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On Bergman's Diamond Lemma for Ring Theory

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Abstract
This expository and review paper deals with the Diamond Lemma for ring theory, which is proved in the first section of G. M. Bergman, The Diamond Lemma for Ring Theory, Advances in Mathematics, 29 (1978), pp. 178–218. No originality of the present note is claimed on the part of the author, except for some suggestions and figures. Throughout this paper, I shall mostly use Bergman's expressions in his paper. In Remarks and Notes, the reader will find some useful information on this topic.

Keywords: Diamond Lemma, Ring Theory, Rewriting, Algebras, Associative Algebras, 2020 Mathematics Subject Classification: 16S15, 16-02.

1. Introduction

This is an expository and review paper which deals with the Diamond Lemma for ring theory, which is proved in the first section of G. M. Bergman, The Diamond Lemma for Ring Theory, Advances in Mathematics, 29 (1978) [1,2]. No originality of the present note is claimed on the part of the author, except for some suggestions and figures. Throughout this paper, I shall mostly use Bergman's expressions in his paper. In Remarks and Notes, the reader will find some useful information on this topic.

Suppose that R is an associative algebra with 1 over the commutative ring k, and that we have a presentation of R by a family X of generators and a family S of relations. Suppose that each relation $\sigma \in S$ has been written in the form $W_\sigma = f_\sigma$, where W_σ is a monomial (a product of elements of X) and f_σ is a k-linear combination of monomials, and that we want to use these relations as instructions for reducing expressions r for elements of R. That is, if any of the monomials occurring in the expression r contains one of the W_σ as a subword, we substitute f_σ for that subword, and we iterate this procedure as long as possible. In general, this process is not always well defined: at each step we must choose *which* reduction to apply to *which* subword of *which* monomial. Etcetera. So we are naturally led to the following questions:

(1) Under what conditions will such a procedure bring every expression to a unique irreducible form?
(2) Suppose that we have a set of suitable conditions satisfying (1). Does this yield then a canonical form for elements of R?

The Diamond Lemma is a general result of this sort due to Newman which was obtained in a graph-theoretic context [3]. Let G be an oriented graph. Here the vertices of G may be thought as expressions for the elements of some algebraic object (in our case, an associative algebra with 1 over the commutative ring k) and the edges as reduction steps (in our case, reductions using such a rule as $W_\sigma = f_\sigma$) going from one such expression to another one. Newman's result is the following [1]. suppose that
(i) The oriented graph satisfies *the descending chain condition*. That is, all positively oriented path in G terminate; and
(ii) Whenever two edges, e and e', proceed from one vertex a of G, there exists positively oritened paths p, p' in G leading from the end points b, b' of these edges to a common vertex c. (This condition is called the diamond condition.)

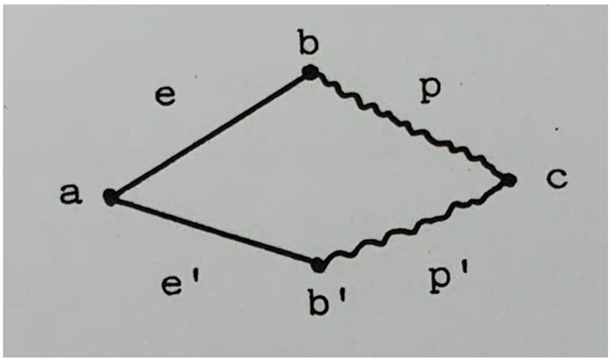


Figure 1: Does it look like a diamond?

Then every connected component C of G has a unique minimal vertex m_C.
This means that every maximal positively oriented path beginning at a point of C will terminate at m_C; is other words (in our context) that the given reduction procedures yield unique canonical forms for elements of the original algebraic object. The main theorem to be proved in the third section, namely the Diamond

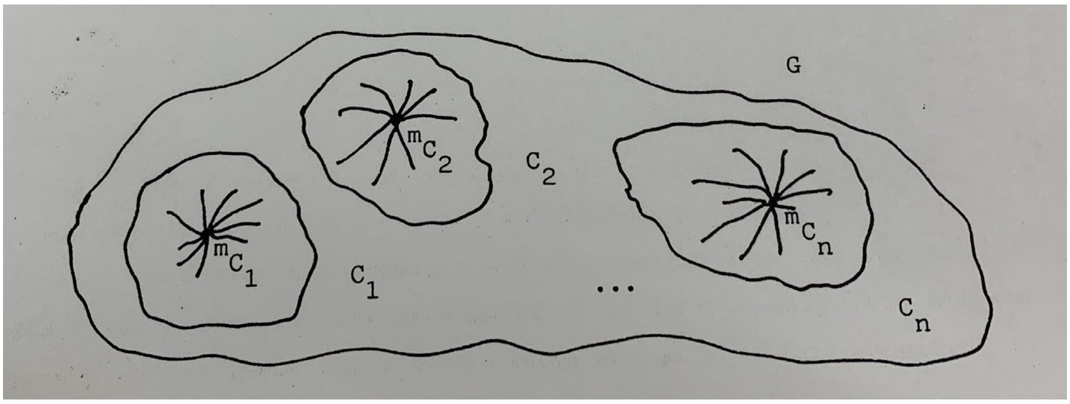


Figure 2: The connected components of G

Lemma for Ring Theory, is an analogue of the above observations for the case of associative rings, with reduction procedures of the form mentioned earlier. (For our argument in the sequel, we do not follow Newman's graph-theoretic formulation).

In the following section 2, we introduce a lot of definitions and prove some lemmas and propositions used for the proof of the Diamond Lemma. In the last fourth section, we give some suggestions on literatures and so on. We have Notes and Appendix at the end of this paper.

2. Preliminaries

Let k be a commutative associative ring with 1, X a set, $< X >$ the free semigroup with 1 on X, and $k < X >$ the free associative k-algebra on X, which is the semigroup algebra of $< X >$ over k.3
Let S be a set of pairs of the form $\sigma = (W_\sigma, f_\sigma)$

where $W_\sigma \in < X >, f_\sigma \in k < X >$. For any $\sigma \in S$ and $A, B \in < X >$, let $r_{A\sigma B}$ denote the *k-module* endomorphism of $k < X >$ that fixes all elements of $< X >$ other than $A_{W\sigma B}$, and that sends this basis element to $A_{f\sigma B}$. We call the given set S a reduction system, and the maps $r_{A\sigma B} : k < X > \to k < X >$ reductions.

We say that a reduction $r_{A\sigma B}$ acts trivially on an element a $\in k < X >$ if the coefficient of $r_{A\sigma B}$ in a is zero. An element a $\in k < X >$ is said to be irreducible if every reduction act trivially on a.

Proposition 2.1 *The irreducible elements of $k < X >$ form a k-submodule of $k < X >$, denoted by $k < X >_{irr}$.*
Proof. Let *a, b* be any irreducible elements of $k < X >$ and λ any element of k. Let r be a reduction, say $r = r_{A\sigma B}$. The coefficient of $r_{A\sigma B}$ in a and b is zero, respectively. Thus so is that of $r_{A\sigma B}$ in $a - b$ and λa. Trivially 0 is irreducible. This completes the proof.

A finite sequence of reduction $r_1,...,r_n (r_i = r_{A i \sigma i B i})$ is said to be final on a $\in k < X >$ if $r_n \cdots r_l(a) \in k < X > i_{rr}$.
An element a of $k < X >$ is called *reduction-finite* if for every infinite sequence $r_1, r_2, ...$ of reduction, ri acts trivially on $r_{i-1} \cdots r_1(a)$ for all sufficiently large i. If a is reduction-finite, then any maximal sequence of reductions ri acts nontrivially on $r_{i-1} \cdots r_1(a)$ is finite, and hence a final sequence.

Proposition 2.2 *The reduction-finite elements of $k < X >$ form a k-submodule of $k < X >$.*
Proof. Suppose that a and b are reduction-finite elements and λ an element of k. Then there are natural number i and i such that for every infinite sequence $r_1, r_2, ...$ of reductions, r_i and r_j act trivially on $r_{i-1} \cdots r_1(a)$ and $r_{j-1} \cdots r_1(b)$, respectively. Take $l = max(i,j)$. For every infinite sequence $r_1, r_2, ...$ of reductions, r_1 and r_i act trivially on $r_{l-1} \cdots r_1(a - b)$ and $r_{j-1} \cdots r_1(\lambda a)$, respectively. Thus, $a - b$ and λa are reduction-finite. 0 is clearly reduction-finite. This completes the proof.
We call an element $a \in k < X >$ *reduction-unique* if
(1) it is reduction-finite; and
(2) its images under all final sequences are the same. (This common value is denoted by $r_s(a)$).

Lemma 2.3 *(i) The set of reduction-finite elements of $k < X >$ form a k-submodule of $k < X >$, and rs is a k-linear map of this submodule into $k < X > i_{rr}$.*
(iii) Suppose $a,b,c \in k < X >$ are such that for all monomials A,B,C occurring with nonzero coefficient in a,b,c, respectiely, the product ABC is reduction-unique. (In particular this implies that abc is reduction-unique.) Let r be any finite composition of reductions. Then ar(b)c is reduction-unique, and $r_s(ar(b)c) = r_s(abc)$. (Note that 'finite' means 0 or ≥ 2. When r is a single reduction, ar(b)c should have the same property as that of abc.)

Proofr. (i) Suppose that $a,b \in k < X >$ are reduction-unique, and $\alpha \in k$. By Proposition 2.2, $\alpha a + b$ is reduction-finite. Let r be any composition of (finite) reductions final on $\alpha a + b$. Since a is reduction-unique, we can find a composition of (finite) reduction r' such that $r'r(a) = r_s(a)$, and similarly there is a composition of reductions r'' such that $r''r'r(b) = r_s(b)$. Because $r(\alpha a + b) \in k < X > i_{rr}$, we have

$$\begin{aligned} r(\alpha a + b) &= r''r'r(\alpha a + b) \\ &= \alpha r''r'r(a) + r''r'r(b) \\ &= \alpha r''r_s(a) + r_s(b) \\ &= \alpha r_s(a) + r_s(b) \end{aligned}$$

That is, images of $\alpha a + b$ under all such final sequenes of reductions are the same, i.e. $\alpha r_s(a) + r_s(b)$. Thus, $\alpha a + b$ is reduction-unique and so is $b - a$ with $\alpha = -1$. 0 is clearly reduction-unique. Therefore, the set of reduction-unique elements of $k < X >$ forms a k-submodule of $k < X >$. Since $r_s(\alpha a + b) = r(\alpha a + b)$, $r_s(\alpha a + b) = \alpha r_s(a) + r_s(b)$. For any reduction-unique elements of $k < X >$, $r_s(s) \in k < X >_{irr}$ is clear. Thus, r_s is a k-linear map of the module into $k < X >_{irr}$.

(ii) Suppose that the assumption of (ii) holds. And say

$$a = \sum_i \alpha_i A_i, \ \ b = \sum_j \beta_j B_i, \ \ c = \sum_l \gamma_l C_l.$$

So $abc = \sum_{i,j,l} \alpha_i \beta_j \gamma_l A_i B_j C_l$ and for any triple (i, j, l), $A_i B_j C_l$ is reduction-unique. So abc is reduction-unique and $r_s(abc) = \sum_{i,j,l} \alpha_i \beta_j \gamma_l r_s(A_i B_j C_l)$. Let r be any finite composition of reductions. It is sufficient to consider the case where a, b, c are monomials and r is a single reduction $r_{D\sigma E}$. In this case, $Ar_{D\sigma E}(B)C = r_{AD\sigma EC}$, which is the image of ABC under a reduction, hence is reduction-unique if ABC is so, with the reduced form. □

By an *overlap ambiguity* of S we mean a 5-tuple (σ, τ, A, B, C) with $\sigma, \tau \in S$ and $A, B, C \in < X > -\{1\}$, such that $W_\sigma = AB$, $W_\tau = BC$. We say that the overlap ambiguity (σ, τ, A, B, C) is *resolvable* if there exist compositions of reductions, r, r', such that $r(f_\sigma C) = r'(Af_\tau)$: in other words, $f_\sigma C$ and Af_τ can be reduced to a common expression (This corresponds to the diamond condition seen in the introduction).

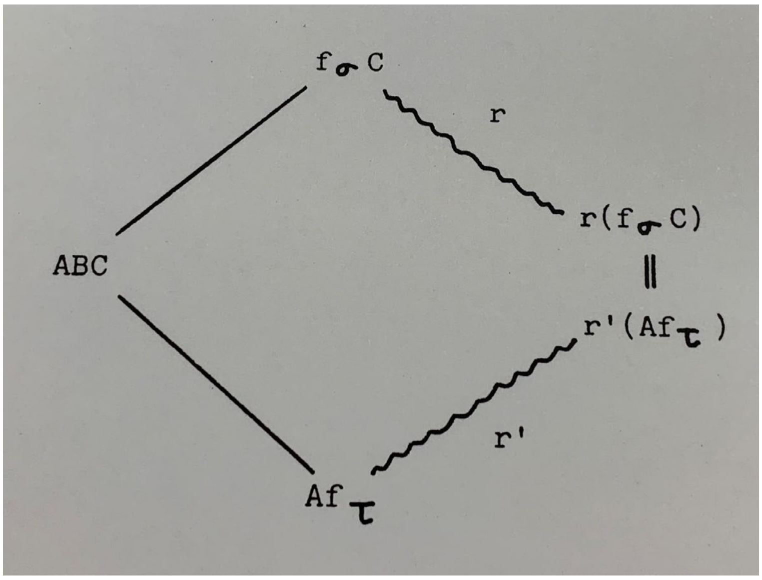


Figure 3: The diamond condition?

Similarly, a 5-tuple (σ, τ, A, B, C) with $\sigma \neq \tau \in S$ and $A, B, C \in < X >$ is called and *inclusion ambiguity*[4] if $W_\sigma = B$, $W_\tau = ABC$. The inclusion ambiguity is called *resolvable* if there exists compositions of reductions, r and r', such that $r(Af_\sigma B) = r'(f_\tau)$.

By a *semigroup partial ordering on* $< X >$, we mean a partial order ”$<$” such that $B < B' \Leftrightarrow ABC < AB'C$ for any $A, B, B', C \in k < X >$, and it is called *compatible with* S if for all $\sigma \in S$, f_σ is a (finite) linear combination of monomials $< W_\sigma$.

If $\leq$ is a semigroup partial ordering on $< X >$ compatible with the reduction system S, and A is any element of $< X >$, let I_A denote the submodule of $k < X >$ spanned by all elements $B(W_\sigma - f_\sigma)C$ such that $BW_\sigma C < A$. We say that an overlap (inclusion) ambiguity (σ, τ, A, B, C) is *resolvable relative to* $\leq$ if $f_\sigma C - Af_\tau \in I_{ABC}$ $(Af_\sigma C - f_\tau \in I_{ABC})$.

The following lemma is trivial. But it will be useful in what follows.

Lemma 2.4 *Let $a \in k < X >$. Suppose that a contains a monmial of the form $AW_\sigma B$ with a coefficient $\lambda (\neq 0) \in k$. Then we have*

$$r_{A\sigma B}(a) = a - \lambda A(W_\sigma - f_\sigma)B.$$

Proof. The lemma immediately follows from the following observation:

$$r_{A\sigma B}(\lambda AW_\sigma B) = \lambda Af_\sigma B = \lambda AW_\sigma B - \lambda A(W_\sigma - f_\sigma)B.$$

□

Let I denote the two-sided ideal of $k < X >$ generated by th elements $W_\sigma - f_\sigma (\sigma \in S)$. As a k-module, I is spanned by the products $A(W_\sigma - f_\sigma)B$.

Proposition 2.5 *Let $a \in k < X >$. Suppose that a is reduction-unique. Then, if $r_s(a) = 0$, then a is an element of I.*

Proof. If $a = 0$, the $0 \in I$ is trivial. So assume $a \neq 0$. Suppose that r is the composition of a sequence final on a and say,

$$r = r_{A_n \sigma_n B_n} \cdots r_{A_2 \sigma_2 B_2} r_{A_1 \sigma_1 B_1}.$$

Then, By Lemma 2.4, $r(a)$ is of the form $a - \sum \lambda_i A_i(W_{\sigma_i} - f_{\sigma_i})B_i$ with $\lambda_i(\neq 0) \in k$ for all i. Because a is reduction-unique, $r(a) = r_s(a)$, which implies $a = \sum \lambda_i A_i(W_{\sigma_i} - f_{\sigma_i})B_i$. Thus, $a \in I$. $\square$

3. The Diamond Lemma

The following theorem is called the Diamond Lemma for Ring Theory.

Theorem 3.1 *Let S be a reduction system for a free associative algebra $k < X >$ (a subset of $< X > \times k < X >$), and $\leq$ a semigroup partial ordering on $< X >$, compatible with S, and satisfies the descending chain condition. Then the following conditionsa are equivalent:*

(a) All ambiguities of S are resolvable.

(a') All ambiguities of S are resolvable relative to $\leq$.

(b) All elements of $k < X >$ are reduction-unique under S.

(c) A set of representatives in $k < X >$ for the elements of the algebra $R = k < X > /I$ determined by the generators and the relations $W_\sigma = f_\sigma$ ($\sigma \in S$) is given by the k-submodule $k < X >_{irr}$ sapanned by the S-irreducible monomials of $< X >$.

When these conditions hold, R may be identified with the k-module $k < X >_{irr}$, made a k-algebra by the multiplication $a \cdot b = r_S(ab)$.

Proof. First we see from our general hypothesis, that every element of $< X >$ is reduction-finite. We can prove this formally by induction with respect to the partial ordering with the descending chain condition $\leq$. But here we prove it informally to make the situation clearer. For illustrative puposes, suppose that $a \in < X >$ has a monomial of the form $AW_\sigma B$ ($\sigma \in S$). By a reduction $r_{A\sigma B}$, $r_{A\sigma B}(a)$ has a monomial of the form $Af_\sigma B$. Since $\leq$ is compatible with S, f_σ is of the form $\sum \lambda_i W_i^\sigma$ with $W_i^\sigma < W_\sigma$ for any i. So every monomial of $Af_\sigma B$ is of the form $AW_i^\sigma B$ for some i. If the monomial $AW_i^\sigma B$ contains a subword W_τ ($\tau \in S$), say $AW_i^\sigma B = A'W_\tau B'$, by a reduction $r_{A'\tau B'}$, $r_{A'\tau B'}r_{A\sigma B}(a)$ has a monomial of the form $A'f_\tau B'$. By compatibility of $\leq$, f_τ is again a (finite) linear combination of monomials $< W_i^\sigma$, say $f_\tau = \sum \mu_j W_j^\tau$ with $W_j^\tau < W_i^\sigma$ for any j. So $r_{A'\tau B'}r_{A\sigma B}(a)$ has a monomial of the form $A'W_j^\tau B'$ for all j. If we iterate this process, we will get a sequence of monomials, for example $W_\sigma > W_i^\sigma > W_j^\tau > \cdots$. All of such sequences must be finite because of the descending chain condition. It is also clear that the number of all the possible sequences is finite. (See the following figure.) Therefore, a is reduction-finite. Since every element of $< X >$ is reduction-finite, hence so is every element of $k < X >$.

Next we prove (b) $\Leftrightarrow$ (c). We note first that (c) simply says

$$k < X > \simeq \; k < X >_{irr} \oplus I.$$

Assuming (b), we show that the following sequence,

$$(*) \quad 0 \longrightarrow I \xrightarrow{i} k < X > \xrightarrow{r_S} k < X >_{irr} \longrightarrow 0,$$

is a short exact sequence k-module homomorphisms, where i is inclusion map. If so, it is immediate to conclude $k < X > \simeq \; k < X >_{irr} \oplus I$. That is, it follows from $r_S i = id_{k<X>_{irr}}$ (see Appendix).

Case 1: inclusion map i is injective. So $0 \to I \to k < X >$ is exact.

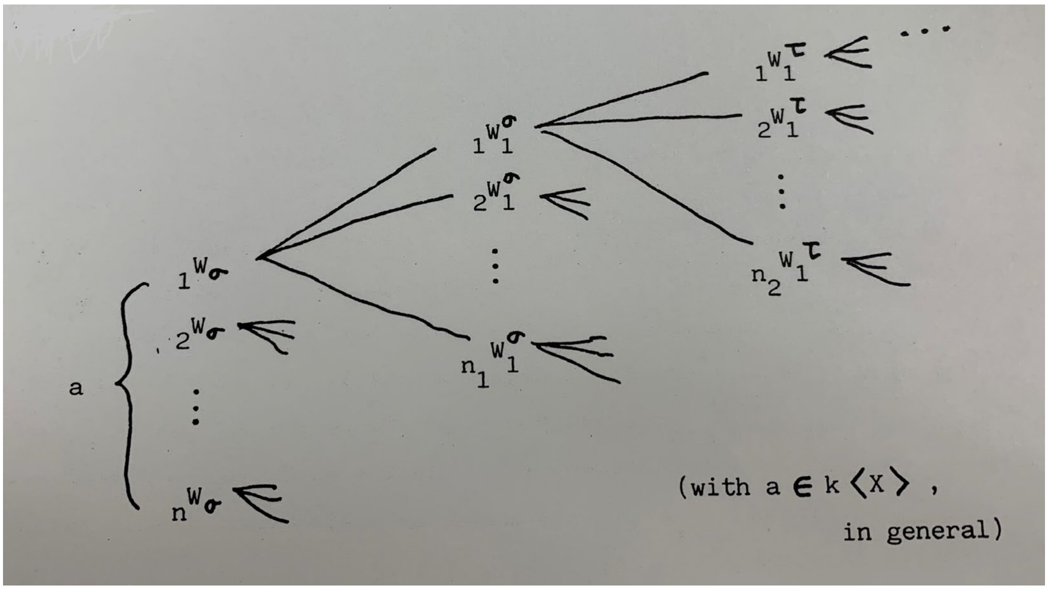


Figure 4: The number of all the possible sequences is finite.

Case 2: $I = im(i) \subseteq ker(r_S)$ is easily seen because for all A, B, σ,

$$\begin{aligned} r_S(A(W_\sigma - f_\sigma)B) &= r_S(AW_\sigma B) - r_S(Af_\sigma B) \\ &(r_S \text{ is } k\text{-linear. (Lemma 2.3.(i))} \\ &= r_S(AW_\sigma B) - r_S(Ar_{1\sigma 1}(1W_\sigma 1)B) = 0 \\ &(\text{By Lemma 2.3.(ii), } A, W_\sigma, B \text{ are reduction-unique by (b).)} \end{aligned}$$

By Proposition 2.5, $ker(r_S) \subseteq im(i) = I$ is obvious. Thus, $im(i) = ker(r_s)$. This is nothing but the exactness of $I \to k < X > \to k < X >_{irr}$.

Case 3: r_S is surjective, since for any $x \in k < X >_{irr}$, $r_S(x) = x$. This means that $k < X > \to k < X >_{irr} \to 0$ is exact.

Conversely, we assume (c) and suppose $a \in k < X >$ can be reduced to either of $b, b' \in K < X >_{irr}$. Then we have $b - b' \in k < X >_{irr} \cap I = \{0\}$, which proves (b).

The final comment in the statement of the theorem is now clear. From the above, $R = k < X > /I \simeq k < X >_{irr}$ is immediate and $k < X >_{irr}$ is a k-algebra with the multiplication $a \cdot b = r_S(ab)$ by the following (1) and (2):

(1) $k < X >_{irr}$ is a k-submodule of $k < X >$ by Lemma 2.3.(i). Thus it is a k-module.

(2) For any $a, b \in k < X >_{irr}$ and any $\alpha \in k$,

$$\alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b),$$

holds, because we see that

$$\begin{aligned} (\alpha a) \cdot b &= r_S((\alpha a) \cdot b) \\ &= r_S(\alpha(ab)) \\ &((\alpha a)b = \alpha(ab) \text{ holds since } k < X > \text{ is a } k\text{-algebra.)} \\ &= \alpha r_S(ab) \ \ (r_S \text{ is } k\text{-linear.)} \\ &= \alpha(ab) \end{aligned}$$

and with similar remarks, $a \cdot (\alpha b) = \alpha(a \cdot b)$.

We next deal with the proof of (b) $\Leftrightarrow$ (a). Suppose (b). We consider only the case of overlap ambiguities, because those of inclusion ones are similarly taken case of. Let (σ, τ, A, B, C) be any overlap ambiguity. $f_\sigma C$, Af_τ are reduction-unique by (b). So we may take compositions of reductions r and r' which are final on $f_\sigma C$ and Af_τ, respectively. By (b), ABC is reduction-unique. Moreover, $rr_{1\sigma C}$ and $r'r_{A\tau 1}$ are obviously final on ABC. So we see that

$$r(f_\sigma C) = rr_{1\sigma C}(ABC) = r_S(ABC) = r' r_{A\tau 1}(ABC) = r(Af_\tau).$$

This means that the ambiguity is resolvable.

In this paragraph, we porve (a) $\Rightarrow$ (a'). We assume (a). First we consider the case of overlap ambiguities.

Let (σ, τ, A, B, C) be any overlap ambiguity of S. By (a), it is resolvable. That is, there are compositions of reductions r and r' such that $r(f_\sigma C) = r'(Af_\tau)$, say

$$r = r_{D_n \sigma_n E_n} \cdots r_{D_1 \sigma_1 E_1} \text{ and } r' = r_{D'_m \tau_m E'_m} \cdots r_{D'_1 \tau_1 E'_1}.$$

By Lemma 2.4, we obtain,

$$r(f_\sigma C) = f_\sigma C - \sum_{i=1}^{n} \lambda_i D_i (W_{\sigma_i} - f_{\sigma_i}) E_i$$

and

$$r'(Af_\tau) = Af_\tau - \sum_{i=1}^{m} \mu_i D'_i (W_{\tau_i} - f_{\tau_i}) E'_i$$

with $\lambda_i(\neq 0), \mu_j(\neq 0) \in k$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Under the diamond condition in our sense, we may have then,

$$f_\sigma C - Af_\tau = \sum_{i=1}^{n} \lambda_i D_i (W_{\sigma_i} - f_{\sigma_i}) E_i - \sum_{i=1}^{m} \mu_i D'_i (W_{\tau_i} - f_{\tau_i}) E'_i.$$

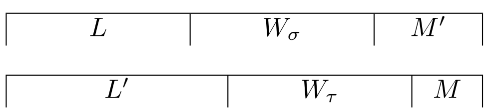
Further, it is not so difficult to (I) $D_i W_{\sigma_i} E_i < W_\sigma C = ABC$ and (II) $D'_j W_{\tau_j} E'_j < W_\tau = ABC$ for any $1 \leq i \leq n$ and any $1 \leq j \leq m$. For the verification of (I), we show only the case of $i = 1$. The rest of the proof is taken care of tby induction. Since $\leq$ is compatible with S, if f_σ is of the form $\sum \alpha_i Z_i$, then $Z_1 < W_\sigma$ holds for any i. Further, $Z_i < W_\sigma$ leads to $Z_i C < W_\sigma C = ABC$ for all i. So we must have $D_1 W_{\sigma_1} E_i = Z_i C$ for some i. Hence, $D_1 W_{\sigma_1} E_1 < ABC$. We can verify (II) similarly. So we omit the verification. For inclusion ambiguity, we can also show resolvability relative to $\leq$ in a completely similar way. So this is left to the reader.

In this paragraph, we take care of the last implication to be shown, i.e. (a') $\Leftarrow$ (b). It suffices to prove all monomials $D \in < X >$ reduction-unique, since the reduction-unique elements of $k < X >$ form a submodule (Lemma 2.3.(i)). That is, if every monomial of $< X >$ is reduction-unique, then $k < X >_{irr} = k < X >$. We assume inductively that all monomials $< D$ are reduction-unique. Thus the domain of r_S includes the submodule spanned by all these monomials, so the kernel of r_S contains I_D. That is, if $a \in ker(r_S)$, then by Proposition 2.5, a is of the form $\sum \lambda_i A_i (W_{\sigma_i} - f_{\sigma_i}) B_i$ with $A_i W_{\sigma_i} B_i < D$ for any i, which means $a \in I_D$. We must now show that given any two reductions $r_{L\sigma M'}$ and $r_{L'\tau M}$ each acting nontrivially on D (and hence each sending D to a linear combination of monomials $< D$), we will have

$$r_S(r_{L\sigma M'}(D)) = r_S(r_{L'\tau M}(D)).$$

We have to check three case for that, according to the relative locations of the subwords W_σ and W_τ in the monomial D. We may assume without loss of generality that $length(L) \leq length(L')$, in other words, that the indicated copy of W_σ in D begins no later than that of W_τ.

Case 1: The subwords W_σ and W_τ overlap in D, neither contains the other, figured as follows: under the condition $length(L) \leq length(L')$,



Tnen $D = LABCM$, where (σ, τ, A, B, C) is an overlap ambiguity of S, i.e. $W_\sigma = AB, W_\tau = BC, \sigma, \tau \in S, A, B, C \in < X > -\{1\}$. Then,

$$
\begin{aligned}
F &:= r_{L\sigma M'}(D) - r_{L'\tau M}(D)\\
&= Lf_\sigma CM - LAf_\tau M\\
&= L(f_\sigma CM - Af_\tau M)\\
&= L(f_\sigma C - Af_\tau)M. \quad \cdots\cdots \quad (1.1)
\end{aligned}
$$

By (a') every overlap ambiguity is resolvable relative to $\geq$. So we have $f_\sigma C - Af_\tau \in I_{ABC}$ by definition. That is,

$$f_\sigma C - Af_\tau = \sum \lambda_i D_i(W_{\sigma_i} - f_{\sigma_i})E_i$$

with $D_i W_{\sigma_i} E_i < ABC$ for any i. Substitute this to (1.1). Then we get,

$$F = \sum \lambda_i LD_i(W_{\sigma_i} - f_{\sigma_i})E_i M. \quad \cdots\cdots \quad (1.2)$$

Since $\geq$ is a semigroup ordering, the following inequality,

$$LD_i(W_{\sigma_i})E_i M < LABCM. \quad \cdots\cdots \quad (1.3)$$

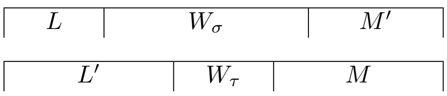
holds by $D_i(W_{\sigma_i})E_i < ABC$. From (1.2) and (1.3), it follows that $F \in I_{LABCM}$. Thus $r_S(F) = 0$, in other words,

$$r_S(r_{L\sigma M'}(D) - r_{L'\tau M}(D)) = 0,$$

so $r_S(r_{L\sigma M'}(D)) = r_S(r_{L'\tau M}(D))$.

The next case is similarly dealt with as the case 1. But we shall work it out for the sake of the reader.

Case 2: One of the subwords W_σ, W_τ is contained in the other. By $length(L) \leq length(L')$, we have the following case where W_σ contains W_τ, figured below.



Then $D = LABCM'$, $CM' = M$ and $L' = LA$, where (σ, τ, A, B, C) is an inclusion ambiguity of S, i.e. $W_\tau = B, W_\sigma = ABC$ with $\tau \neq \sigma \in S, A, B, C \in < X >$. Then,

$$
\begin{aligned}
F &:= r_{L\sigma M'}(D) - r_{L'\tau M}(D)\\
&= Lf_\sigma M' - LAf_\tau CM'\\
&= L(f_\sigma - Af_\tau C)M'. \quad \cdots\cdots \quad (2.1)
\end{aligned}
$$

By (a') we know that every inclusion ambiguity is resolvable relative to $\geq$. So we get $Af_\tau C - f_\sigma \in I_{ABC}$ by definition. That is,

$$Af_\tau C - f_\sigma = \sum \lambda_i D_i(W_{\sigma_i} - f_{\sigma_i})E_i$$

with $D_i W_{\sigma_i} E_i < ABC$ for any i. By substituing this to (2.1), we obtain,

$$F = \sum (-\lambda_i) LD_i(W_{\sigma_i} - f_{\sigma_i})E_i M'. \quad \cdots\cdots \quad (2.2)$$

Because $\geq$ is a semigroup ordering, the following inequality,

$$LD_i(W_{\sigma_i})E_i M' < LABCM. \quad \cdots\cdots \quad (2.3)$$

holds by $D_i(W_{\sigma_i})E_i < ABC$. From (2.2) and (2.3), we get $F \in I_{LABCM'} = I_D$, so $r_S(F) = 0$. That is,

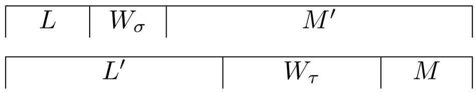
$$r_S(r_{L\sigma M'}(D) - r_{L'\tau M}(D)) = 0,$$

thus

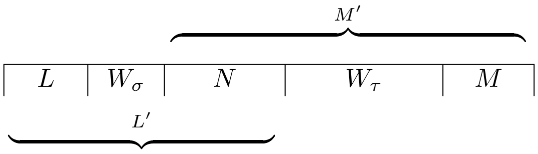
$$r_S(r_{L\sigma M'}(D)) = r_S(r_{L'\tau M}(D)).$$

The following is our last case to check, with which we complet the whole proof of the Diamond Lemma.

Case 3: We consider the case where W_σ and W_τ is disjoint. By the condition on thelength of L and L', $length(L) \leq length(L')$, the case is figured below.



So we may assume $D = LW_\sigma NW_\tau M$, i.e.,



is our present case with

$$r_{L\sigma M'}(D) = Lf_\sigma NW_\tau M, \quad (M' = NW_\tau M)$$

and

$$r_{L'\tau M}(D) = LW_\sigma Nf_\tau M. \quad (L' = LW_\tau N)$$

By the general assumption, the ordering $\leq$ is compatible with S. So f_σ can be witten as a linear combination of monomials $< W_\sigma$, say $f_\sigma = \sum \lambda_i Z_i$ with $Z_i < W|_\sigma$, $\lambda_i (\neq 0) \in k$. The ordering is a semigroup ordering, so for any i, we have $LZ_iNW_\tau M < LW_\sigma NW_\tau M = D$ from $Z_i < W_\sigma$. By induction hypothesis, $LZ_iNW_\tau M$ is reduction=unique for all i. Let $a = 1, c = 1$ and $b = Lf_\sigma NW_\tau M = \sum \lambda_i LZ_iNW_\tau M$. Then, for all monomials A, B, C occurring with non-zero coefficient in a, b, c, i.e. $1, LZ_iNW_\tau M(\forall i), 1$, respectively, the product ABC, namely $LZ_iNW_\tau M(\forall i)$ is reduction-unique. Apply now Lemma 2.3.(ii) to such a, b, c with $r = r_{(Lf_\sigma N)\tau M}$. Then we have $ar(b)c$ is reduction-unique and $r_S(ar(b)c) = r_S(abc)$. In other words, $Lf_\sigma Nf_\tau M$ is reduction-unique and

$$r_S(Lf_\sigma Nf_\tau M) = r_S(Lf_\sigma NW_\tau M). \quad \cdots\cdots \quad (3.1)$$

Similarly, we can obtain,

$$r_S(Lf_\sigma Nf_\tau M) = r_S(LW_\sigma Nf_\tau M). \quad \cdots\cdots \quad (3.2)$$

From (3.1) and (3.2), it follows that $r_S(Lf_\sigma NW_\tau M) = r_S(LW_\tau Nf_\tau M)$, which implies $r_S(r_{L\sigma M'}(D)) = r_S(r_{L'\tau M}(D))$. □

The following corollary may be reserved for the reader to prove.

Corollary 3.2 *Let $k < X >$ be a free associative algebra, and "$\leq$" a semigroup partial ordering of $< X >$ with the descending chain condition.*

If S is a reduction system on $k < X >$ compatible with $\leq$ and having no ambiguities, then the set of k-algebra relations $W_\sigma = f_\sigma$ ($\sigma \in S$) is independent.

More generally, if $S_1 \subseteq S_2$ are reduction systems, such that S_2 is compatible with $\leq$ and all its ambiguities are resolvable, and if S_2 contains some σ such that W_σ is irreducible with respect to S_1, then the inclusion of ideals associated with these systems, $I_1 \subseteq I_2$, is strict.

4. Remarks

First I would like to recommend the reader to read the original paper [2] of Bergman, because it is written with a broad perspective over a lot of algebraic structures, where the reader will find many interesting materials.

I have to mention at least that there is the correction and updates for the paper. Refer to Bergman [3].

The Diamond Lemma has another origin, although Newman [10] is already mentioned. For that, refer to Bokut et al [4] and Shirshov [11]. Also see Matveev [9]. Historically, Shirshov [11] gave the present lemma first for Lie algebras. Someone calls the Diamond Lemma Schirshov-Bergman's diamond lemma.

To get more recent trends for the Diamond Lemma, I would like to cite, among others, Chenavier [5], Chenavier and Lucas [6], Elias [7] and Tsuchioka [12]. There the reader will find much more information on the lemma and see some practices as an application to representation theory, as an example.

Recently, there is a trend about Composition-Diamond Lemma. But I do not touch on it here. I shall have an opportunity in another occasion.

I think the Diamond Lemma and its techniques can be applied not only to mathematics but also to many scientific fields.

Notes

[1]As a remark for logicians, Newman's paper [10] is closely related to the theory of λ-calculi. This article also contains an interesting observation about a relation of weak Church-Rosser and Church-Rosser properties (see Barendregt [1, p. 58]).

[2]Newman's original formulation and terminology for the Diamond Lemma is different from the one in the introduction of the present note (see Newman [10] for the details.)

[3]Given such k, X in the context, the semigroup algebra of $< X >$ over k is called the *free* (associative) k-*algebra* on X.

[4]Inclusion ambiguities are, in a sense, always avoidable. Suppose that S is a reduction system for a free algebra $k < X >$. Let us construct a subset $S' \subseteq S$ by (1) deleting all $\sigma \in S$ such that W_σ contains a *proper* subword of the form $W_\tau (\tau \in S)$, and (2) whenever more than one element $\sigma_1, \sigma_2, \cdots \in S$ act on the same monomial (i.e. $W_{\sigma_1} = W_{\sigma_2} = \dots$) dropping all but one of the σ_i from S. Then $S' \subseteq S$ will have the property that $a \in k < X >$ is reducible under S. But from this it follows that if $a \in k < X >$ is reduction-unique under S, then so is a under S' and $r'_S(a) = r_S(a)$. Hence if S is such that every element of $k < X >$ is reduction-unique under it, then S' has the same property and $r_{S'} = r_S$. Therefore, S', which has no inclusion ambiguities, defines the same ring and the same canonical form as S. (This remark is due to Bergman [2, p. 192].)

[5]As an application of the lemma, Bergman shows, for example an alternative proof of Poincareé-Birkoff-Witt Theorem in [2, p.186], which gives a basis of the universal enveloping algebra $U(g)$ of g if we know a basis of a Lie algebra g. Also see Varadarajan [13].

Appendix

In this appendix, we show a well-known basic theorem for modules without a proof, which is used, in the proof of Theorem 3.1, namely the Diamond Lemma, in the third section.

Theorem 4.1 *Let R be a ring and $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ a short exact sequence of R-module homomorphism. Then the following statements are equivalent.*

(1) *There is an R-module homomorphism $h : C \longrightarrow B$ with $gh = id_C$.*

(2) *There is an R-module homomorphism $k : B \longrightarrow A$ with $kf = id_A$.*

(3) *The given sequence is isomorphic (with identity maps on A and C) to the direct sum short exact sequence*

$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0,$$

in particular $B \simeq A \oplus C$.

Proof. See for example Hungerford [8]. □

Ackowledgemant
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