

Odd Log-Logistic Rayleigh Distribution with Applications

Sadiq Abubakar Ismaila^{1*} and Bello Ibrahim Monday²

¹Ahmadu Bello University Zaria, Kaduna

²University of Calabria, Kaduna

*Corresponding Author

Sadiq Abubakar Ismaila, Ahmadu Bello University Zaria, Kaduna.

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Abstract

In this article, a new extended Rayleigh distribution called the Odd Log-Logistic Rayleigh (OLLR) Distribution is developed. A two parameter distribution derived by incorporating the Rayleigh distribution in the Odd Log-Logistic – Generalized (OLL-G) family of distributions introduced by Gleaton and Lynch (2006). Various statistical properties of the proposed distribution are derived. The flexibility of the OLLR distribution is assessed using two real-life datasets. Results show that the OLLR provides a superior fit relative to its competitors across both datasets indicating it is a competitive and flexible alternative for lifetime data modelling.

Keywords: Scale Parameter, Monte Carlo Simulation, Hazard Function, Survival Function

1. Introduction

Statistics Distributions plays a central role in modelling real world phenomena as real-life datasets exhibits different characteristics such as skewness, kurtosis, and varying hazard rate shapes. The choice of an adequate distribution to represent such data therefore has a direct effect on the accuracy of any statistical inference drawn from it. Classical lifetime distributions are often rigid to capture the diverse hazard rate behaviours observed in applied fields such as reliability engineering, survival analysis, and biomedical research. To address this limitation, several methods for generating flexible distributions have been developed, through techniques such as compounding, and convolution. These generalized families extend classical distributions, allowing them to better capture skewness and varying hazard shapes. One such baseline distribution commonly used in reliability and lifetime modelling is the Rayleigh distribution (Rayleigh, 1880), valued for its simplicity and single-scale parameter, though its lack of a shape parameter limits its flexibility in fitting datasets with more complex hazard behavior. To overcome this, generator families such as the Odd Log-Logistic Generalized (OLL-G) family (Gleaton & Lynch, 2006) have been used to extend baseline distributions by introducing an additional shape parameter, thereby improving tail behaviour and adaptability to a wider range of empirical data patterns. Related work, such as the Logistic Rayleigh distribution (Chaudhary & Kumar, 2020), reflects the ongoing interest in extending the Rayleigh distribution for improved applicability. Motivated by this, this study proposes the Odd Log-Logistic Rayleigh (OLLR) distribution, obtained by incorporating the Rayleigh distribution as the baseline in the OLL-G family. The resulting two parameter distribution is examined structurally and statistically, with its mathematical properties, estimation procedure, and practical performance compared against existing competitors to establish its usefulness for real-world lifetime data applications.

1.1. The Rayleigh Distribution

The Cumulative Distribution Function (CDF) and Probability Density Function (PDF) of the Rayleigh distribution is given as:

$$F(x; \theta) = 1 - e^{-\frac{x^2}{2\theta^2}}; \quad x > 0, \theta > 0 \quad (1)$$

$$f(x; \theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}; x > 0, \theta > 0 \quad (2)$$

where Θ is the Scale parameter.

2. The Odd Log-Logistic Generalized (OLL-G) Family of Distributions

The Odd Log-Logistic Generalized (OLL-G) Family of distributions was introduced by Gleaton and Lynch (2006). The CDF and PDF of the family are expressed as:

$$F(x; \beta, \xi) = \frac{G(x; \xi)^\beta}{G(x; \xi)^\beta + \bar{G}(x; \xi)^\beta}; x \in \mathbb{R}, \beta > 0 \quad (3)$$

$$f(x; \beta, \xi) = \frac{\beta g(x; \xi) G(x; \xi)^{\beta-1} \bar{G}(x; \xi)^{\beta-1}}{(G(x; \xi)^\beta + \bar{G}(x; \xi)^\beta)^2}; x \in \mathbb{R}, \beta > 0 \quad (4)$$

where $G(x; \xi)$, and $g(x; \xi)$ is the CDF and PDF of the baseline distribution respectively. The function $\bar{G}(x; \xi)$ defined by $\bar{G}(x; \xi) = 1 - G(x; \xi)$ is the survival function of the baseline distribution, ξ is a $Q \times 1$ vector of parameters that is dependent on the parameters of the baseline distribution and β is the positive shape parameter.

3. The Odd Log-Logistic Rayleigh (OLLR) Distribution

By incorporating the Rayleigh distribution as the baseline distribution into the OLL generator, the CDF and PDF of the OLLR distribution are obtained in Equation (5) and (6) respectively as:

$$F(x; \beta, \theta) = \frac{\left(1 - e^{-\frac{x^2}{2\theta^2}}\right)^\beta}{\left(1 - e^{-\frac{x^2}{2\theta^2}}\right)^\beta + e^{-\frac{\beta x^2}{2\theta^2}}}; x > 0, \beta > 0, \theta > 0 \quad (5)$$

$$f(x; \beta, \theta) = \frac{\beta x e^{-\frac{\beta x^2}{2\theta^2}} \left(1 - e^{-\frac{x^2}{2\theta^2}}\right)^{\beta-1}}{\theta^2 \left(\left(1 - e^{-\frac{x^2}{2\theta^2}}\right)^\beta + e^{-\frac{\beta x^2}{2\theta^2}} \right)^2}; x > 0, \beta > 0, \theta > 0 \quad (6)$$

where β and θ are the shape and scale parameters that determine the tail behavior and spread of the distribution respectively. The survival function and hazard rate function are demonstrated respectively below as:

$$S(x; \beta, \theta) = \frac{e^{-\frac{\beta x^2}{2\theta^2}}}{\left(1 - e^{-\frac{x^2}{2\theta^2}}\right)^\beta + e^{-\frac{\beta x^2}{2\theta^2}}}; x > 0, \beta > 0, \theta > 0 \quad (7)$$

$$\tau(x; \beta, \theta) = \frac{\beta x \left(1 - e^{-\frac{x^2}{2\theta^2}}\right)^{\beta-1}}{\theta^2 \left(\left(1 - e^{-\frac{x^2}{2\theta^2}}\right)^\beta + e^{-\frac{\beta x^2}{2\theta^2}} \right)}; x > 0, \beta > 0, \theta > 0 \quad (8)$$

Validity Test: For $f(x; \beta, \theta)$ to be a valid PDF, the integral $\int_0^\infty f(x; \beta, \theta) = 1$ must hold TRUE.
Recall from the fundamental theorem of Calculus;

$$\int_0^\infty f(x; \beta, \theta) = \lim_{x \rightarrow \infty} F(x; \beta, \theta) - \lim_{x \rightarrow 0} F(x; \beta, \theta) \quad (9)$$

substituting the value as x reaches infinity ($x \rightarrow \infty$) and as x tends towards zero ($x \rightarrow 0$) into the CDF in Equation (5) as follows:

$$\int_0^\infty f(x; \beta, \theta) = \lim_{x \rightarrow \infty} \left[\frac{\left(1 - e^{-\frac{x^2}{2\theta^2}}\right)^\beta}{\left(1 - e^{-\frac{x^2}{2\theta^2}}\right)^\beta + e^{-\frac{\beta x^2}{2\theta^2}}} \right] - \lim_{x \rightarrow 0} \left[\frac{\left(1 - e^{-\frac{x^2}{2\theta^2}}\right)^\beta}{\left(1 - e^{-\frac{x^2}{2\theta^2}}\right)^\beta + e^{-\frac{\beta x^2}{2\theta^2}}} \right] \quad (10)$$

as, $x \rightarrow \infty$, $e \rightarrow 0$ and also as $x \rightarrow 0$, $e^0 \rightarrow 1$. Then;

$$= \left[\frac{(1 - e^{-\infty})^\beta}{(1 - e^{-\infty})^\beta + e^{-\infty}} \right] - \left[\frac{(1 - e^0)^\beta}{(1 - e^0)^\beta + e^0} \right] = \left[\frac{(1 - 0)^\beta}{(1 - 0)^\beta + 0} \right] - \left[\frac{(1 - 1)^\beta}{(1 - 1)^\beta + 1} \right] = 1 \quad (11)$$

Hence, the OLLR is a valid probability distribution since the integral of the PDF with respect to the range space integrates to 1.

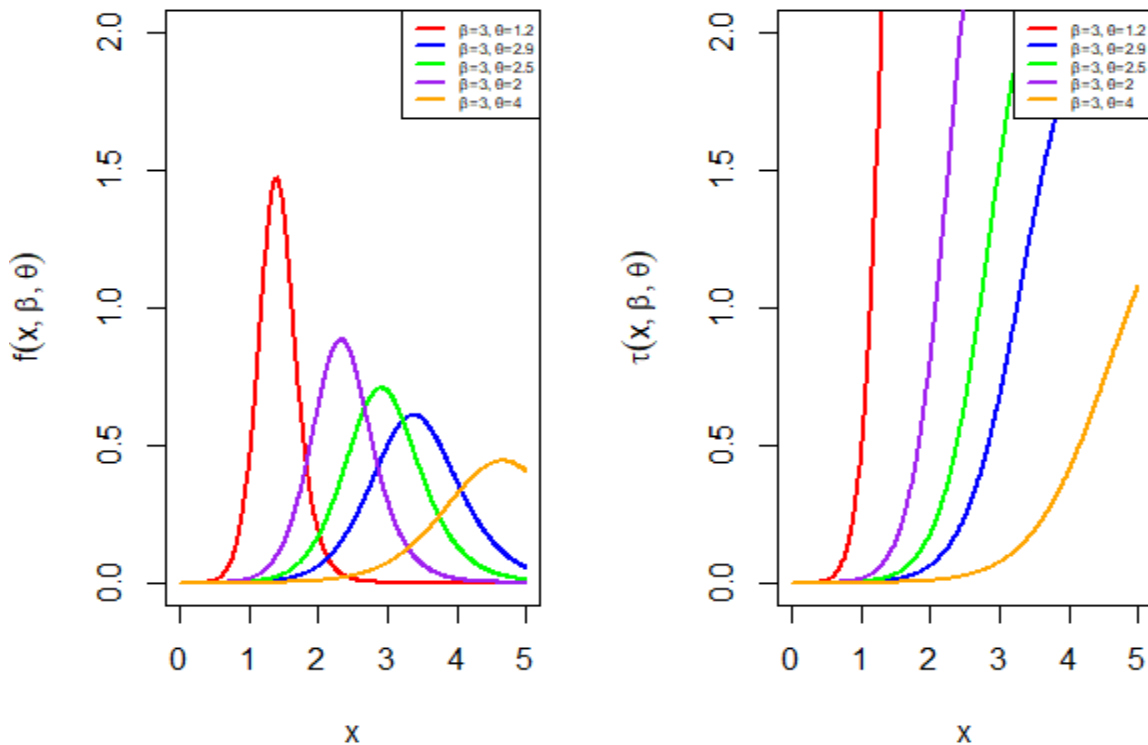


Figure 1: PDF and Hazard Rate Function Plots for Some Selected Values of the Parameters

4. Quantile Function

The quantile function is useful for simulation and random variate generation. The quantile function of the OLLR distribution denoted as $Q(u; \beta, \theta)$ is obtained by inverting Equation (5) thereafter solving for x and is obtained as:

$$Q(u; \beta, \theta) = \theta \sqrt{2 \ln \left(1 + \left(\frac{u}{1-u} \right)^{\frac{1}{\beta}} \right)}; \quad 0 < u < 1, \beta > 0, \theta > 0 \quad (12)$$

The median of the distribution is obtained by substituting $u = 0.5$ in Equation (12) and is obtained as:

$$Q(0.5; \beta, \theta) = \theta \sqrt{2 \ln(2)}; \quad \theta > 0 \quad (13)$$

5. Infinite Representation of the Model

The CDF and PDF of the OLLR distribution can be obtained in closed form by applying successive generalized binomial identity to Equation (5) and Equation (6) respectively.

Applying the binomial identity $(1+c)^{-n} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+k)}{k! \Gamma(n)} c^k$ infinite representation of the CDF of OLLR as:

$$F(x; \beta, \theta) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{k+l} \Gamma(\beta k + l)}{l! \Gamma(\beta k)} e^{-\frac{x^2(\beta k + l)}{2\theta^2}}; \quad x > 0, \beta > 0, \theta > 0 \quad (14)$$

differentiating the mixture representation of the CDF to obtain the PDF in Equation (15) as:

$$f(x; \beta, \theta) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{k+l+1} x \Gamma(\beta k + l + 1)}{\theta^2 l! \Gamma(\beta k)} e^{-\frac{x^2(\beta k + l)}{2\theta^2}}; \quad x > 0, \beta > 0, \theta > 0 \quad (15)$$

6. Moments and Moment Generating Function

If x is a random variable that follows the OLLR (β, θ) , then:

6.1. Moments

The raw moments of the OLLR distribution is obtained using:

$$\mu_r' = E(x^r) = \int_0^{\infty} x^r f(x; \beta, \theta) dx \quad (16)$$

substituting Equation (15) into Equation (16) and simplifying, we obtained the r th raw moments of the OLLR distribution as follows:

$$\begin{aligned} &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{k+l+1} \Gamma(\beta k + l + 1)}{\theta^2 l! \Gamma(\beta k)} \int_0^{\infty} x^{r+1} e^{-\frac{x^2(\beta k + l)}{2\theta^2}} dx \\ \mu_r' &= (2\theta^2)^{\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1\right) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{k+l+1} \Gamma(\beta k + l)}{l! \Gamma(\beta k) (\beta k + l)^{\frac{r}{2}}} \end{aligned} \quad (17)$$

6.2. Moment Generating Function (MGF)

The MGF of the OLLR distribution can be written in terms of the moments and is given as:

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x; \beta, \theta) dx \quad (18)$$

applying the Exponential identity $e^{tx} = \sum_{r=0}^{\infty} \frac{(tx)^r}{r!}$ to Equation (18), the MGF is re-written as:

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x; \beta, \theta) dx$$

$$M_x(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} (2\theta^2)^{\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1\right) \frac{t^r (-1)^{k+l+1} \Gamma(\beta k + l)}{l! r! \Gamma(\beta k) (\beta k + l)^{\frac{r}{2}}} \quad (19)$$

6.3. Distribution of the Order Statistics

Order statistics are necessary for extreme value analysis. The PDF of the r th order statistics for the OLLR distribution is given as:

$$fX_{(r)}(x) = \frac{1}{B(r, n-r+1)} \sum_{m=0}^{n-r} \frac{(-1)^m \Gamma(n-r+1)}{m! \Gamma(n-r+1-m)} f(x; \beta, \theta) F(x; \beta, \theta)^{r+m-1} \quad (20)$$

Substituting Equation (14) and Equation (15) into Equation (20), the distribution of the r th order statistics is obtained as:

$$fX_{(r)}(x) = \frac{1}{B(r, n-r+1)} \sum_{m=0}^{n-r} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{m+k+l+1} x \Gamma(\beta k + l + 1) \Gamma(n-r+1)}{\theta^2 m! l! \Gamma(\beta k) \Gamma(n-r+1-m)} e^{-\frac{x^2(\beta k + l)}{2\theta^2}}$$

$$X \left(\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{k+l} \Gamma(\beta k + l)}{l! \Gamma(\beta k)} e^{-\frac{x^2(\beta k + l)}{2\theta^2}} \right)^{r+m-1} \quad (21)$$

where the first and largest order statistics are obtained by substituting $r = 1$ and $r = n$ into Equation (21) respectively.

7. Parameter Estimation using Maximum Likelihood Estimation

If $X_1, X_2, \dots, X_n$ are random samples of size n drawn from the OLLR distribution, the parameters β and θ are then estimated using the method of MLE. The likelihood function $L(\beta, \theta)$ is given by:

$$L(\beta, \theta) = \frac{\beta^n}{\theta^{2n}} \prod_{i=1}^n \left(\frac{x_i e^{-\frac{\beta x_i^2}{2\theta^2}} \left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right)^{\beta-1}}{\left(\left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right)^{\beta} + e^{-\frac{\beta x_i^2}{2\theta^2}} \right)^2} \right) \quad (22)$$

The Log-likelihood function of the OLLR distribution defined by $l(\beta, \theta)$ is obtained by taking the natural logarithm of $l(\beta, \theta)$ and is expressed as:

$$l(\beta, \theta) = n \ln \beta - 2n \ln \theta + \sum_{i=1}^n \ln x_i - \frac{\beta}{2\theta^2} \sum_{i=1}^n x_i^2 + (\beta - 1) \sum_{i=1}^n \ln \left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right)$$

$$- 2 \sum_{i=1}^n \ln \left(\left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right)^{\beta} + e^{-\frac{\beta x_i^2}{2\theta^2}} \right) \quad (23)$$

for the MLE estimates, we differentiate Equation (23) with respects to β and θ thereafter setting the derivative to be equal to zero and is given as:

$$\frac{\partial(\iota(\beta, \theta))}{\partial \beta} = \frac{n}{\beta} - \frac{1}{2\theta^2} \sum_{i=1}^n x_i^2 + \sum_{i=1}^n \ln \left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right) - 2 \sum_{i=1}^n \left(\frac{\left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right)^\beta \ln \left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right) - \frac{x_i^2}{2\theta^2} e^{-\frac{\beta x_i^2}{2\theta^2}}}{\left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right)^\beta + e^{-\frac{\beta x_i^2}{2\theta^2}}} \right) = 0 \quad (24)$$

$$\frac{\partial(\iota(\beta, \theta))}{\partial \theta} = \frac{-2n}{\theta} + \frac{\beta}{\theta^3} \sum_{i=1}^n x_i^2 + \frac{(\beta-1)}{\theta^3} \sum_{i=1}^n \left(\frac{x_i^2 e^{-\frac{x_i^2}{2\theta^2}}}{1 - e^{-\frac{x_i^2}{2\theta^2}}} \right) + \frac{2\beta}{\theta^3} \sum_{i=1}^n \left(\frac{x_i^2 e^{-\frac{x_i^2}{2\theta^2}} \left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right)^{\beta-1} - x_i^2 e^{-\frac{\beta x_i^2}{2\theta^2}}}{\left(\left(1 - e^{-\frac{x_i^2}{2\theta^2}} \right)^\beta + e^{-\frac{\beta x_i^2}{2\theta^2}} \right)} \right) = 0 \quad (25)$$

The estimates $\hat{\beta}$ and $\hat{\theta}$ are obtained by solving Equation (24) and Equation (25) numerically using iterative methods.

8. Simulation Study

To assess the consistency of the parameter estimates, Markov Chain Monte Carlo (MCMC) Simulation study with 1,000 replications was carried out for the parameter setting $\mathcal{G} = (\beta = 5, \theta = 0.5)$ under various sample sizes ($n=50, 100, 200, 500$). The performance measures utilized includes average estimate, Bias and Root Mean Squared Error (RMSE). The statistics are given as:

$$Bias(\mathcal{G}) = \hat{\mathcal{G}} - \mathcal{G}, \quad RMSE(\mathcal{G}) = \sqrt{\frac{1}{1000} \sum_{i=1}^{1,000} (\mathcal{G}_i - \hat{\mathcal{G}}_i)^2} \quad (26)$$

n	Avg. (β)	Bias (β)	RMSE(β)	Avg. (θ)	Bias (θ)	RMSE(θ)
50	5.1523	0.1523	0.6754	0.4995	-0.00004	0.0088
100	5.0575	0.0575	0.4486	0.5002	0.0002	0.0061
200	5.0237	0.0237	0.3091	0.5001	0.0001	0.0045
500	5.0095	0.0095	0.1852	0.4999	-0.0001	0.0027

Table 1: Simulation Result for $\mathcal{G} (\beta = 5, \theta = 0.5)$

Table 1 confirms the consistency of the parameter estimates as the sample size gets larger. The bias and RMSE decays to zero indicating the average estimate converges to the actual parameters for large n .

9. Real-life Applications

In order to determine the practical superiority and flexibility of the OLLR distribution, the model is fitted to two dataset and compared with some competing distributions including the Logistic Rayleigh (LR) and Rayleigh distributions using information criteria (AIC, BIC, AICc) with K-S test for goodness of fits and the model with the least information criteria is termed the best model.

Dataset 1: Air Conditioning System Failure Times (Proschan, 1963) This dataset records the intervals (in hours) between successive failures of air-conditioning units in aircrafts. It's one of the most famous datasets in reliability literature specifically because it fits an Exponential distribution well. The data set are:

194, 15, 41, 29, 33, 181, 413, 14, 58, 37, 100, 65, 9, 169, 447, 184, 36, 201, 118, 34, 31, 18, 18, 67, 57, 62, 7, 22, 34, 90

Dataset 2: Remission Times of Bladder Cancer Patients (Lee & Wang, 2003)

This dataset gives the complete remission times (in months) for 128 bladder cancer patients, with no censoring i.e every observation is an actual event time. The data set are:

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57,

5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69

10. Estimates with Goodness-of-Fits Statistics for the Datasets For Dataset 1

Models	MLE	LOG-L	AIC	BIC	AICc	K-S statistic
OLLR	$\hat{\beta} = 0.4666$ $\hat{\theta} = 90.9371$	-170.669	345.339	348.142	345.784	0.2993
LR	$\hat{\alpha} = 0.4923$ $\hat{\lambda} = 0.0001$	-178.716	361.432	364.234	361.876	0.4485
Rayleigh	$\hat{\theta} = 100.5004$	-187.263	376.526	377.927	376.669	0.4674

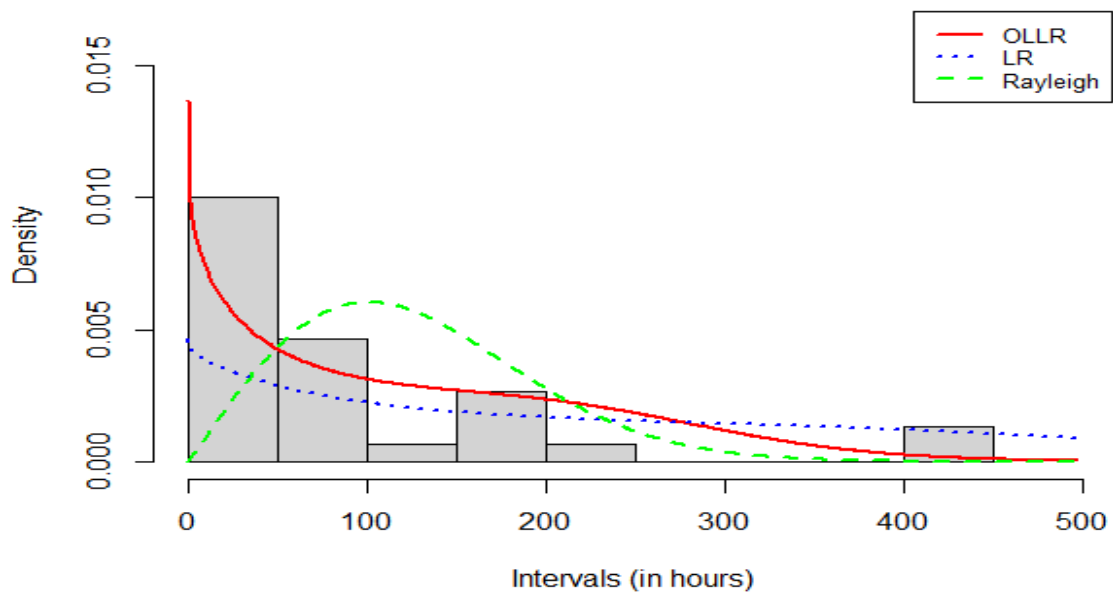


Figure 2: Histogram with Density Plot for Dataset 1

For Dataset 2

Models	MLE	LOG-L	AIC	BIC	AICc	K-S statistic
OLLR	$\hat{\beta} = 0.5097$ $\hat{\theta} = 9.1573$	-440.151	884.304	890.007	884.399	0.2397
LR	$\hat{\alpha} = 0.5099$ $\hat{\lambda} = 0.0119$	-440.152	884.304	890.008	884.400	0.2399
Rayleigh	$\hat{\theta} = 9.9317$	-491.265	984.531	987.383	984.563	0.3521

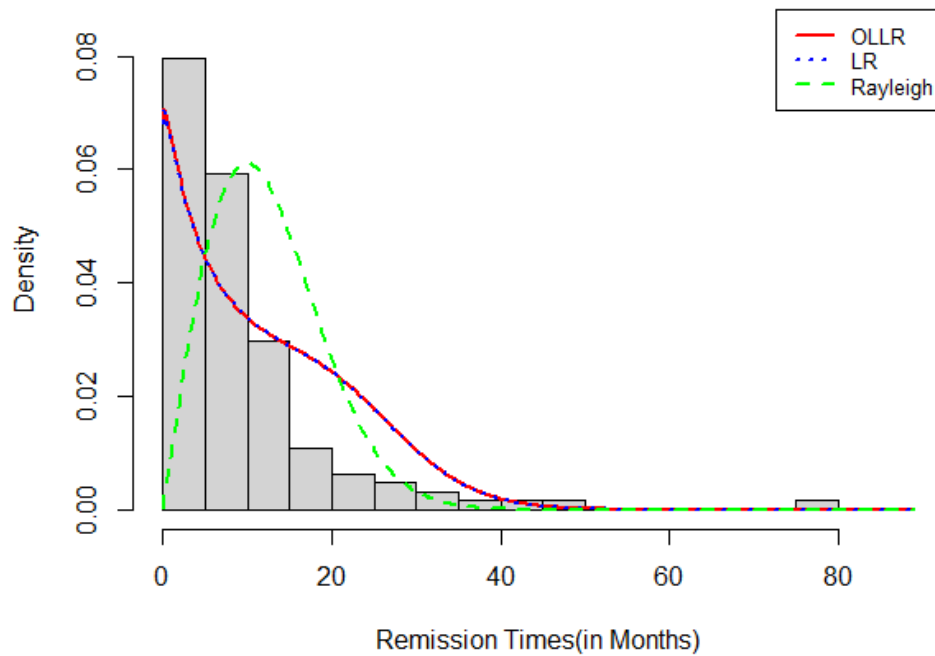


Figure 3: Histogram with Density Plot for Dataset 2

11. Conclusion

The results above verifies the superiority of the OLLR distribution over its competitors via information Criteria having the lowest information criteria, and highest log-likelihood function for both datasets. The proposed distribution also has the lowest K-S statistic indicating that the null hypothesis that the datasets follows the OLLR distribution cannot be rejected at 5%, 10% or 1% significance level as fitted in Figure 2 and Figure 3 respectively [1-5].

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