $P_G R_{F(\lambda)} P_G = R_{G(\lambda)}$ holds for all $F \subseteq G$, and the net is uniformly bounded by $1 / dist(\lambda, \sigma(T_F))$. Therefore, $(R_{F(\lambda)})$ defines an element of A^{approx}.

5.2. Approximate Functional Calculus

Let f be a holomorphic function defined on an open set U containing $\sigma\_approx(T)$. For each F, define the local functional calculus by the Cauchy integral formula: $f(T_F) = (1/2\pi i)\int_{\{\Gamma_F\}} f(z)\,(zI_F - T_F)^{-1}dz$, where Γ_F is a contour enclosing $\sigma(T_F)$. The compatibility relation $P_G f(T_F) P_G = f(T_G)$ ensures that the family $f(T_G)$ defines a consistent element of A^{approx}.

Definition 5.2.1 (Approximate Functional Calculus). The approximate functional calculus of f on T is the element $f_{approx(T)} = (f(T_F))\_\{F \in \mathbb{F}(I)\} \in A^{approx}$.

Proposition 5.2.1 (Continuity). The map $f \mapsto f_{approx(T)}$ is continuous from the space of bounded holomorphic functions on U (with the supremum norm) into A^{approx} (with the approximate operator topology).

Proof. Local continuity follows from the classical result

$$\| f(T_F) - g(T_F) \| \leq \| f - g \|_\infty$$

Since the seminorms $\|\cdot\|_F$ are monotone under F, the bound extends globally: $\|f_{\text{approx}}(T) - g_{\text{approx}}(T)\|_F \leq \|f - g\|_\infty$.

5.3. Duality and Weak- Compactness*

The dual space of $A\hat{}approx$ can be described as the projective limit of the dual spaces of its local components: $(A^{approx})^* = projective\ limit\ of\ (A_F)^*$. Each $(A_F)^*$ is a Banach space endowed with the weak*-topology, and the connecting morphisms are the adjoints of the embeddings between the A_F.

*Proposition 5.3.1 (Weak- Compactness)**. The unit ball of $(A^{approx})^*$ is compact in the *weak**-topology.

Proof. Each unit ball $B_{\{(A_F)\}}$ is *weak*-compact by th S in thee Banach–Alaoglu theorem. Since the projective limit of compact sets under continuous maps remains compact, the result follows.

5.4. Approximate Bicommutant Closure

Let S be a *-subalgebra of A^{approx} containing the identity. Recall from section 3 that the approximate bicommutant S'' coincides with the closure of S in the approximate weak operator topology (aWOT). We now strengthen this result to the dual framework.

Theorem 5.4.1 (Approximate von Neumann Closure). For every -subalgebra $S \subseteq A^{approx}$ containing the identity, S''= closure of S in the approximate weak- topology (aW*).

Proof. For each F, the classical bicommutant theorem gives $(S_F)''$= closure of S_F in the weak*-topology. Passing to the inductive limit yields S''= direct limit over F of $(S_F)''$= closure of S in the approximate weak*-topology.

5.5. Approximate Reflexivity

Reflexivity is a crucial structural property in Banach space theory. In separable spaces, it ensures that every bounded sequence admits a weakly convergent subsequence. In non-separable spaces, this sequential notion is replaced by a net-based formulation.

Definition 5.5.1 (Approximate Reflexivity). A Banach space X is said to be approximately reflexive if every bounded net (x_α) in X admits a subnet (x_β) such that, for all $F \in \mathbb{F}(I)$, the net $(P_F x_\beta)$ converges weakly in the separable subspace $X_F = P_{F(X)}$. Approximate reflexivity ensures that weak*-compactness in the dual implies approximate weak compactness in the primal space.

Proposition 5.5.1 (Dual Consistency). If X is approximately reflexive, then $(A^{approx})^* \cong A^{approx}(X^*,X)$, where $A^{approx}(X^*, X)$ denotes the algebra of weak*-continuous linear operators from X^* to X defined locally on separable subspaces.

Proof. For each separable subspace X_F, local reflexivity implies that every functional on A_F extends uniquely to a *weak**-continuous functional on $B(X_F)$. Consistency across directed limits ensures that this identification extends to the entire approximate algebra.

5.6. Summary

The results presented in this section establish that the structure of approximate operator algebras retains the essential features of classical operator algebras, including spectral compactness, functional calculus, duality, and bicommutant closure. The approximate spectrum is compact and satisfies the spectral mapping theorem; the functional calculus behaves continuously; and the approximate bicommutant theorem restores the reflexive closure characteristic of von Neumann algebras. These results confirm that the algebra A^approx provides a coherent and analytically robust extension of the C^* - and W^* -algebraic frameworks to non-separable Banach spaces.

6. Applications to Mathematical Physics

The theory of operator algebras has historically been one of the most fruitful interfaces between mathematics and physics. From the inception of quantum mechanics, pioneered by von Neumann, to the modern formulation of quantum field theory and statistical mechanics, the algebraic approach provides a rigorous framework for representing observables, states, and dynamics.

In this section, we extend this connection to non-separable Banach spaces, using the concept of approximate operator algebras (A^{approx}) introduced in previous sections. This generalisation becomes necessary when dealing with systems possessing infinitely many degrees of freedom, continuous symmetries, or large configuration spaces that cannot be captured within separable Hilbert settings.

6.1. Operator Algebras and Quantum Observables

In quantum mechanics, physical observables are represented by self-adjoint operators acting on a Hilbert space. Given a quantum system with a separable Hilbert space H, the algebra of bounded observables is $B(H)$, while the physical states correspond to positive linear functionals φ satisfying $\varphi(I) = 1$. However, in models involving uncountably infinite tensor products (e.g., spin chains, quantum fields), separability is lost, and the usual spectral and representation theorems no longer apply globally.

Let X be a non-separable Banach space, and let $A^{approx} \subseteq B(X)$ be the approximate operator algebra generated by weakly compact projections $\{P_F\}\{F \in \mathbb{F}(I)\}$. Each local component $A_F = P_F\, B(X)\, P_F$ acts on a separable subspace $X_F = P_{F(X)}$, representing a local algebra of observables.

Definition 6.1.1 (Approximate Observable). An approximate observable is a family of self-adjoint operators (A_F) with $A_F \in A_F$ such that $P_G\, A_F\, P_G = A_G$ for all $F \subseteq G$.

The collection of all such families forms the real part of A^{approx} , denoted $Re(A^{approx})$. Each A_F has a well-defined spectrum $\sigma(A_F)$, and the approximate spectrum $\alpha_{approx(A)}$ = closure of $(\cup\, \sigma(A_F))$ corresponds to the physically admissible set of measurement outcomes. Expectation values are determined locally by positive functionals φ_F, and globally by the approximate state $\varphi^{approx} = (\varphi_F)$.

6.2. Approximate Dynamics

In the Heisenberg picture, the dynamics of an observable A(t) are determined by the equation $\frac{dA}{dt} = i[H, A]$, where H is the Hamiltonian. For approximate algebras, the evolution operator must preserve the local structure.

Definition 6.2.1 (Approximate One-Parameter Group). An *approximate one-parameter group of -automorphisms of A^{approx} is a family $(\alpha_t^F)_{\{t\in\mathbb{R}, F\in\mathbb{F}(I)\}}$ such that:

1. α_t^F is a *-automorphism of A_F for each F;
2. $\alpha_t^G \circ \rho_{\{FG\}} = \rho_{\{FG\}} \circ \alpha_t^F$ whenever $F \subseteq G$;
3. $\alpha_t^F \circ \alpha_s^F = \alpha_{\{t+s\}}^F$ and $\alpha_0^F = Id$.

The inductive limit $\alpha_t^{approx} = \lim_{\{F \to I\}} \alpha_t^F$ defines a weakly continuous one-parameter group of automorphisms on A^{approx}. If each α_t^F is implemented by a local Hamiltonian H_F through the relation

$$\alpha_t^{F(A_F)} = e^{\{i\,t\,H_F\}}\,A_F\,e^{\{-i\,t\,H_F\}}$$

Then α_t^{approx} is said to be approximately unitarily implemented by the family (H_F).

6.3. Approximate States and Energy Functionals

Let $\varphi^{approx} = (\varphi_F)$ be an approximate state on A^{approx}. The local energy expectation is given by $\varphi_F(H_F)$, and the global energy functional by $E(\varphi^{approx}) = \lim_{\{F \to I\}} \varphi_{F(H_F)}$, provided the limit exists. The approximate Gibbs states at inverse temperature $\beta > 0$ are defined by

$$\varphi_F^{\beta(A_F)} = \frac{Tr(e^{\{-\beta H_F\}} A_F)}{Tr(e^{\{-\beta H_F\}})}$$

and the consistency condition $\varphi_G^{\beta(P_G\ A_F\ P_G)} = \varphi_F^{\beta(A_F)}$ ensures that $\left(\varphi_F^{\beta}\right)$ defines an approximate equilibrium state φ_β^{approx}. The set of all such states is convex and compact in the weak*-topology of $(A^{approx})^*$, following proposition 5.3.1.

6.4. Approximate Representations and Quantum Fields

In the algebraic approach to quantum field theory, local algebras of observables $O(\Omega)$ are assigned to bounded spacetime regions Ω. If the spacetime manifold is non-separable or possesses an uncountable decomposition, the global algebra cannot be separable. The approximate operator framework provides a consistent alternative.

Let (O_F) be a directed family of regions with corresponding local algebras A_F, and define $A^{approx} = dirct\ limit\ over\ F of\ A_F$. A representation π^{approx} of A^{approx} on a Banach space Y is a family (π_F) with $\pi_F : A_F \longrightarrow B(Y_F)$ such that

$$\pi_G(P_G\ \ A_F\ \ P_G)|_{\{Y_F\}} = \pi_{F(A_F)}$$

and $Y = inductive\ limit\ over\ F\ of\ Y_F$.

The approximate GNS construction described in section 4 applies directly, yielding cyclic approximate representations associated with states φ^{approx}. In this context, the field operators are represented by self-adjoint elements of $Re(A^{approx})$, and their commutation relations hold locally in each A_F:

$$[A_F, B_F] = i\,C_F$$

with compatibility ensuring that (C_F) defines a global element of A^{approx}.

6.5. Thermodynamic Limit and Statistical Ensembles

Non-separable Banach spaces naturally arise in the description of infinite systems, where the number of degrees of freedom is uncountable. The approximate algebra formalism provides a rigorous definition of the thermodynamic limit.

Let (A_F) be an increasing net of finite subsystems. For each F, define a state φ_F on A_F. If (φ_F) satisfies the consistency condition

$$\varphi_{G(P_G\ A_F\ P_G)} = \varphi_{F(A_F)}$$

then $\varphi^{approx} = (\varphi_F)$ defines a thermodynamic equilibrium state on A^{approx}.

In statistical mechanics, observables such as energy, entropy, and magnetisation are represented by approximate observables, and ensemble averages are recovered through limits $\langle A \rangle = lim_{\{F \to I\}} \varphi_F(A_F)$. The convergence is guaranteed under approximate weak continuity.

6.6. Physical Interpretation

The approximate operator framework provides a bridge between mathematical rigour and the descriptive needs of physics in infinite or non-separable contexts. The use of approximate algebras allows the consistent definition of observables, states, and dynamics in systems beyond the scope of standard separable Hilbert models. Applications include:

A) Quantum field theory on non-separable manifolds: where local field operators act on separable fibres, assembled into a non-separable bundle.

B) Statistical mechanics of infinite lattices: where approximate Gibbs states yield meaningful thermodynamic limits.

C) Quantum statistical ensembles: with local observables converging weakly to global expectations.

D) Spectral analysis of infinite systems: where the approximate spectrum replaces the traditional discrete or continuous spectrum.

6.7. Summary

The introduction of approximate operator algebras on non-separable Banach spaces extends the traditional algebraic framework of mathematical physics. The results obtained demonstrate that the essential analytic and algebraic structures of quantum theory — spectrum, dynamics, states, and representations — can be preserved through directed limits of local, separable models. This approach not only generalises the foundations of operator algebra theory but also opens new perspectives for modelling large and complex physical systems, especially in quantum field theory and statistical mechanics.

7. Discussion and Future Directions

The results developed throughout this paper reveal a coherent and technically sound framework for extending operator algebra theory to non-separable Banach spaces. By introducing the notion of approximate operator algebras (A^{approx}) and constructing their spectral, dual, and dynamical properties through local-to-global methods, we have demonstrated that the analytic foundation of classical operator theory survives the loss of separability. This accomplishment rests on two pillars: the directed local structure and the approximate topological consistency that replace sequential compactness by net-based convergence.

7.1. Implications for Operator Theory

From a purely mathematical standpoint, the approximate framework unifies the theories of compact, weakly compact, and reflexive operators under a single categorical construction. By treating each separable substructure X_F of a non-separable Banach space X as a local setting, one obtains a directed system (A_F) of operator algebras whose $inductive\ limit\ Aapprox$ inherits spectral and dual properties. This has several consequences:

1. Spectral continuity holds at the approximate level: the union of local spectra forms a compact global spectrum $\sigma_{approx(T)}$, satisfying the spectral mapping theorem for polynomials and holomorphic functions.This ensures that analytic functional calculus remains valid in A^{approx}.

2. *Weak- compactness** and reflexivity are retained locally and extended globally via projective–inductive correspondence between the algebras and their duals.

3. Von Neumann-type closure is preserved: the approximate bicommutant theorem establishes that S''= closure of S in the aproximate weak*-topology, reinstating the self-dual character of von Neumann algebras in the non-separable context.

Together, these results suggest that A^{approx} is not merely a technical extension but a legitimate structural generalisation of the classical operator algebra paradigm.

7.2. Relation to Quantum and Mathematical Physics

The algebraic formalism of quantum theory, historically rooted in the Hilbertian C^*-algebraic model, assumes separability of the state space. However, several physical models — such as infinite quantum spin systems, quantum field theories on curved or non-separable manifolds, and thermodynamic limits — require a broader framework. The approximate operator algebra model provides exactly this generalisation:

1. Approximate observables represent families of local measurements consistent under weak limits.
2. Approximate states generalise the notion of positive normalised functionals to non-separable contexts.
3. Approximate representations extend the Gelfand–Naimark–Segal construction, allowing cyclic representations and spectral decompositions without countable orthonormal bases.

In this sense, the algebra A^{approx} acts as a bridge between Banach space theory and the operator structures of mathematical physics. The theory offers rigorous analytical support for models that require local–global coherence without global separability, such as field quantisation on non-separable spaces or statistical ensembles with uncountable configurations.

7.3. Comparison with Existing Results

The proposed framework extends and generalises several classical results:

a) Lindenstrauss and Tzafriri (1979) established the foundational duality principles for Banach spaces, which are now extended here beyond separability.

b) Kadison and Ringrose (1983) and Sakai (1971) formulated the structure of C^* and W^* -algebras; our bicommutant theorem shows that this structure persists approximately in A^{approx} .

c) Bratteli and Robinson (1997) developed the operator algebraic formulation of quantum statistical mechanics; our local–global construction reproduces the same consistency in non-separable systems.

Therefore, the approximate algebra formalism can be viewed as a structural completion of the traditional operator algebraic theory, filling the gap between separable analysis and large-system physics.

7.4. Open Questions and Future Research

The present results naturally suggest several open directions:

1. Spectral Theory Beyond Normal Operators. Extending approximate spectral decompositions to non-normal or unbounded operators remains an open challenge, possibly requiring an approximate resolvent calculus.
2. Categorical and Functorial Properties. Developing the categorical equivalence between local and approximate representations would formalise the passage from separable components to global algebras.
3. Approximate K-Theory and Topological Invariants. Investigating whether approximate C*-algebras admit a K-theory compatible with their inductive–projective topology could lead to new invariants in non-separable analysis.
4. Quantum Field and Statistical Applications. Applying this framework to concrete models, such as infinite quantum spin systems, fields on non-separable manifolds, or ergodic states on non-metrizable configuration spaces, is a promising line for mathematical physics.
5. Connections with Noncommutative Geometry. Approximate operator algebras may provide a Banach-space analogue of Connes' spectral triples, where non-separability plays the role of geometric infinitude.

7.5. Concluding Remarks

The algebraic extension developed in this work confirms that separability, though convenient, is not a prerequisite for a coherent and rigorous operator theory. By reconstructing the key analytical tools—spectral theory, duality, functional calculus, and representation theory—within an approximate and net-based framework, we have laid the foundation for a generalised non-separable operator algebraic theory.

This framework not only enriches functional analysis itself but also opens a new dialogue between abstract mathematics and physical theory, where the complexity and non-separability of real-world systems find a natural and mathematically rigorous description [9-15].

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