

New Method to Construct Parabolic Coordinates

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Abstract

In this paper, we consider fully original method to construct the orthogonal parabolic coordinates [1],[2] for any dimension N . Unlike the three-dimensional case that is produced by rotation about the parabolic symmetry axis, we join three paraboloids that possess three-fold rotational symmetry. In N -dimensional case, we combine N paraboloids that have rotational symmetry of order N . Moreover, for any dimension, we can obtain the Laplace-Runge-Lenz vector.

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1. Two-Dimensional Parabolic Coordinates

In the Euclidean space E^2 we will use the Cartesian coordinates (x, y) , where the origin of the Euclidean space is the point $(x = 0, y = 0)$.

Consider an arbitrary directed line $l \in E^2$ and the segment $[a, b] \in l$ that passes through the point $(0, 0)$ (the origin of the plane E^2). Note also that the center of $[a, b]$ seats at $(0, 0)$. Let the points $a(x_a, y_a)$ and $b(x_b, y_b)$, where (x_a, y_a) , (x_b, y_b) are coordinates of the points a, b , be ends of the segment $[a, b]$.

Moreover, if we consider the points a, b as radius-vectors $\vec{a}, \vec{b}$, then we suppose that

$$|\vec{a}| = |\vec{b}| = 1.$$

On the other hand, the radius-vectors $\vec{a}$, and $(\vec{b})$ collapse

$$\vec{a} + \vec{b} = 0. \quad (1)$$

If the vectors are considered component-wise, then we have two equations

$$\begin{aligned} x_a + x_b &= 0, \\ y_a + y_b &= 0. \end{aligned} \quad (2)$$

Now, we can define the orthogonal curves $\mathfrak{A}(\sigma)$, $\mathfrak{B}(\tau)$ as

$$\begin{aligned}\mathfrak{A}(\sigma) : \quad \sigma &= \sqrt{\sqrt{x^2 + y^2} + x_a x + y_a y}, \\ \mathfrak{B}(\tau) : \quad \tau &= \sqrt{\sqrt{x^2 + y^2} + x_b x + y_b y},\end{aligned}\tag{3}$$

Where, in the right-hand sides of equalities (3), the numbers $(x_a, y_a), (x_b, y_b)$ are coordinates of the points a; b and x; y are variables. Further note that $\mathfrak{A}(\sigma), \mathfrak{B}(\tau)$, see (3), are confocal parabolas and the direction of the parabolic symmetry axes coincides with the orientation of the segment [a, b].

Note also that we have the equality

$$x_a x + y_a y + x_b x + y_b y = 0.\tag{4}$$

We can see, if the parabolas $\mathfrak{A}(\sigma), \mathfrak{B}(\tau)$ correspond to identical values of σ, τ , that the union of parabolas $\mathfrak{A}(\sigma) \cup \mathfrak{B}(\tau) = \mathfrak{AB}(\sigma, \tau)$ possesses 2-fold rotational symmetry with respect to the point (0, 0).

Recall that the parabolic foci situate at the point (0, 0). Note that parabolas $\mathfrak{A}, \mathfrak{B}$ are outside of the segment [a, b].

$$\mathfrak{A}(\sigma) \cup \mathfrak{B}(\tau) =: \mathfrak{AB}(\sigma, \tau)$$

In particular, from (4), we have that guarantees the orthogonality of the grid (σ, τ) .

$$\sqrt{x^2 + y^2} = \frac{1}{2} (\sigma^2 + \tau^2)\tag{5}$$

2. Three-Dimensional Space

Three-dimensional space $\mathbb{E}^3$ has Cartesian coordinates (x, y, z). Let a two-dimensional plane S be situated anyhow in $\mathbb{E}^3$ and the regular (equilateral) triangle $\Delta ABC \in S$ be oriented arbitrary in S. Moreover, the triangle ΔABC is centered, i.e., the center of the triangle (center of the circle that circumscribes the triangle ΔABC) is located at the origin of the plane S (for simplicity, we suppose that the origin of the plane S coincides with the origin of the space $\mathbb{E}^3$).

Similarly, to the two-dimensional case, consider three points $a(x_a, y_a, z_a); b(x_b, y_b, z_b)$ and $c(x_c, y_c, z_c)$ that are situated at the vertices of the triangle ΔABC . Analogously to the previous case, radius-vectors $\bar{a}, \bar{b}$ and $\bar{c}$ collapse.

$$\begin{aligned}|\bar{a}| &= |\bar{b}| = |\bar{c}| = 1, \\ \bar{a} + \bar{b} + \bar{c} &= 0.\end{aligned}\tag{6}$$

If we rewrite component-wise vector equality (6), we receive

$$\begin{aligned}x_a + x_b + x_c &= 0, \\ y_a + y_b + y_c &= 0, \\ z_a + z_b + z_c &= 0.\end{aligned}$$

We have orthogonal confocal paraboloids

$$\begin{aligned}\mathfrak{A}(\sigma) : \quad \sigma &= \sqrt{\sqrt{x^2 + y^2 + z^2} + x_a x + y_a y + z_a z}, \\ \mathfrak{B}(\tau) : \quad \tau &= \sqrt{\sqrt{x^2 + y^2 + z^2} + x_b x + y_b y + z_b z}, \\ \mathfrak{C}(\chi) : \quad \chi &= \sqrt{\sqrt{x^2 + y^2 + z^2} + x_c x + y_c y + z_c z}.\end{aligned}\tag{7}$$

Note that $\mathfrak{A}(\sigma)$, $\mathfrak{B}(\tau)$, $\mathfrak{C}(x)$ are confocal paraboloids and the directions of the symmetry axes of the paraboloids coincide with the directions of radius-vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$.

Moreover, the symmetry axes of the paraboloids $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ lie on the plane S and all the paraboloids are three-dimensional surfaces, which are outside of the plane S.

If we join three confocal paraboloids (7)

$$\mathfrak{A}(\sigma) \cup \mathfrak{B}(\tau) \cup \mathfrak{C}(\chi) =: \mathfrak{ABC}(\sigma, \tau, \chi), \quad (8)$$

that correspond to the identical values of σ , τ , x we obtain that a surface (8), formed by paraboloids (7), possesses rotational symmetry of order three.

Similarly, to the two-dimensional case, we will have the following useful equality

$$x_a x + y_a y + z_a z + x_b x + y_b y + z_b z + x_c x + y_c y + z_c z = 0. \quad (9)$$

Note also that from (9) we obtain

$$\sqrt{x^2 + y^2 + z^2} = \frac{1}{3} (\sigma^2 + \tau^2 + \chi^2).$$

3. Four-Dimensional Space

Consider also the four-dimensional case. Note that we must take an arbitrary three-dimensional space $F \in \mathbb{E}^4$ and a tetrahedron $ABCD \in F$ and we must consider four radius-vectors that go from the center to vertices of the tetrahedron. Note that four-dimensional paraboloids are outside of the space F.

References

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