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Numerical Integral Approximation to Estimate Matusita Overlapping Coefficient for Normal Distributions

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Abstract
The overlapping coefficients are defined as the measures of similarity and agreement between two or more distributions. Estimation of the well-known overlapping coefficient; Matusita ρ under normal distributions is the main aim of this paper. Given that we have two independent random samples each following a normal distribution, a new method is proposed to estimate ρ without using any assumptions about the equality of the location or scale parameters. Three numerical integration methods are suggested and used to achieve our main objective. The maximum likelihood estimation method is used to estimate the interesting parameters. The properties of the resulting estimators are investigated and compared with some corresponding estimators that have been developed in the literature by using the simulation method. The simulation results show the effectiveness of the proposed technique over the existing ones for almost all considered cases in this study.

Keywords: Normal Distribution; Numerical Integration Methods; Matusita Overlapping Coefficient; Maximum Likelihood Method; Relative Bias; Relative Mean Square Error.

Introduction

There are three well known overlapping (OVL) coefficients, namely Matusita (1955) coefficient (ρ), Morisita (1959) coefficient (λ) and Weitzman (1970) coefficient (Δ). These coefficients measure the similarity or agreement between two distributions which are very useful and used in several applications such as a measure of distinctness clusters (Sneath, 1977); reliability analysis (Dhaker et al., 2019) and the goodness of fit test for two independent distributions [1]. In this paper we interest with the Matusita coefficient ρ. Given that we have two continuous probability density functions $f_1(x)$ and $f_2(x)$, the formula of ρ is defined as follow:

$$\rho = \int \sqrt{f_1(x) f_2(x)} dx .$$

Computing the overlapping coefficient ρ between two densities gives an idea about the degree of similarity or closeness of the two phenomena that follow these densities. It should be noticed that the values of above overlapping measures is bounded by [0,1]. If the value of ρ is closed to zero, this indicates no common area between the two densities .On the other hand, if its value is close to 1, this indicates complete matching of the two densities.

In general, there are two methods used for OVL estimation, the parametric method and the nonparametric method. A known probability density functions with unknown parameter(s) ϴ is the assumption of parametric method. The unknown parameters ϴ can be estimated by any rigorous statistical point estimation methods like the method of moments or the maximum likelihood (ML) estimation method. Unlike the parametric method, the nonparametric method does not require any assumptions about the shape or the formula of the underlying statistical distributions (see for example, [1, 2]). The parametric method has been considered by many authors to estimate the various OVL coefficients. Inman and Bradly derived the ML estimator of Weitzman coefficient measure (Δ) under the assumption that the two densities are normal with different means and equal variances [3]. Mulekar and Mishra addressed the problem of estimation OVL coefficients in the case of two normal densities with equal means but different variances [4]. Mulekar and Mishra compared the confidence intervals for the overlapping coefficients using re-sampling technique, namely, Jacknife, bootstrap and transformation methods under the normal densities with equal means [5]. Reiser and Faraggi constructed generalized confidence intervals for the OVL of two normal distributions with equal variances [6]. The problem of estimating the OVL coefficients for other densities have been addressed in the literature. For example, see [7, 8, 9, 10, 11, 12, 13].

As pointed out above, the studies that used the normal distributions assumed either the means of two distributions or the variances of the two distributions are equal. This assumption is necessary to determine the closed form of the OVL coefficient that we interest to estimate it. Without this assumption, the researcher cannot find a formula for the parameter to be estimated and therefore the researcher cannot perform the estimation process. To overcome this problem, recently, Eidous and Al-Daradkeh suggested a new method to estimate [14]. Their technique depends basically on the writing the integral in the formula of ρ as expected value of some functions and then estimate this expected value by using the method of moments. Despite that, their method leads to a new estimator for ρ with good properties obtained by simulation technique, we expect that there is a scope of improvements. In this paper, another new technique based on the numerical integration methods is suggested to estimate ρ under the normal distribution. This suggested technique leads to three new estimators for ρ, which will compare via simulation method with the estimator obtained by Eidous and Al-Daradkeh [14].

Estimation of ρ Under the Normal Distributions

Let X_1 and X_2 be two independent random variables such that $X_i \sim N(\mu_i, \sigma_i^2)$, i=1,2. On one hand, if $\mu_1=\mu_2=(\mu$, say) then Mulekar and Mishra derived the value of ρ, which is given by [4],

$$\rho = \sqrt{\frac{2C}{1+C^2}}$$

where $C = \sigma_1/\sigma_2$. To estimate ρ, let $(X_{11,}X_{12}, \ldots. X_{1n_1})$ and $(X_{21,}X_{22}, \ldots. X_{2n_2})$ be two independent random samples taken from $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$ respectively. The ML estimators of μ, σ_1^2 and σ_2^2 are $\hat{\mu} = \frac{\sum_{i=1}^{n_1} X_{1i} + \sum_{i=1}^{n_2} X_{2i}}{n_1+n_2}$, $\hat{\sigma}_1^2 = \frac{\sum_{i=1}^{n_1}(X_{1i}-\hat{\mu})^2}{n_1}$ and $\hat{\sigma}_2^2 = \frac{\sum_{i=1}^{n_2}(X_{2i}-\hat{\mu})^2}{n_2}$ respectively. Therefore and by using the invariance property, the ML estimator of ρ is (Mulekar and Mishra,1994),

$$\rho = \sqrt{\frac{2\hat{C}}{1+\hat{C}^2}}$$

Where $\hat{C} = \hat{\sigma}_1/\hat{\sigma}_2$. On the other hand, if $\sigma_1 = \sigma_2 = (\sigma$, say) then the formula of ρ is given by,

$$\rho = e^{-(\mu_1-\mu_2)^2/(8\sigma^2)}.$$

The ML estimator of ρ is,

$$\hat{\rho} = e^{-(\hat{\mu}_1-\hat{\mu}_2)^2/(8\hat{\sigma}^2)},$$

Where $\hat{\mu}_1 = \bar{X}_1$, $\hat{\mu}_2 = \bar{X}_2$ and $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n_1}(X_{1i}-\bar{X}_1)^2 + \sum_{i=1}^{n_2}(X_{2i}-\bar{X}_2)^2}{n_1+n_2}$ are the ML estimators of μ_1, μ_2 and σ^2 respectively.

Eidous and Al-Daradkeh expressed the formula of ρ as follows [14],

$$\rho_{ED} = \frac{1}{2}\left[E\left(\frac{f_2(X_1)}{f_1(X_1)}\right)^{\frac{1}{2}} + E\left(\frac{f_1(X_2)}{f_2(X_2)}\right)^{\frac{1}{2}}\right]$$

They gave the following estimator for ρ under two normal distributions without using any assumptions about the equality of their means or variances,

$$\hat{\rho}_{ED} = \frac{1}{2}\left[\frac{1}{n_1}\sum_{i=1}^{n_1}\left(\frac{\hat{f}_2(X_{1i})}{\hat{f}_1(X_{1i})}\right)^{1/2} + \frac{1}{n_2}\sum_{i=1}^{n_2}\left(\frac{\hat{f}_1(X_{2i})}{\hat{f}_2(X_{2i})}\right)^{1/2}\right],$$

Where $\widehat{f_1} = N(\hat{\mu}_1, \hat{\sigma}_1^2)$ and $\widehat{f_2} = N(\hat{\mu}_2, \hat{\sigma}_2^2)$. Also, $\hat{\mu}_1 = \bar{X}_1$, $\hat{\mu}_2 = \bar{X}_2$, $\hat{\sigma}_1^2 = \sum_{i=1}^{n_1}(X_{1i} - \bar{X}_1)^2/n_1$ and $\hat{\sigma}_2^2 = \sum_{i=1}^{n_2}(X_{2i} - \bar{X}_2)^2/n_2$ are the ML estimators of μ_1, μ_2, σ_1^2 and σ_2^2 respectively.

New Approximation Expressions for Matusita ρ

In this section, we give a new approximation expression for the Matusita coefficient ρ under the two normal distributions without using any assumptions about their parameters. Let $u(x)=\sqrt{(f_1(x;\mu_1,\sigma_1^2) f_2(x;\mu_2,\sigma_2^2)}$ then,

$$\rho = \int_{-\infty}^{\infty} \sqrt{f_1(x;\mu_1,\sigma_1^2) f_2(x;\mu_2,\sigma_2^2)}\,dx$$

$$= \int_{-\infty}^{\infty} u(x)\,dx$$

Let $G(x)$ be any continuous increasing function in x and consider the transformation $y=G(x)$, then $x=G^{-1}(y)=w(y)$,say). Therefore,

$$\rho = \int_{G(-\infty)}^{G(\infty)} u(w(y))w'(y)\,dy$$

In particular, we interest the case where $G(x)$ is a continuous cumulative distribution function. In this case, $G(-\infty)=0$ and $G(\infty)=1$ and

$$\rho = \int_0^1 u(w(y))w'(y)\,dy.$$

To approximate the last integral, the interval [0,1] is divided into k subintervals each of length $1/k$ as follows,
$0=y_0< y_1< y_2< \ldots <y_k=1$,

Now, we consider the three numerical integrals rules; trapezoidal, Simpson 1/3 and Simpson 3/8, which are briefly is given as the following [15]:
For more simplicity, let $h_1(y)=\sqrt{(f_1(w(y);\mu_1,\sigma_1^2) f_2(w(y);\mu_2,\sigma_2^2)}\; w'(y)$, then $\rho=\int_0^1 h_1(y)\,dy$ can be approximated by using the three interested numerical integral rules as given below,

- The approximation of ρ by using the trapezoidal rule (denoted by ρ_1) is

$$\rho_{Trap} \cong \frac{1}{2k}\left[h_1(0) + 2\sum_{j=1}^{k-1} h_1(y_j) + h_1(1)\right].$$

- The approximation of ρ by using the Simpson 1/3 rule (denoted by ρ_2) is

$$\rho_{Simp1} \cong \frac{1}{3k}\left[h_1(0) + 4\sum_{j=1}^{k/2} h_1(y_{2j-1}) + 2\sum_{j=1}^{k/2-1} h_1(y_{2j}) + h_1(1)\right].$$

- The approximation of ρ by using the Simpson 3/8 rule (denoted by ρ_3) is

$$\rho_{Sim2} \cong \frac{3}{8k}\left\{h_1(0) + 3\sum_{\substack{j=1\\ j\neq 3m}}^{k-1} h_1(y_j) + 2\sum_{j=1}^{k/3-1} h_1(y_{3j})\right\}, m \in N_0$$

Estimation of Matusita Measures and Practical Implementation

Let $f_1(x;\hat{\mu_1},\hat{\sigma_1})$ and $f_2(x;\hat{\mu_2},\hat{\mu_2}^2)$ be the ML estimators of $f_1(x;\mu_1,\sigma_1^2)$ and $f_2(x;\mu_2,\sigma_2^2)$ respectively. Also, let $\sqrt{\hat{f}_1(w(y);\hat{\mu}_1,\hat{\sigma}_1^2)\hat{f}_2(w(y);\hat{\mu}_2,\hat{\sigma}_2^2)}\,dw'(y)$ then the proposed estimators of ρ_{Trap}, ρ_{Sim1} and ρ_{Sim2} are given below:

- **Trapezoidal rule is**

$$\hat{\rho}_{Trap} = \frac{1}{k}\sum_{j=1}^{k-1}\hat{h}_1(j/k).$$

- **Simpson 1/3 rule is**

$$\hat{\rho}_{Sim1} = \frac{1}{3k}\left[4\sum_{j=1}^{k/2}\hat{h}_1((2j-1)/k) + 2\sum_{j=1}^{k/2-1}\hat{h}_1(2j/k)\right].$$

- **Simpson 3/8 rule is**

$$\hat{\rho}_{Sim2} = \frac{3}{8k}\left\{3\sum_{\substack{j=1\\ j\neq 3m}}^{k-1}\hat{h}_1(j/k) + 2\sum_{j=1}^{k/3-1}\hat{h}_1(3j/k)\right\}, m\in N_0$$

To use the above three estimators in practice, two quantities are to be determined. The first one is the transformation $G(x)$ and hence $w(y)$. The second quantity is the number of partitions k. In this paper, our special interest is to take G to be any continuous cumulative distribution function with support $(-\infty,\infty)$. Let T be a continuous random variable with cumulative distribution function $G_T(t)$ given by,

$$G_T(x) = 1 - \frac{1}{(1+e^x)^{\alpha}}, -\infty < x < \infty,\ \alpha > 0.$$

That is, T has a generalized Logistic distribution with *pdf,*

$$g_T(x) = \frac{\alpha e^{-\alpha x}}{(1+e^x)^{\alpha+1}}, -\infty < x < \infty, \alpha > 0.$$

In this case, $y=1-(1+e^x)^{-\alpha}$ with inverse transformation $x=w(y)=ln\left((1-y)^{-\frac{1}{\alpha}}-1\right)$ and $w'(y) = \frac{1}{\alpha(1-y)\left(1-(1-y)^{1/\alpha}\right)}dy.$

The parameter α in the above transformation is under the user control. Mathematically, any choice of $\alpha>0$ is possible. However, to study its practice effect on the performances of the proposed estimators, the two values $\alpha=0.5$ and $=1.0$ are considered in our simulation study in the next section. The second quantity that need to be determine is k. The results of simulation study in the next section are obtained based on the suggested choice $k=\min\{n_1,n_2\}$.

Simulation Study and Results

In this simulation study, the four estimators $\hat{\rho}_{ED}$,$\hat{\rho}_{Trap}$,$\hat{\rho}_{simp1}$ and $\hat{\rho}_{simp2}$ of ρ are considered. The estimator that developed by Eidous and Al-Daradkeh is considered in this study for sake of comparison [14]. Assume that we have two independent random samples $x_{11},x_{12},\ldots,x_{1n1}$) of size n_1 and $x_{21},x_{22},\ldots,x_{2n2}$) of size n_2, where the two samples are generated from $N(0,1)$ and $N(\mu_2,\sigma_2^2)$,(μ_2,σ_2^2)=(-0.2,1.1),(2.5,4),(3.5,1.5),(10,2.5) respectively. These selection values of μ_1,σ_1^2 μ_2,σ_2^2 are chosen to vary the exact values of ρ between 0 and 1 From each pair of distributions, 1000 samples of sizes (n_1,n_2)= (24,30), (54,54),(96,180) were simulated independently from the two normal distributions $N(\mu_1,\sigma_1^2)$ and $N(\mu_2,\sigma_2^2)$ with selected parameters given in Table (1).

The empirical results given in Table (1) were calculated based on one thousand replications (R=1000). For each estimator, we compute the relative bias (RB), relative root mean square error (RRMSE) and efficiency (EFF). These measures are defined as follows:

Let $\hat{\theta}$ be a specific estimator for a parameter θ (exact value), and let $\hat{\theta}_{(j)}$ be the observed value of $\hat{\theta}$ based on iteration j,j=1,2,…,R=1000, then,

$$RB = \frac{\hat{E}(\hat{\theta}) - \theta}{\theta},$$

$$RRMSE = \frac{\sqrt{\widehat{MSE}(\hat{\theta})}}{\theta}$$

and the efficiency of the proposed estimator (Prop-Est) with respect to Eidous and Al-Daradkeh estimator ($\hat{\rho}_{ED}$) is defined by [14],

$$EFF = \frac{\widehat{MSE}(\hat{\rho}_{ED})}{\widehat{MSE}\,(Prop - Est)},$$

where $\hat{E}(\hat{\theta}) = \frac{\sum_{j=1}^{R} \hat{\theta}_{(j)}}{R}$ and $\widehat{MSE}(\hat{\theta}) = \frac{\sum_{j=1}^{R} (\hat{\theta}_{(j)} - \theta)^2}{R}$.

All simulation results are calculated by using Mathematica, Version 11. Based on the simulation results, which presented in Table (1), the general conclusions are:

1. The values of |RB|s and RRMSE for all estimators of OVL measures decrease as the sample sizes increase. This result is very clear if we compare the values of |RB|s and RRMSE for the different estimators when (n_1,n_2)=(24,30) with their values when (n_1,n_2)=(96,180) in all Tables. This indicates that the various estimators are consistent estimators.

2. Regardless which OVL measure is to be estimated and for all simulated pair distributions, it is evidence that the proposed estimators are more efficient than the estimator $\hat{\rho}_{ED}$ that suggested by Eidous and Al-Daradkeh in most considered cases [14]. This is clearly appear if one examines the corresponding values of RRMSE and EFF especially when the exact values of the OVL measure get small toward 0.

3. By comparing the performances of the proposed estimators of ρ together, it is clear that their performances are close to each other and their values are coincide for large sample sizes. This indicates that the three numerical integration rules; trapezoidal, Simpson 1/3, and Simpson 3/8 rules give similar results in estimating the OVL measures ρ. In other words, the use of any one of these three rules gives the same results as the use of the other two rules in estimating ρ.

4. By examining and comparing the results corresponding to the two transformations $1-(1+e^x)^{-1}$ (i.e. $\alpha = 1$) and $1-(1+e^x)^{-1/2}$ (i.e. $\alpha = ½$), it appears that the proposed estimators are not sensitive to the transformation choice. At least and depending on the simulation results, the above two transformations work well in estimating the Matusita OVL measure ρ.

Discussion

This paper has been suggested and developed a new technique for estimating the Matusita coefficient (measure) ρ under pair normal distributions without using any restrictions about the equality of any two parameters. In addition, the proposed technique in this paper can be used under different parametric distributions like exponential, Lomax distributions etc. The numerical results showed the effectiveness of this proposed technique.

Table 1. The RB, RRMSE and EFF of the estimators $\hat{\rho}_{ED}, \hat{\rho}_{Trap}$, $\hat{\rho}_{Sim1}$ and $\hat{\rho}_{Sim2}$ when the data are simulated from pair normal distributions a) $N(0,1)$ and $N(-0.2,1.1)$ b) $N(0,1)$ and $N(2.5,4)$ c) $N(0,1)$ and $N(3.5,1.5)$ and d) $N(0,1)$ and $N(10,2.5)$.

a) The exact $\rho = 0.9932$.

		α = 1				α = 1/2			
(n_1, n_2)		$\hat{\rho}_{ED}$	$\hat{\rho}_{Trap}$	$\hat{\rho}_{Sim1}$	$\hat{\rho}_{Sim2}$	$\hat{\rho}_{Dar}$	$\hat{\rho}_{Trap}$	$\hat{\rho}_{Sim1}$	$\hat{\rho}_{Sim2}$
(24, 30)	RB	-0.020	-0.019	-0.019	-0.018	-0.020	-0.019	-0.017	-0.016
	RRMSE	0.034	0.031	0.031	0.031	0.032	0.030	0.030	0.029
	EFF	**1.000**	**1.170**	**1.170**	**1.180**	**1.000**	**1.150**	**1.200**	**1.230**
(54, 54)	RB	-0.010	-0.009	-0.009	-0.009	-0.009	-0.009	-0.009	-0.008
	RRMSE	0.019	0.018	0.018	0.018	0.017	0.016	0.016	0.016
	EFF	**1.000**	**1.100**	**1.100**	**1.100**	**1.000**	**1.130**	**1.130**	**1.140**
(96, 180)	RB	-0.003	-0.003	-0.003	-0.003	-0.003	-0.003	-0.003	-0.003
	RRMSE	0.009	0.009	0.009	0.009	0.009	0.008	0.008	0.008
	EFF	**1.000**	**1.090**	**1.090**	**1.090**	**1.000**	**1.110**	**1.110**	**1.110**

b) The exact ρ =0.6258.

		α = 1				α = 1/2			
(n_1, n_2)		$\hat{\rho}_{ED}$	$\hat{\rho}_{Trap}$	$\hat{\rho}_{Sim1}$	$\hat{\rho}_{Sim2}$	$\hat{\rho}_{Dar}$	$\hat{\rho}_{Trap}$	$\hat{\rho}_{Sim1}$	$\hat{\rho}_{Sim2}$
(24, 30)	RB	-0.030	-0.022	-0.016	-0.017	-0.026	-0.023	-0.016	-0.017
	RRMSE	0.137	0.092	0.093	0.093	0.130	0.092	0.092	0.092
	EFF	**1.000**	**2.217**	**2.136**	**2.159**	**1.000**	**1.996**	**1.963**	**1.983**
(54, 54)	RB	-0.014	-0.008	-0.007	-0.007	-0.012	-0.008	-0.006	-0.006
	RRMSE	0.101	0.067	0.068	0.068	0.097	0.065	0.065	0.065
	EFF	**1.000**	**2.220**	**2.190**	**2.190**	**1.000**	**2.250**	**2.230**	**2.230**
(96, 180)	RB	-0.009	-0.002	-0.001	-0.001	-0.010	-0.005	-0.005	-0.004
	RRMSE	0.063	0.042	0.042	0.042	0.064	0.042	0.042	0.042
	EFF	**1.000**	**2.230**	**2.210**	**2.210**	**1.000**	**2.280**	**2.270**	**2.270**

c) The exact $\rho = 0.3744$.

		α = 1				α = 1/2			
(n_1, n_2)		$\hat{\rho}_{ED}$	$\hat{\rho}_{Trap}$	$\hat{\rho}_{Sim1}$	$\hat{\rho}_{Sim2}$	$\hat{\rho}_{Dar}$	$\hat{\rho}_{Trap}$	$\hat{\rho}_{Sim1}$	$\hat{\rho}_{Sim2}$
(24, 30)	RB	-0.150	-0.105	-0.073	-0.080	-0.147	-0.044	-0.043	-0.043
	RRMSE	0.419	0.353	0.354	0.354	0.431	0.374	0.376	0.376
	EFF	**1.000**	**1.410**	**1.400**	**1.404**	**1.000**	**1.325**	**1.313**	**1.311**
(54, 54)	RB	-0.101	-0.051	-0.040	-0.041	-0.076	-0.022	-0.022	-0.022
	RRMSE	0.293	0.247	0.249	0.248	0.294	0.254	0.254	0.254
	EFF	**1.000**	**1.406**	**1.378**	**1.387**	**1.000**	**1.337**	**1.335**	**1.335**
(96, 180)	RB	-0.041	-0.016	-0.014	-0.014	-0.048	-0.022	-0.022	-0.022
	RRMSE	0.216	0.185	0.186	0.186	0.218	0.179	0.179	0.179
	EFF	**1.000**	**1.365**	**1.356**	**1.357**	**1.000**	**1.476**	**1.476**	**1.476**

d) The exact $\rho = 0.0264$.

		α = 1				α = 1/2			
(n_1, n_2)		$\hat{\rho}_{ED}$	$\hat{\rho}_{Trap}$	$\hat{\rho}_{Sim1}$	$\hat{\rho}_{Sim2}$	$\hat{\rho}_{Dar}$	$\hat{\rho}_{Trap}$	$\hat{\rho}_{Sim1}$	$\hat{\rho}_{Sim2}$
(24, 30)	RB	-0.199	0.097	0.120	0.115	-0.241	0.165	0.166	0.166
	RRMSE	1.482	0.845	0.862	0.858	1.272	0.900	0.901	0.901
	EFF	**1.000**	**3.076**	**2.962**	**2.986**	**1.000**	**1.996**	**1.991**	**1.991**
(54, 54)	RB	-0.188	0.049	0.056	0.055	-0.104	0.081	0.081	0.081
	RRMSE	1.490	0.609	0.612	0.612	1.611	0.638	0.639	0.639
	EFF	**1.000**	**5.986**	**5.915**	**5.925**	**1.000**	**6.366**	**6.364**	**6.364**
(96, 180)	RB	-0.052	0.025	0.027	0.027	-0.083	0.025	0.025	0.025
	RRMSE	1.303	0.355	0.356	0.356	1.164	0.353	0.353	0.353
	EFF	**1.000**	**13.466**	**13.389**	**13.394**	**1.000**	**10.835**	**10.835**	**10.835**

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