ISSN: 2994- 9459

Research Article

Current Research in Statistics & Mathematics

New Formula for The Riemann Zeta Function

Zaroual Ayoub*

Engineer, HASSAN II Agronomic and Veterinary Institute, Morocco

*Corresponding Author
Zaroual Ayoub, Engineer, HASSAN II Agronomic and Veterinary Institute, Morocco.

Submitted: 2024, Feb 01; **Accepted**: 2024, Feb 26; **Published**: 2024, Mar 15

Citation: Ayoub, Z. (2024). New Formula for The Riemann Zeta Function. *Curr Res Stat Math, 3*(1), 01-26.

Abstract

In this paper, we prove that Reimann's zeta function $\zeta(s)=\sum_{n=1}^{+\infty}\frac{1}{n^s} : s=a+ib\,, s\neq 1$ *on the complex plane is a spiral of radius r :*

$$r = \lim_{N\to+\infty}\sqrt{\left(\frac{bN^{1-a}}{(1-a)^2+b^2}+\sum_{j=1}^{+\infty}\left(\frac{b_{2j}}{(2j)!}\times\frac{A_{2j-1}}{N^{a+2j-1}}\right)\right)^2+\left(\frac{1}{2N^a}+\frac{(1-a)N^{1-a}}{(1-a)^2+b^2}+\sum_{j=1}^{+\infty}\left(\frac{b_{2j}}{(2j)!}\times\frac{B_{2j-1}}{N^{a+2j-1}}\right)\right)^2}$$

With :

- $A_{2j-1}=\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)+1}K_{2j-1}^{2p}(a)$
- $B_{2j-1}=\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)}K_{2j-1}^{2p+1}(a)$
- $a\to K_n^p(a),\ (a\in\mathbb{R}),\ n\in\mathbb{N}^*, p\in\mathbb{N},$ *is the function that sums the multiplications between all the elements of the non-repeating combinations of the n elements* $\{a;a+1;a+2;\dots;a+n-1n: n\in\mathbb{N}^*\}$ *taken p to p* (C_n^p)

And coordinate center :

$$\left(\left(\frac{1}{2}-\frac{1-a}{(1-a)^2+b^2}-\sum_{j=1}^{+\infty}\frac{b_{2j}}{(2j)!}B_{2j-1}\right),\left(\frac{-b}{(1-a)^2+b^2}-\sum_{j=1}^{+\infty}\frac{b_{2j}}{(2j)!}A_{2j-1}\right)\right)$$

We then show that Reimann's zeta function can be extended analytically on all the complex plane except in s =1 by :

$$\boldsymbol{\zeta(s)}=\left[\frac{1}{2}-\frac{1-a}{(1-a)^2+b^2}-\sum_{j=1}^{+\infty}\frac{b_{2j}}{(2j)!}B_{2j-1}\right]-\boldsymbol{i}\times\left[\frac{b}{(1-a)^2+b^2}+\sum_{j=1}^{+\infty}\frac{b_{2j}}{(2j)!}A_{2j-1}\right]$$

$$\equiv$$

$$\boldsymbol{\zeta(s)}=\left[\frac{1}{2}-\frac{1-a}{(1-a)^2+b^2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]\right]$$

$$+\boldsymbol{i}\times\left[\frac{-b}{(1-a)^2+b^2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]\right]$$

And finally, we conclude that all non-trivial zeros of the zeta function have a real part a and an imaginary part b that satisfies the equation :

$$\frac{1}{2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}\left[K_{2j-1}^{2(j-n)-1}(a)-(1-a)K_{2j-1}^{2(j-n-1)}(a)\right]\right]=0$$

Keywords: Riemann zeta function, Complex analysis

1. Introduction

Reimann's zeta function, often referred to as $\zeta(s)$ is a special mathematical function that plays an essential role in the study of the distribution of prime numbers and in number theory in general. It was introduced by the German mathematician Bernhard Reimann in the mid-19th century [1-5]. The Reimann zeta function is defined for complex numbers s of the form $s = a + bi$ where a and b are real numbers, and i is the imaginary unit ($i^2 = -1$). The formula for Reimann's zeta function is as follows:

$$\zeta(s)=\sum_{n=1}^{+\infty}\frac{1}{n^s}=\sum_{n=1}^{+\infty}\frac{1}{n^{a+bi}}$$

When the real part of s is strictly greater than 1 ($a > 1$)the series converges, giving a finite value to the Reimann zeta function.

For $s = 1$ it has a simple pole and for any real part s strictly smaller than 1 ($a < 1$) the series diverges and can be analytically extended using the functional identity :

$$\zeta(s)=2^s\pi^{s-1}\ \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\,\zeta(1-s),$$

Where Γ is Euler's Gamma function. It then becomes possible to use this formula to define zeta for any negative real part s (with $\zeta(0)=-\frac{1}{2}$.

We deduce that strictly negative even integers are zeros of zeta (called trivial zeros). $\zeta(-2n)=0$ and that non-trivial zeros are symmetrical about the axis $a=\frac{1}{2}$ axis and all have a real part between 0 and 1; this region of the complex plane is called the critical band.
As a result, Reimann's hypothesis can be reformulated as follows:

$$\zeta(s)=0\ \text{ And }\ 0<a<1\ \text{ implies } a=\frac{1}{2}$$

2. Literature review

2.1. Analytical extension principle

Theorem: Let U be a connected open of $\mathbb{C}$ *let* f and g *be* two holomorphic functions on U, and let A be a part of U admitting an accumulation point that belongs to U. Then

$$f=g \text{ on } A \Leftrightarrow f=g \text{ on } U.$$

In particular, if $f=g$ in a neighborhood of a point a of U, then $f=g$ on U

This theorem is used to prove many uniqueness results for holomorphic functions. For example, the only holomorphic function f:ℂ→ℂ that verifies $f(\frac{1}{n}) = \frac{1}{n}$ for all $n \geq 1$ is the function $f(z)$=z. We apply the previous theorem to A={$\frac{1}{n}$: n≥1}, U=ℂnoting that 0∈ℂ is an accumulation point of A.

Definition[2] - Let U be an open of the set ℂ of complex numbers and f an application of U in ℂ.

● We say that f is derivable (in the complex sense) or holomorphic at a point z_0 of U if the following limit, called the derivative of f at z_0 exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

● We say that f is holomorphic on U if it is holomorphic at any point of U.

● In particular, a holomorphic function on ℂ.

2.2. Bernoulli numbers

The Bernoulli numbers, noted B_n (or sometimes b_n so as not to confuse them with Bernoulli polynomials), are a series of rational numbers.

These numbers were first studied by Jacques Bernoulli[3] (which led Abraham de Moivre to give them the name we know today), looking for formulas to express sums of the type :

$$\sum_{k=0}^{n-1} k^m = 0^m + 1^m + 2^m + \cdots + (n-1)^m.$$

For integer values of m, this sum is written as a polynomial of variable n whose first terms are:

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1}\left(n^{m+1} - \frac{1}{2}\binom{m+1}{1}n^m + \frac{1}{6}\binom{m+1}{2}n^{m-1} - \frac{1}{30}\binom{m+1}{4}n^{m-3} + \frac{1}{42}\binom{m+1}{6}n^{m-5} + \dots\right).$$

With :

$$\binom{n}{k} = C_n^k = \frac{n!}{k!(n-k)!}.$$

The first Bernoulli numbers are given in the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$	0	$\frac{7}{6}$

Jacques Bernoulli knew a few formulas like :

$$\begin{aligned}
1 + 2 + 3 + \cdots + (n-1) &= \frac{1}{2}n^2 - \frac{n}{2} &&= \frac{n(n-1)}{2};\\
1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 &= \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{n}{6} &&= \frac{n(n-1)(2n-1)}{6};\\
1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 &= \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2 &&= \frac{n^2(n-1)^2}{4};\\
1^4 + 2^4 + 3^4 + \cdots + (n-1)^4 &= \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{n}{30} &&= \frac{n(n-1)(2n-1)(3n^2-3n-1)}{30};\\
1^5 + 2^5 + 3^5 + \cdots + (n-1)^5 &= \frac{1}{6}n^6 - \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 &&= \frac{n^2(n-1)^2(2n^2-2n-1)}{12}.
\end{aligned}$$

Bernoulli observed that the expression :

$$S_m(n)=\sum_{k=0}^{n-1}k^m=0^m+1^m+2^m+\cdots+(n-1)^m$$

Is always a polynomial in n, of degree $m+1$, and defines the Bernoulli numbers B_k by :

$$S_m(n)=\sum_{k=0}^{m}\frac{m!}{(m+1-k)!}\frac{B_k}{k!}n^{m+1-k}=\frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k\,n^{m+1-k}=\sum_{k=0}^{m}\binom{m}{k}B_k\frac{n^{m+1-k}}{m+1-k}.$$

In particular, the coefficient of n in the polynomial $S_m(n)$ is the number B_k.

2.3. Special values of the Reimann zeta function4

2.3.1. In 0 and 1

In zero, we have : $\zeta(0)=-\frac{1}{2}$

In 1 there is a pole, so $\zeta(1)$ is not finite but the limit is $-\infty$on the left and $+\infty$right :

$$\lim_{s\to1^{\pm}}\zeta(s)=\pm\infty$$

2.3.2. Positive Even Integers

The exact values of the zeta function at even positive integers can be expressed from Bernoulli numbers:

$$\forall n\in\mathbb{N},\ \zeta(2n)=(-1)^{n+1}\frac{(2)^{2n-1}B_{2n}}{(2n)!}\pi^{2n}.$$

2.3.3 Odd Positive Integers

There is no general formula for calculating the zeta function for odd positive integers.

The sum of the harmonic series is infinite:

$$\zeta(1)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots=\infty$$

The value $\zeta(3)$ is also known as Apéry's constant (1.202..) and appears in the electron's gyromagnetic ratio. The value $\zeta(5)$ appears in Planck's law (1.036...).

2.3.4 Negative Integers

In general, for any negative integer, we have :

$$\zeta(-n)=(-1)^n\frac{B_{n+1}}{n+1}$$

Trivial" zeros are negative even integers:

$$\zeta(-2n)=0\qquad(B_{2n+1}=0:n>0)$$

3. Graphic Observations

We introduce the function $s\to Z(s)$ for all s in the set of complex numbers $s=a+bi$such that :

$$Z(s)=\sum_{n=1}^{N}\frac{1}{n^s}=\sum_{n=1}^{N}\frac{1}{n^{a+bi}}$$

We have the following equality:

$$\lim_{N\to+\infty}Z(s)=\zeta(s)$$

And since :

$$\frac{1}{n^{a+bi}} = n^{-a} \times e^{-ibln(n)} = \frac{\cos(bln(n))}{n^a} - i \times \frac{\sin(bln(n))}{n^a}$$

This gives :

$$\zeta(s) = \lim_{N\to+\infty} \sum_{n=1}^{N} \frac{\cos(bln(n))}{n^a} - i \times \lim_{N\to+\infty} \sum_{n=1}^{N} \frac{\sin(bln(n))}{n^a} \quad (1)$$

Figure 1 shows the graphical representation of $Z(s)$ in the complex plane when N varies between $N1=10^3$ a $N_2=6\times10^5$ for some real numbers (a,b).

Indications :

We know that :

- $\zeta\left(\frac{1}{2} \pm \text{ i} \times 49.773832478 \ldots\right) = 0$
- $\zeta\left(\frac{1}{2} \pm \text{ i} \times 101.317851006 \ldots\right) = 0$

49.773832478…; 101.317851006…are estimates of the imaginary part of some non-trivial zeros of the zeta function.

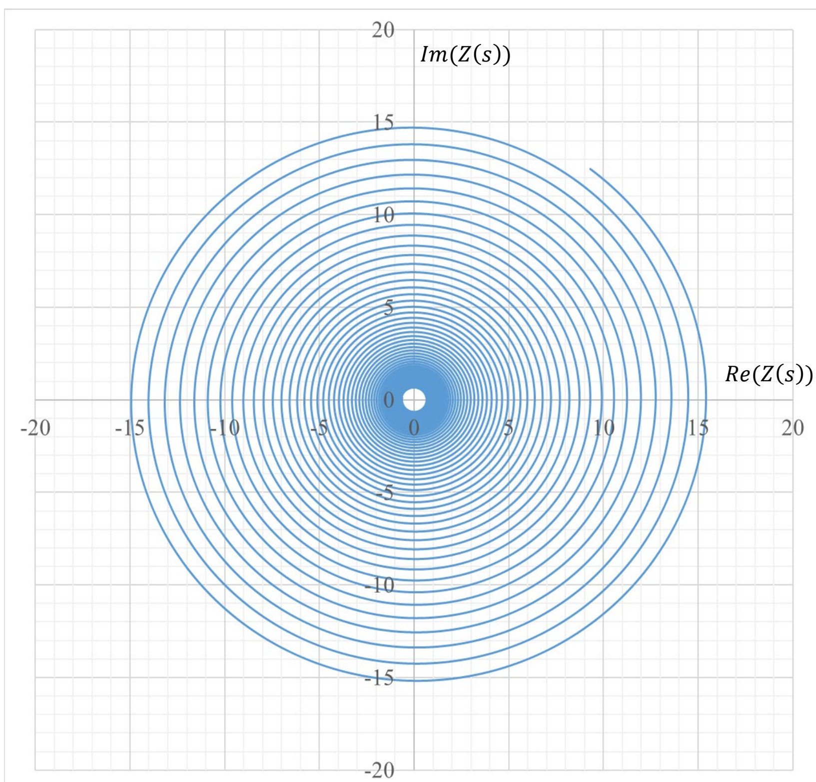

Figure 1 Graphical representation of $Z(s)$ of $N_1 = 10^3$ a $N_2 = 6 \times 10^5$ in the complex plane when $s = \frac{1}{2} + i \times 49.773832478 \ldots$

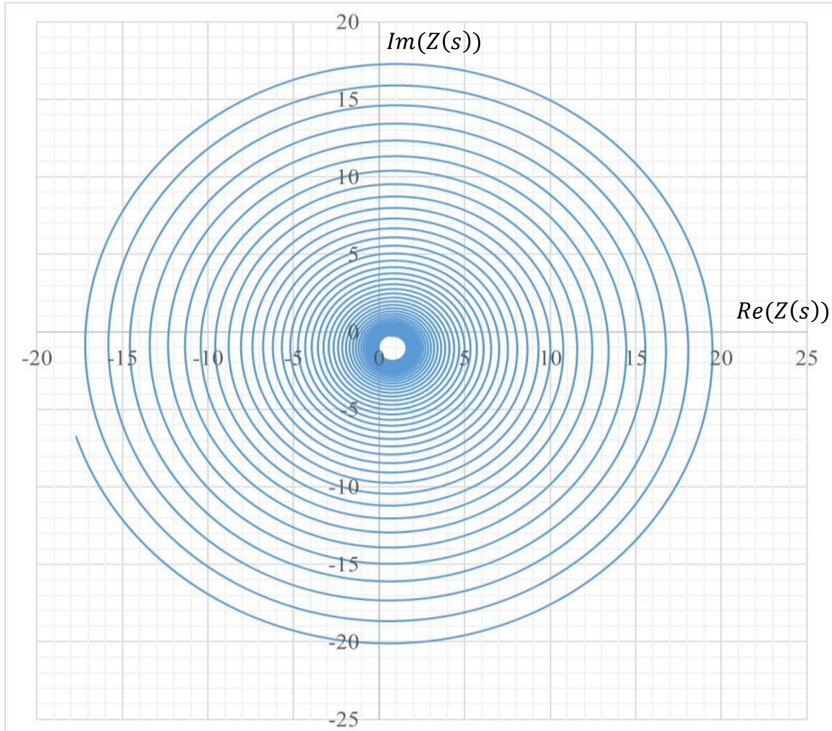

Figure 2: Graphical representation of $Z(s)$ of $N_1 = 10^3$ a $N_2 = 6 \times 10^5$ in the complex plane when $s = \frac{1}{2} + i \times 40$

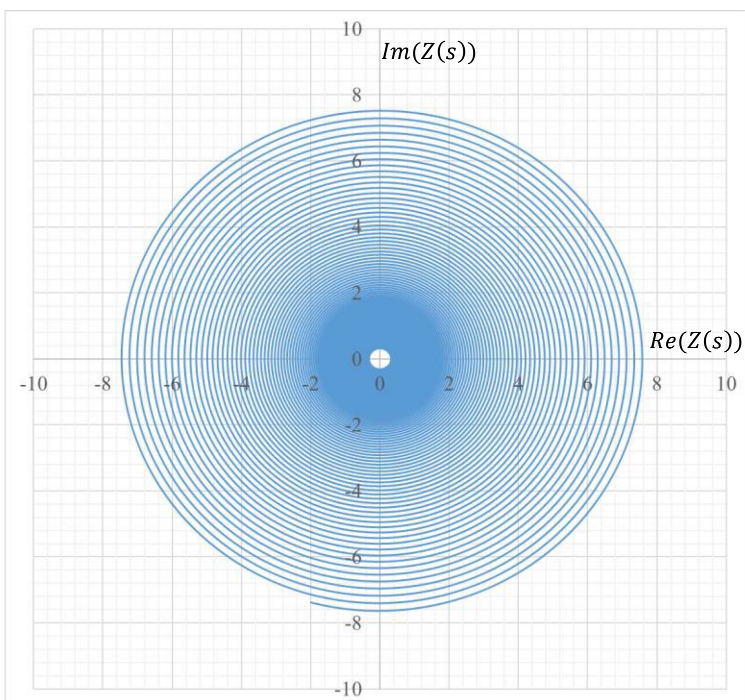

Figure 3: Graphical representation of $Z(s)$ of $N_1 = 10^3$ a $N_2 = 6 \times 10^5$ in the complex plane when $s = \frac{1}{2} + i \times 101.317851006 \ldots$

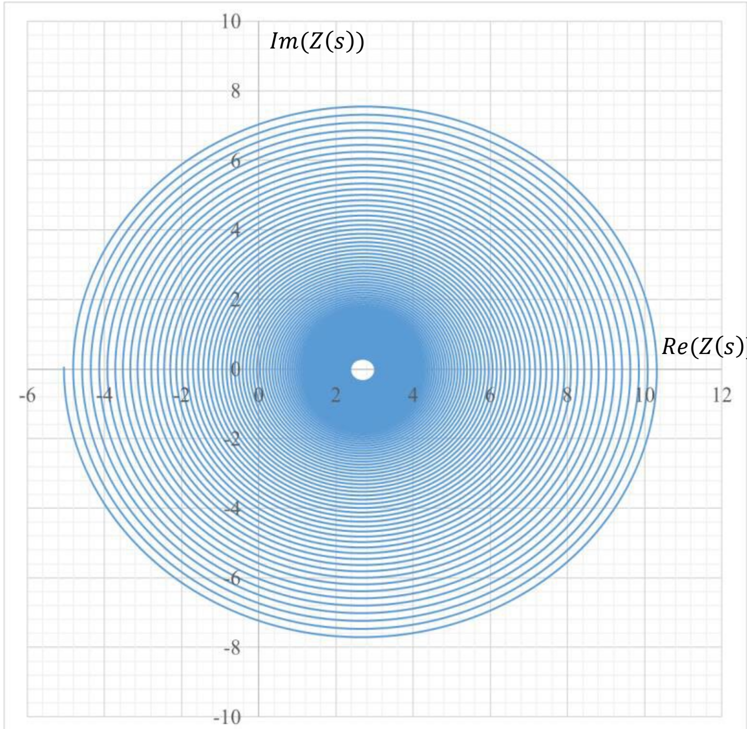

Figure 4: Graphical representation of $Z(s)$ of $N_1 = 10^3$ a $N_2 = 6 \times 10^5$ in the complex plane when $s = \frac{1}{2} + i \times 100$

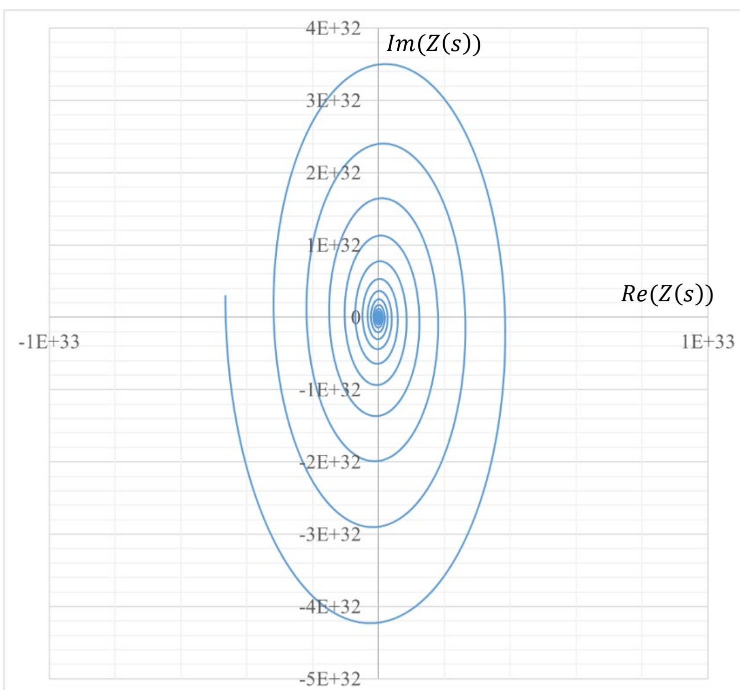

Figure 5: Graphical representation of $Z(s)$ of $N_1 = 10^3$ a $N_2 = 6 \times 10^5$ in the complex plane when $s = -5 + i \times 100$

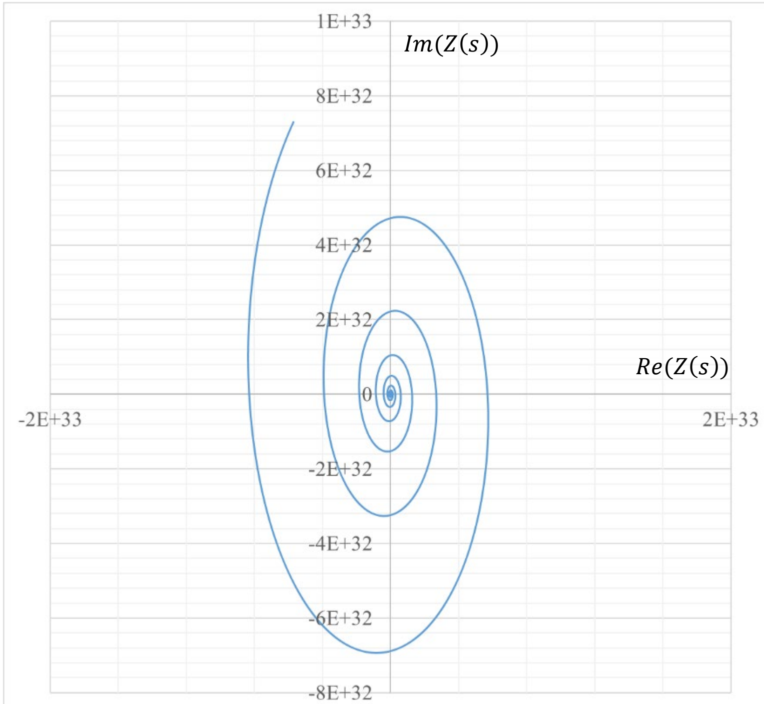

Figure 6: Graphical representation of $Z(s)$ of $N_1 = 10^3$ a $N_2 = 6 \times 10^5$ in the complex plane when $s = -5 + i \times 50$

From figures 1, 2, 3, 4, 5 and 6, we can draw the following conclusions:

- When $N \to +\infty$, $Z(s)$:$s=a+bi$in the complex plane is a spiral with radius $r \in \mathbb{R}$ depending on N, a and b and a center with coordinates $(u,v) \in \mathbb{R}^2$ depending on a and b ;
- When $a=1/2$ and b takes one of the values of the imaginary part of the non-trivial zeros of the zeta function, the spiral appears to have the origin of the reference frame as its center (0,0).

4. Hypothesis (0)

Based on graphical observation, hypothesis (0) can be formulated:
The analytical extension of Reimann's zeta function can be written as ζ(s)=$u+iv$ and (u,v) are the coordinates of the center of the $Z(s)$ spiral towards infinity.

5. Demonstration

From (1) we have :

$$\zeta(s) = \lim_{N \to +\infty} \sum_{n=1}^{N} \frac{\cos(bln(n))}{n^a} - i \times \lim_{N \to +\infty} \sum_{n=1}^{N} \frac{\sin(bln(n))}{n^a}$$

Posing $f(n) = \frac{\cos(bln(n))}{n^a}$ and $g(n) = \frac{\sin(bln(n))}{n^a}$

By applying the Euler-Maclaurin formula, which can be expressed as follows:
For a function f continuously differentiable $2k$ times on the segment $[p, q]$ (with $k \geq 1$) we have:

$$\sum_{n=1}^{N} f(n) = \frac{f(1) + f(N)}{2} + \int_1^N f(x)dx + \sum_{j=1}^{k} \frac{b_{2j}}{(2j)!}\left(f^{(2j-1)}(N) - f^{(2j-1)}(1)\right) + R_k$$

The numbers b_2j denote Bernoulli numbers, and the remainder R_k is expressed using the Bernoulli polynomial B_2k :

$$R_k = -\frac{1}{(2k)!}\int_p^q f^{(2k)}(x)B_{2k}(x-\lfloor x\rfloor)\,\mathrm{d}x.$$

The functions f and g are continuously differentiable $2k$ times on the segment $[1, N]$ (with $k \geq 1$), then :

$$\sum_{n=1}^{N} f(n) = \frac{f(1)+f(N)}{2} + \int_1^N f(x)dx + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}(f^{(2j-1)}(N) - f^{(2j-1)}(1))$$

$$\sum_{n=1}^{N} g(n) = \frac{g(1)+g(N)}{2} + \int_1^N g(x)dx + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}(g^{(2j-1)}(N) - g^{(2j-1)}(1))$$

$$\lim_{k\to+\infty} R_k = 0$$

Calculation of $\int_1^N f(x)dx$ et $\int_1^N g(x)dx$:

$$\int_1^N \frac{\cos(bln(x))}{x^a} = \int_1^N \left[\frac{x^{1-a}}{1-a}\right]' \cos(bln(x)) = \left[\frac{x^{1-a}}{1-a}\cos(bln(x))\right]_1^N + \frac{b}{1-a}\int_1^N \frac{\sin(bln(x))}{x^a}$$

$$= \frac{N^{1-a}}{1-a}\cos(bln(N)) - \frac{1}{1-a} + \frac{b}{1-a}\int_1^N \frac{\sin(bln(x))}{x^a} \qquad (a \neq 1)$$

$$\int_1^N \frac{\sin(bln(x))}{x^a} = \int_1^N \left[\frac{x^{1-a}}{1-a}\right]' \sin(bln(x)) = \left[\frac{x^{1-a}}{1-a}\sin(bln(x))\right]_1^N - \frac{b}{1-a}\int_1^N \frac{\cos(bln(x))}{x^a}$$

$$= \frac{N^{1-a}}{1-a}\sin(bln(N)) - \frac{b}{1-a}\int_1^N \frac{\cos(bln(x))}{x^a}$$

Substitute $\int_1^N \frac{\cos(bln(x))}{x^a}$ and $\int_1^N \frac{\sin(bln(x))}{x^a}$ in both ties :

$$\int_1^N \frac{\cos(bln(x))}{x^a} = \frac{N^{1-a}}{1-a}\cos(bln(N)) - \frac{1}{1-a} + \frac{b}{1-a} \times \left[\frac{N^{1-a}}{1-a}\sin(bln(N)) - \frac{b}{1-a}\int_1^N \frac{\cos(bln(x))}{x^a}\right]$$

$$\equiv \left[1 + \left(\frac{b}{1-a}\right)^2\right] \times \int_1^N \frac{\cos(bln(x))}{x^a} = \frac{N^{1-a}}{1-a} \times \left[\cos(bln(N)) + \frac{b}{1-a}\sin(bln(N))\right] - \frac{1}{1-a}$$

$$\equiv \int_1^N \frac{\cos(bln(x))}{x^a} = \frac{\left[\frac{N^{1-a}}{1-a}\times\left[\cos(bln(N)) + \frac{b}{1-a}\sin(bln(N))\right] - \frac{1}{1-a}\right]}{\left[1+\left(\frac{b}{1-a}\right)^2\right]}$$

$$\int_1^N \frac{\sin(bln(x))}{x^a} = \frac{N^{1-a}}{1-a}\sin(bln(N)) - \frac{b}{1-a} \times \left[\frac{N^{1-a}}{1-a}\cos(bln(N)) - \frac{1}{1-a} + \frac{b}{1-a}\int_1^N \frac{\sin(bln(x))}{x^a}\right]$$

$$\equiv \left[1 + \left(\frac{b}{1-a}\right)^2\right] \times \int_1^N \frac{\sin(bln(x))}{x^a} = \frac{N^{1-a}}{1-a} \times \left[\sin(bln(N)) - \frac{b}{1-a}\cos(bln(N))\right] + \frac{b}{(1-a)^2}$$

$$\equiv \int_1^N \frac{\sin(bln(x))}{x^a} = \frac{\left[\frac{N^{1-a}}{1-a}\times\left[\sin(bln(N))-\frac{b}{1-a}\cos(bln(N))\right]+\frac{b}{(1-a)^2}\right]}{\left[1+\left(\frac{b}{1-a}\right)^2\right]}$$

This gives :

$$\int_1^N \boldsymbol{f(x)dx} = \int_1^N \frac{\cos(bln(x))}{x^a} = \frac{\left[\frac{N^{1-a}}{1-a}\times\left[\cos(bln(N))+\frac{b}{1-a}\sin(bln(N))\right]-\frac{1}{1-a}\right]}{\left[1+\left(\frac{b}{1-a}\right)^2\right]} \qquad (2)$$

$$\int_1^N \boldsymbol{g(x)dx} = \int_1^N \frac{\sin(bln(x))}{x^a} = \frac{\left[\frac{N^{1-a}}{1-a}\times\left[\sin(bln(N))-\frac{b}{1-a}\cos(bln(N))\right]+\frac{b}{(1-a)^2}\right]}{\left[1+\left(\frac{b}{1-a}\right)^2\right]}$$

Calculation of $\boldsymbol{f^{(2j-1)}(N)}$ *et* $\boldsymbol{f^{(2j-1)}(1)}$:

Calculating the first and second derivatives of *f* :

$$\left[\frac{\cos(bln(x))}{x^a}\right]' = \frac{-ax^{a-1}\cos(bln(x))-bx^{a-1}\sin(bln(x)}{x^{2a}} = \frac{-a\cos(bln(x))-bsin(bln(x))}{x^{a+1}}$$

$$\left[\frac{\cos(bln(x))}{x^a}\right]'' = \left[\frac{-a\cos(bln(x))-bsin(bln(x))}{x^{a+1}}\right]' = -a\times\left[\frac{\cos(bln(x))}{x^{a+1}}\right]' - b\times\left[\frac{\sin(bln(x))}{x^{a+1}}\right]'$$

$$= -a\times\left[\frac{-(a+1)\cos(bln(x))-bsin(bln(x))}{x^{a+2}}\right] - b\times\left[\frac{bcos(bln(x))-(a+1)\sin(bln(x))}{x^{a+2}}\right]$$

$$= \frac{(a(a+1)-b^2)\cos(bln(x)) + (b(a+1)+ab)\sin(bln(x))}{x^{a+2}}$$

We can see that the derivative of order *k* of *f* can be written as :

$$f^{(k)} = \frac{A_k\sin(bln(x)) + B_k\cos(bln(x))}{x^{a+k}}$$

A_k *et* B_k Are factors that depend on *a* and *b* and the derivation order *k*.

$$A_1 = -b \quad B_1 = -a$$

$$A_2 = b(a+a+1) \quad B_2 = a(a+1) - b^2$$

$$f^{(k+1)} = \left[\frac{A_k\sin(bln(x))+B_k\cos(bln(x))}{x^{a+k}}\right]' = A_k\times\left[\frac{\sin(bln(x))}{x^{a+k}}\right]' + B_k\times\left[\frac{\cos(bln(x))}{x^{a+k}}\right]'$$

$$= A_k\times\left[\frac{bcos(bln(x))-(a+k)\sin(bln(x))}{x^{a+k+1}}\right] + B_k\times\left[\frac{-bsin(bln(x))-(a+k)\cos(bln(x))}{x^{a+k+1}}\right]$$

$$= \frac{(A_k\times b-B_k\times(a+k))\cos(bln(x))-(A_k\times(a+k)+B_k\times b)\sin(bln(x))}{x^{a+k+1}}$$

And so we have the following two equalities:

$$A_{k+1} = -(A_k \times (a+k) + B_k \times b)$$
$$B_{k+1} = (A_k \times b - B_k \times (a+k)) \quad (3)$$

They are used to calculate the remainder of the factors:

$$A_3 = -(A_2 \times (a+2) + B_2 \times b)$$
$$= -(b(a+a+1) \times (a+2) + (a(a+1) - b^2) \times b)$$
$$= b^3 - b(a(a+1) + a(a+2) + (a+1)(a+2))$$

$$B_3 = (A_2 \times b - B_2 \times (a+2))$$
$$= b(a+a+1) \times b - (a(a+1) - b^2) \times (a+2)$$
$$= b^2(a+a+1+a+2) - a(a+1)(a+2)$$

We know that the combination without repetition of *n* elements taken *p* by *p* is equal to :

$$C_n^p = \frac{n!}{p! \times (n-p)!}$$

Considering the function $a \to K_n^p(a)$, $(a \in \mathbb{R})$, $n \in \mathbb{N}*, p \in \mathbb{N}$ *et* $n \geq p+1$ which sums the multiplications between all the elements of the non-repeating combinations of the n elements $\{a; a+1; a+2; ....; a+n-1 : n \in \mathbb{N}*\}$ taken *p* by *p* :

$K_2^1(a) = (a) + (a+1)$; $\quad C_2^1 = \frac{2!}{1! \times (2-1)!} = 2$

$K_2^2(a) = a(a+1)$; $\quad C_2^2 = \frac{2!}{2! \times (2-2)!} = 1$

$K_3^2(a) = a(a+1) + a(a+2) + (a+1)(a+2)$; $\quad C_3^2 = \frac{3!}{2! \times (3-2)!} = 3$

$K_3^1(a) = (a) + (a+1) + (a+2)$; $\quad C_3^1 = \frac{3!}{1! \times (3-1)!} = 3$

$K_3^3(a) = a(a+1)(a+2)$; $\quad C_3^3 = \frac{3!}{3! \times (3-3)!} = 1$

We'll have :

$A_1 = -bK_1^0(a)$; $\quad B_1 = -K_1^1(a)$

$A_2 = bK_2^1(a)$; $\quad B_2 = -(b^2K_2^0(a) - K_2^2(a))$

$A_3 = b^3K_3^0(a) - bK_3^2(a)$; $\quad B_3 = b^2K_3^1(a) - bK_3^3(a)$

$A_4 = -(b^3K_4^1(a) - bK_4^3(a))$; $\quad B_4 = b^4K_4^0(a) - b^2K_4^2(a) + K_4^4(a)$

$A_5 = -(b^5K_5^0(a) - b^3K_5^2(a) + bK_5^4(a))$; $B_5 = -(b^4K_5^1(a) - b^2K_5^3(a) + K_5^5(a))$

$A_6 = b^5K_6^1(a) - b^3K_6^3(a) + bK_6^5(a)$; $B_6 = -(b^6K_6^0(a) - b^4K_6^2(a) + b^2K_6^4(a) - K_6^6(a))$

We observe that A_k and B_k can be written in terms of the parity of the derivation order k :
$\forall\, k \in \mathbb{N}* si\ k=2n+1 : n \in \mathbb{N}$:

$$A_{2n+1} = \sum_{p=0}^{n} (-1)^{p+n+1} b^{2(n-p)+1} K_{2n+1}^{2p}(a)$$

$$B_{2n+1} = \sum_{p=0}^{n} (-1)^{p+n+1} b^{2(n-p)} K_{2n+1}^{2p+1}(a) \quad (4)$$

$\forall\, k \in \mathbb{N}^*\ si\ k = 2n\ : n \in \mathbb{N}^*$:

$$A_{2n} = \sum_{p=1}^{n} (-1)^{p+n} b^{2(n-p)+1} K_{2n}^{2p-1}(a)$$

$$B_{2n} = \sum_{p=0}^{n} (-1)^{p+n} b^{2(n-p)} K_{2n}^{2p}(a)$$

Demonstration by recurrence :

For $k = 1$:

$$A_1 = \sum_{p=0}^{0} (-1)^{p+n+1} b^{2(n-p)+1} K_{2n+1}^{2p}(a) = -b\, K_1^0(a) = -b$$

$$B_1 = \sum_{p=0}^{0} (-1)^{p+n+1} b^{2(n-p)} K_{2n+1}^{2p+1}(a) = -K_1^1(a) = -a$$

For $k = 2$:

$$A_2 = \sum_{p=1}^{1} (-1)^{p+n} b^{2(n-p)+1} K_{2n}^{2p-1}(a) = bK_2^1(a) = b(a + a + 1)$$

$$B_2 = \sum_{p=0}^{1} (-1)^{p+n} b^{2(n-p)} K_{2n}^{2p}(a) = -(b^2K_2^0(a) - K_2^2(a)) = a(a+1) - b^2$$

Assuming : $\forall\, k \in \mathbb{N}* si\ k=2n : n \in \mathbb{N}*$:

$$A_{2n} = \sum_{p=1}^{n} (-1)^{p+n} b^{2(n-p)+1} K_{2n}^{2p-1}(a)$$

$$B_{2n} = \sum_{p=0}^{n} (-1)^{p+n} b^{2(n-p)} K_{2n}^{2p}(a)$$

From (3) we have :

$$A_{k+1} = -(A_k \times (a+k) + B_k \times b)$$

$$B_{k+1} = (A_k \times b - B_k \times (a+k))$$

So :

$$A_{2n+1} = -(A_{2n} \times (a+2n) + B_{2n} \times b)$$

$$= -(a+2n)\sum_{p=1}^{n}(-1)^{p+n}b^{2(n-p)+1}K_{2n}^{2p-1}(a) - b\sum_{p=0}^{n}(-1)^{p+n}b^{2(n-p)}K_{2n}^{2p}(a)$$

$$= (a+2n)\sum_{p=1}^{n}(-1)^{p+n+1}b^{2(n-p)+1}K_{2n}^{2p-1}(a) + \sum_{p=0}^{n}(-1)^{p+n+1}b^{2(n-p)+1}K_{2n}^{2p}(a)$$

$$= \sum_{p=1}^{n}(-1)^{p+n+1}b^{2(n-p)+1}(a+2n)K_{2n}^{2p-1}(a) + \sum_{p=1}^{n}(-1)^{p+n+1}b^{2(n-p)+1}K_{2n}^{2p}(a) + (-1)^{n+1}b^{2n+1}K_{2n+1}^{0}(a)$$

$$= \sum_{p=1}^{n}(-1)^{p+n+1}b^{2(n-p)+1}\left[(a+2n)K_{2n}^{2p-1}(a) + K_{2n}^{2p}(a)\right] + (-1)^{n+1}b^{2n+1}K_{2n+1}^{0}(a)$$

$$= \sum_{p=1}^{n}(-1)^{p+n+1}b^{2(n-p)+1}\,K_{2n+1}^{2p}(a) + (-1)^{n+1}b^{2n+1}K_{2n+1}^{0}(a)$$

$$= \sum_{p=0}^{n}(-1)^{p+n+1}b^{2(n-p)+1}\,K_{2n+1}^{2p}(a)$$

Because $(a+2n)K_{2n}^{2p-1}(a) + K_{2n}^{2p}(a) = K_{2n+1}^{2p}(a)$ (Appendix 1)

$$B_{2n+1} = (A_{2n} \times b - B_{2n} \times (a+2n))$$

$$= b\sum_{p=1}^{n}(-1)^{p+n}b^{2(n-p)+1}K_{2n}^{2p-1}(a) - (a+2n)\sum_{p=0}^{n}(-1)^{p+n}b^{2(n-p)}K_{2n}^{2p}(a)$$

$$= \sum_{p=0}^{n-1}(-1)^{p+n+1}b^{2(n-p)}K_{2n}^{2p+1}(a) + \sum_{p=0}^{n}(-1)^{p+n+1}b^{2(n-p)}(a+2n)K_{2n}^{2p}(a)$$

$$= \sum_{p=0}^{n-1}(-1)^{p+n+1}b^{2(n-p)}\left[K_{2n}^{2p+1}(a) + (a+2n)K_{2n}^{2p}(a)\right] + (-1)^{2n+1}(a+2n)K_{2n}^{2n}(a)$$

$$= \sum_{p=0}^{n-1}(-1)^{p+n+1}b^{2(n-p)}K_{2n+1}^{2p+1}(a) + (-1)^{2n+1}(a+2n)K_{2n}^{2n}(a)$$

$$= \sum_{p=0}^{n}(-1)^{p+n+1}b^{2(n-p)}K_{2n+1}^{2p+1}(a)$$

Because :

$$K_{2n}^{2p+1}(a) + (a+2n)K_{2n}^{2p}(a) = K_{2n+1}^{2p+1}(a)$$

et $(a+2n)K_{2n}^{2n}(a) = K_{2n+1}^{2n+1}(a)$ (Appendix 1)

And since we're interested in derivatives of order $(2j-1)$:

$$f^{(2j-1)}(x) = \frac{A_{2j-1}\sin\left(b\ln(x)\right) + B_{2j-1}\cos(b\ln(x))}{x^{a+2j-1}}$$

Replacing n by ($j-1$) in (4) :

$$A_{2j-1} = \sum_{p=0}^{j-1} (-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)$$

$$B_{2j-1} = \sum_{p=0}^{j-1} (-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a) \qquad (5)$$

$$f^{(2j-1)}(N) = \frac{A_{2j-1}\sin\big(bln(N)\big) + B_{2j-1}\cos\big(bln(N)\big)}{N^{a+2j-1}}$$

$$f^{(2j-1)}(1) = \frac{A_{2j-1}\sin\big(bln(1)\big) + B_{2j-1}\cos\big(bln(1)\big)}{1^{a+2j-1}} = B_{2j-1} \qquad (6)$$

Calculation of $g^{(2j-1)}(N)$ et $g^{(2j-1)}(1)$:

By doing the same for the function $g(x) = \frac{\sin(bln(x))}{x^a}$:

$$\left[\frac{\sin(bln(x))}{x^a}\right]' = \frac{bx^{a-1}\cos(bln(x)) - ax^{a-1}\sin(bln(x)}{x^{2a}} = \frac{b\cos(bln(x)) - a\sin(bln(x))}{x^{a+1}}$$

$$g^{(k)} = \frac{A'_k \sin\big(bln(x)\big) + B'_k \cos\big(bln(x)\big)}{x^{a+k}}$$

$$A'_1 = -a = B_1 \quad B'_1 = b = -A_1$$

We show by recurrence that : $\forall k \in \mathbb{N}^* \; A'_k = B_k \;\; et \; B'_k = -A_k$

For $k = 1$ it's true.

Assuming that $\forall k \in \mathbb{N}^* \; A'_k = B_k \;\; et \; B'_k = -A_k$

$$g^{(k)} = \frac{B_k \sin(bln(x)) - A_k \cos(bln(x))}{x^{a+k}}$$

$$g^{(k+1)} = \left[\frac{B_k \sin(bln(x)) - A_k \cos(bln(x))}{x^{a+k}}\right]'$$

$$= B_k \times \left[\frac{\sin(bln(x))}{x^{a+k}}\right]' - A_k \times \left[\frac{\cos(bln(x))}{x^{a+k}}\right]'$$

$$= B_k \times \left[\frac{bcos(bln(x)) - (a+k)\sin(bln(x))}{x^{a+k+1}}\right] - A_k \times \left[\frac{-bsin(bln(x)) - (a+k)\cos(bln(x))}{x^{a+k+1}}\right]$$

$$= \frac{(B_k \times b + A_k \times (a+k))\cos(bln(x)) + (A_k \times b - B_k \times (a+k))\sin(bln(x))}{x^{a+k+1}}$$

$A'_{k+1} = A_k \times b - B_k \times (a+k)$

$B'_{k+1} = B_k \times b + A_k \times (a+k)$

We know that :

$A_{k+1} = -(A_k \times (a+k) + B_k \times b)$

$B_{k+1} = (A_k \times b - B_k \times (a+k))$

We therefore have equality :

$A'_{k+1} = B_{k+1}$ and $B'_{k+1} = -A_{k+1}$

As a result : $\forall k \in \mathbb{N}^*\ A'_k = B_k\ \ et\ B'_k = -A_k$

And the derivatives of order (2j−1) of g can be calculated as follows:

$$g^{(2j-1)} = \frac{A'_{2j-1}\sin(bln(x)) + B'_{2j-1}\cos(bln(x))}{x^{a+2j-1}}$$

$$= \frac{B_{2j-1}\sin(bln(x)) - A_{2j-1}\cos(bln(x))}{x^{a+2j-1}}$$

Consequently :

$$g^{(2j-1)}(N) = \frac{B_{2j-1}\sin(bln(N)) - A_{2j-1}\cos(bln(N))}{N^{a+2j-1}}$$

$$g^{(2j-1)}(1) = \frac{B_{2j-1}\sin(bln(1)) - A_{2j-1}\cos(bln(1))}{1^{a+2j-1}} = -A_{2j-1} \qquad (7)$$

Calculation of $\sum_{n=1}^{N} f(n)$ and $\sum_{n=1}^{N} g(n)$:

According to Euler-Maclaurin we have :

$$\sum_{n=1}^{N} f(n) = \frac{f(1)+f(N)}{2} + \int_1^N f(x)dx + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}(f^{(2j-1)}(N) - f^{(2j-1)}(1))$$

$$\sum_{n=1}^{N} g(n) = \frac{g(1)+g(N)}{2} + \int_1^N g(x)dx + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}(g^{(2j-1)}(N) - g^{(2j-1)}(1))$$

According to relationships (5); (6) and (7) :

$$\sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}\left(f^{(2j-1)}(N) - f^{(2j-1)}(1)\right) = \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}\left(\frac{A_{2j-1}\sin(bln(N)) + B_{2j-1}\cos(bln(N))}{N^{a+2j-1}} - B_{2j-1}\right)$$

$$\sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}\left(g^{(2j-1)}(N) - g^{(2j-1)}(1)\right) = \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}\left(\frac{B_{2j-1}\sin(bln(N)) - A_{2j-1}\cos(bln(N))}{N^{a+2j-1}} + A_{2j-1}\right)$$

Posing :

- $U = \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times \frac{A_{2j-1}}{N^{a+2j-1}})$
- $V = \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times \frac{B_{2j-1}}{N^{a+2j-1}})$
- $W_1 = \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times B_{2j-1})$
- $W_2 = \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times A_{2j-1})$

We'll have :

$$\sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}\left(f^{(2j-1)}(N) - f^{(2j-1)}(1)\right) = U\sin\left(bln(N)\right) + V\cos\left(bln(N)\right) - W_1$$

$$\sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!}\left(g^{(2j-1)}(N) - g^{(2j-1)}(1)\right) = V\sin\left(bln(N)\right) - U\cos\left(bln(N)\right) + W_2$$

From (2) we have :

$$\frac{f(1)+f(N)}{2} + \int_1^N f(x)dx = \frac{1}{2} + \frac{\cos\left(bln(N)\right)}{2N^a} + \frac{\left[\frac{N^{1-a}}{1-a}\times\left[\cos\left(bln(N)\right)+\frac{b}{1-a}\sin\left(bln(N)\right)\right]-\frac{1}{1-a}\right]}{\left[1+\left(\frac{b}{1-a}\right)^2\right]}$$

$$\frac{g(1)+g(N)}{2} + \int_1^N g(x) = \frac{\sin\left(bln(N)\right)}{2N^a} + \frac{\left[\frac{N^{1-a}}{1-a}\times\left[\sin\left(bln(N)\right)-\frac{b}{1-a}\cos\left(bln(N)\right)\right]+\frac{b}{(1-a)^2}\right]}{\left[1+\left(\frac{b}{1-a}\right)^2\right]}$$

Posing :

- $U' = \frac{\frac{bN^{1-a}}{(1-a)^2}}{\left[1+\left(\frac{b}{1-a}\right)^2\right]} = \frac{bN^{1-a}}{(1-a)^2+b^2}$
- $V' = \frac{1}{2N^a} + \frac{\frac{N^{1-a}}{1-a}}{\left[1+\left(\frac{b}{1-a}\right)^2\right]} = \frac{1}{2N^a} + \frac{(1-a)N^{1-a}}{(1-a)^2+b^2}$
- $W_1' = \frac{1}{2} - \frac{\frac{1}{1-a}}{\left[1+\left(\frac{b}{1-a}\right)^2\right]} = \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2}$
- $W_2' = \frac{\frac{b}{(1-a)^2}}{\left[1+\left(\frac{b}{1-a}\right)^2\right]} = \frac{b}{(1-a)^2+b^2}$

We'll have :

$$\frac{f(1)+f(N)}{2}+\int_1^N f(x)dx = U'\sin\big(bln(N)\big)+V'\cos\big(bln(N)\big)+W_1'$$

$$\frac{g(1)+g(N)}{2}+\int_1^N g(x)dx = V'\sin\big(bln(N)\big)-U'\cos\big(bln(N)\big)+W_2'$$

This gives :

$$\sum_{n=1}^{N} f(n) = U'\sin\big(bln(N)\big)+V'\cos\big(bln(N)\big)+W_1'+U\sin\big(bln(N)\big)+ V\cos\big(bln(N)\big)-W_1$$

$$=(U'+U)\sin\big(bln(N)\big)+(V'+V)\cos\big(bln(N)\big)+W_1'-W_1)$$

$$\sum_{n=1}^{N} g(n) = V'\sin\big(bln(N)\big)-U'\cos\big(bln(N)\big)+W_2'+V\sin\big(bln(N)\big) - U\cos\big(bln(N)\big)+W_2$$

$$=(V'+V)\sin\big(bln(N)\big)-(U'+U)\cos\big(bln(N)\big)+W_2'+W_2$$

$$\sum_{n=1}^{N} g(n) = (V'+V)\sin\big(bln(N)\big)-(U'+U)\cos\big(bln(N)\big)+W_2'+W_2$$

$$\sum_{n=1}^{N} f(n) = (U'+U)\sin\big(bln(N)\big)+(V'+V)\cos\big(bln(N)\big)+W_1'-W_1 \quad (8)$$

Note that :

$$\left[\sum_{n=1}^{N} f(n)-(W_1'-W_1)\right]^2+\left[\sum_{n=1}^{N} g(n)-(W_2'+W_2)\right]^2 = (U'+U)^2+(V'+V)^2$$

Since we always have :

$$[A\sin(x)+B\cos(x)]^2+[B\sin(x)-A\cos(x)]^2 = A^2+B^2$$

We know that the general equation of a spiral of variable radius r and center (u,v) in the Cartesian plane can be written as follows:

$$[x-u]^2+[y-v]^2=r^2$$

Replacing x by $\sum_{n=1}^{N} f(n)$ and y by $-\sum_{n=1}^{N} g(n)$ in the equation, we conclude that :

$\forall (a,b)\in\mathbb{R}^2, a\neq 1, b\neq 0, Z(s)=\sum_{n=1}^{N} f(n) - i\times\sum_{n=1}^{N} g(n)$ in the complex plane is a spiral of radius $r=\sqrt{(U'+U)^2+(V'+V)^2}$ and center

$(W_1'-W_1, -(W_2'+W_2))$. This corroborates our observation.

$f(n) = \frac{\cos(bln(n))}{n^a}$ and $g(n) = \frac{\sin(bln(n))}{n^a}$

$$r = \sqrt{(U' + U)^2 + (V' + V)^2}$$

$$=$$

$$\sqrt{\left(\frac{bN^{1-a}}{(1-a)^2+b^2} + \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times \frac{A_{2j-1}}{N^{a+2j-1}})\right)^2 + \left(\frac{1}{2N^a} + \frac{(1-a)N^{1-a}}{(1-a)^2+b^2} + \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times \frac{B_{2j-1}}{N^{a+2j-1}})\right)^2}$$

$$(W_1' - W_1 , -(W_2' + W_2))$$

$$=$$

$$(\frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} B_{2j-1}) , \frac{-b}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} A_{2j-1}))$$

From (1) we have :

$$\zeta(a+ib) = \lim_{N\to+\infty} \sum_{n=1}^{N} \frac{\cos(bln(n))}{n^a} - i \times \lim_{N\to+\infty} \sum_{n=1}^{N} \frac{\sin(bln(n))}{n^a}$$

$$= \lim_{N\to+\infty} \sum_{n=1}^{N} f(n) - i \times \lim_{N\to+\infty} \sum_{n=1}^{N} g(n)$$

$$\lim_{N\to+\infty} \sum_{n=1}^{N} f(n) = \lim_{N\to+\infty} (U' + U)\sin\big(bln(N)\big) + (V' + V)\cos\big(bln(N)\big) + W_1' - W_1$$

$$\lim_{N\to+\infty} \sum_{n=1}^{N} g(n) = \lim_{N\to+\infty} (V' + V)\sin\big(bln(N)\big) - (U' + U)\cos\big(bln(N)\big) + W_2' + W_2$$

We know that :

- $-1 < \sin\big(bln(N)\big) < 1$ and $-1 < \cos\big(bln(N)\big) < 1$
- $U' + U = \frac{bN^{1-a}}{(1-a)^2+b^2} + \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times \frac{A_{2j-1}}{N^{a+2j-1}})$
- $V' + V = \frac{1}{2N^a} + \frac{(1-a)N^{1-a}}{(1-a)^2+b^2} + \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times \frac{B_{2j-1}}{N^{a+2j-1}})$
- $W_1 = \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times B_{2j-1})$
- $W_1' = \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2}$

- $W_2 = \sum_{j=1}^{+\infty}(\frac{b_{2j}}{(2j)!} \times A_{2j-1})$
- $W_2' = \frac{b}{(1-a)^2+b^2}$
- $A_{2j-1} = \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)$
- $B_{2j-1} = \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)$

When $a > 1$:

$$\lim_{N\to+\infty}(U' + U) = 0$$

$$\lim_{N\to+\infty}(V' + V) = 0$$

So :

$$\lim_{N\to+\infty} \sum_{n=1}^{N} f(n) = W_1' - W_1$$

$$= \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times B_{2j-1}\right]$$

$$= \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$$

$$\lim_{N\to+\infty} \sum_{n=1}^{N} g(n) = W_2' + W_2$$

$$= \frac{b}{(1-a)^2+b^2} + \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times A_{2j-1}\right]$$

$$= \frac{b}{(1-a)^2+b^2} + \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)\right]$$

We know that the zeta function when $a > 1$ converges.

So $\sum_{n=1}^{N} f(n)$ and $\sum_{n=1}^{N} g(n)$ also converge, and the zeta function can be written as :

$$\zeta(a+ib) = u(a,b) + iv(a,b)$$

with $u(a,b) = \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$

and $v(a,b) = \frac{-b}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)\right]$

This result can be interpreted as :

● When $N \to +\infty$ the radius of the spiral tends towards 0, producing a point on the complex plane with coordinates $(u(a,b), v(a,b))$.

When $a < 1$:

$\sum_{n=1}^{N} f(n)$ and $\sum_{n=1}^{N} g(n)$ do not admit a specific limit, but as the function *Z(s)* diverges it graphically represents on the complex plane a spiral of radius $r = \sqrt{(U' + U)^2 + (V' + V)^2}$ tending towards $+\infty$ and with a coordinate center $(u(a,b), v(a,b))$:

- $u(a,b) = \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$
- $v(a,b) = \frac{-b}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)\right]$

We define the function $s \to Y(s), \forall s \in \mathbb{C} - \{1\}, s = a + ib$, tel que :

$$Y(s) = u(a,b) + iv(a,b) \qquad (9)$$

- $u(a,b) = \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$
- $v(a,b) = \frac{-b}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)\right]$

Note that :

- $\zeta(s)$ et $Y(s)$ are holomorphic on $\mathbb{C} - \{1\}$
- $\zeta(s) = Y(s)$ on $\{s = a + ib \in \mathbb{C} - \{1\} : a > 1\}$
- $\{s = a + ib \in \mathbb{C} - \{1\} : a > 1\}$ is a part of $\mathbb{C} - \{1\}$

So according to the principle of analytical extension $\zeta(s) = Y(s)$ on $\mathbb{C} - \{1\}$.

This means that assumption (0) is true and we can write :

$$\forall s \in \mathbb{C} - \{1\}, s = a + ib : \zeta(s) = u(a,b) + iv(a,b)$$

- $u(a,b) = \frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)} K_{2j-1}^{2p+1}(a)\right]$
- $v(a,b) = \frac{-b}{(1-a)^2+b^2} - \sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!} \times \sum_{p=0}^{j-1}(-1)^{p+j} b^{2(j-p-1)+1} K_{2j-1}^{2p}(a)\right]$

Calculation of $\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!}\times\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)}K_{2j-1}^{2p+1}(a)\right]$:

$$\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!}\times\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)}K_{2j-1}^{2p+1}(a)\right]$$

$$=$$

$$-\frac{b_2}{2!}K_1^1(a)+\frac{b_4}{4!}b^2K_3^1(a)-\frac{b_4}{4!}K_3^3(a)-\frac{b_6}{6!}b^4K_5^1(a)+\frac{b_6}{6!}b^2K_5^3(a)-\frac{b_6}{6!}K_5^5(a)+$$
$$\frac{b_8}{8!}b^6K_7^1(a)-\frac{b_8}{8!}b^4K_7^3(a)+\frac{b_8}{8!}b^2K_7^5(a)-\frac{b_8}{8!}K_7^7(a)\dots$$

Calculation of $\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!}\times\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)}K_{2j-1}^{2p+1}(a)\right]$:

$$\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!}\times\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)+1}K_{2j-1}^{2p}(a)\right]$$

$$=$$

$$-\frac{b_2}{2!}bK_1^0(a)-\frac{b_4}{4!}bK_3^2(a)+\frac{b_4}{4!}b^3K_3^0(a)-\frac{b_6}{6!}bK_5^4(a)+\frac{b_6}{6!}b^3K_5^2(a)-\frac{b_6}{6!}b^5K_5^0(a)-$$
$$\frac{b_8}{8!}bK_7^6(a)+\frac{b_8}{8!}b^3K_7^4(a)-\frac{b_8}{8!}b^5K_7^2(a)+\frac{b_8}{8!}b^7K_7^0(a)\dots$$

We rearrange the terms of the two expressions:

$$\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!}\times\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)}K_{2j-1}^{2p+1}(a)\right]$$

$$=$$

$$-\left[\frac{b_2}{2!}K_1^1(a)+\frac{b_4}{4!}K_3^3(a)+\frac{b_6}{6!}K_5^5(a)+\frac{b_8}{8!}K_7^7(a)\dots\right]$$

$$+b^2\left[\frac{b_4}{4!}K_3^1(a)+\frac{b_6}{6!}K_5^3(a)+\frac{b_8}{8!}K_7^5(a)\dots\right]$$

$$-b^4\left[\frac{b_6}{6!}K_5^1(a)+\frac{b_8}{8!}K_7^3(a)\dots\right]$$

$$+b^6\left[\frac{b_8}{8!}K_7^1(a)\dots\right]$$

$$\vdots$$

$$\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!}\times\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)+1}K_{2j-1}^{2p}(a)\right]$$

$$=$$

$$-b\left[\frac{b_2}{2!}K_1^0(a)+\frac{b_4}{4!}K_3^2(a)+\frac{b_6}{6!}K_5^4(a)+\frac{b_8}{8!}K_7^6(a)\dots\right]$$

$$+b^3\left[\frac{b_4}{4!}K_3^0(a)+\frac{b_6}{6!}K_5^2(a)+\frac{b_8}{8!}K_7^4(a)\dots\right]$$

$$-b^5\left[\frac{b_6}{6!}K_5^0(a)+\frac{b_8}{8!}K_7^2(a)\dots\right]$$

$$+b^7\left[\frac{b_8}{8!}K_7^0(a)\dots\right]$$

$$\vdots$$

We observe that :

$$\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!}\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)}K_{2j-1}^{2p+1}(a)\right]$$

$$=$$

$$-\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]$$

And :

$$\sum_{j=1}^{+\infty}\left[\frac{b_{2j}}{(2j)!}\sum_{p=0}^{j-1}(-1)^{p+j}b^{2(j-p-1)+1}K_{2j-1}^{2p}(a)\right]$$

$$=$$

$$-\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]$$

Then :

$\forall s\in\mathbb{C}-\{1\}, s=a+ib$:

(10)

$$\boldsymbol{\zeta(s)}=\left[\frac{1}{2}-\frac{1-a}{(1-a)^2+b^2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]\right]$$

$$+\boldsymbol{i}\times\left[\frac{-b}{(1-a)^2+b^2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]\right]$$

6. Consequences

6.1. *b* = 0

From (10), when $b = 0$:

$$\zeta(a) = \frac{1}{2} - \frac{1}{1-a} + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2j-1}(a) = \frac{1}{2} - \frac{1}{1-a} + \sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!} \prod_{p=0}^{2(j-1)} (a+p)$$

We can check that :

- $\zeta(0) = -\frac{1}{2}$
- $\zeta(-1) = \frac{1}{2} - \frac{1}{2} - \frac{b_2}{2} = -\frac{1}{12} \quad (b_2 = \frac{1}{6})$
- $\zeta(-2) = \frac{1}{2} - \frac{1}{3} - b_2 = 0$
- Generally, we find that $\zeta(-2n) = 0$ (trivial zeros)

We know that $\forall n \in \mathbb{N}$:

$$\zeta(2n) = (-1)^{n+1} \frac{2^{2n-1} b_{2n}}{(2n)!} \pi^{2n} \quad \text{and} \quad \zeta(-n) = (-1)^n \frac{b_{n+1}}{n+1}$$

So we have the following two equalities $\forall n \in \mathbb{N}$:

$$\sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!} \prod_{p=0}^{2(j-1)} (2n+p) = (-1)^{n+1} \frac{2^{2n-1} b_{2n}}{(2n)!} \pi^{2n} - \frac{1}{2n-1} - \frac{1}{2}$$

$$\sum_{j=1}^{+\infty} \frac{b_{2j}}{(2j)!} \prod_{p=0}^{2(j-1)} (p-n) = (-1)^n \frac{b_{n+1}}{n+1} + \frac{1}{n+1} - \frac{1}{2}$$

6.2. *b*≠0

The equation : $\zeta(s)=0$

From (10), Implies :

- $\frac{1}{2} - \frac{1-a}{(1-a)^2+b^2} + \sum_{n=0}^{+\infty} \left[(-1)^n b^{2n} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n)-1}(a) \right] = 0$
- $\frac{-b}{(1-a)^2+b^2} + \sum_{n=0}^{+\infty} \left[(-1)^n b^{2n+1} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n-1)}(a) \right] = 0$

Implies :

- $\sum_{n=0}^{+\infty} \left[(-1)^n b^{2n} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n)-1}(a) \right] = \frac{1-a}{(1-a)^2+b^2} - \frac{1}{2}$
- $\sum_{n=0}^{+\infty} \left[(-1)^n b^{2n+1} \sum_{j=n+1}^{+\infty} \frac{b_{2j}}{(2j)!} K_{2j-1}^{2(j-n-1)}(a) \right] = \frac{b}{(1-a)^2+b^2}$

Implies, by dividing by b in the second equation and replacing $\frac{1}{(1-a)^2+b^2}$ in the first equation:

$$\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]$$

$$=$$

$$(1-a)\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]-\frac{1}{2}$$

Implies :

$$\frac{1}{2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}\left[K_{2j-1}^{2(j-n)-1}(a)-(1-a)K_{2j-1}^{2(j-n-1)}(a)\right]\right]=0$$

So we can say that $s=a+ib$ is a non-trivial zero of the zeta function when a and b are solutions of the equation :

$$\frac{1}{2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}\left[K_{2j-1}^{2(j-n)-1}(a)-(1-a)K_{2j-1}^{2(j-n-1)}(a)\right]\right]=0$$

7. Conclusion

Based on the results of this article, we can define the zeta function on all the complex plane except in s=1 by :

$$\forall s=a+ib\in\mathbb{C}-\{1\}$$

$$\boldsymbol{\zeta}(\boldsymbol{s})=\left[\frac{1}{2}-\frac{1-a}{(1-a)^2+b^2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n)-1}(a)\right]\right]$$

$$+\boldsymbol{i}\times\left[\frac{-b}{(1-a)^2+b^2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n+1}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2(j-n-1)}(a)\right]\right]$$

We conclude that $s=a+ib$ is a non-trivial zero of the zeta function when a and b are solutions of the equation :

$$\frac{1}{2}+\sum_{n=0}^{+\infty}\left[(-1)^n b^{2n}\sum_{j=n+1}^{+\infty}\frac{b_{2j}}{(2j)!}\left[K_{2j-1}^{2(j-n)-1}(a)-(1-a)K_{2j-1}^{2(j-n-1)}(a)\right]\right]=0$$

We can also write :

$$\boldsymbol{\zeta}(\boldsymbol{s})=\sum_{n=0}^{+\infty}(-1)^n\frac{b^{2n}}{(2n)!}L^{(2n)}(a)-\frac{1-a}{(1-a)^2+b^2}++i\left[\sum_{n=0}^{+\infty}(-1)^n\frac{b^{2n+1}}{(2n+1)!}L^{(2n+1)}(a)-\frac{b}{(1-a)^2+b^2}\right]$$

With : $L(a)=\frac{1}{2}+\sum_{j=1}^{+\infty}\frac{b_{2j}}{(2j)!}K_{2j-1}^{2j-1}(a)$

Appendix 1

In this article we have defined the function noted $a\to K_n^p(a)$, $(a\in\mathbb{R})$,$n\in\mathbb{N}*$,$p\in\mathbb{N}$ and $n\geq p+1$which sums the multiplications between all the elements of the non-repeating combinations of the n elements $\{a;a+1;a+2;....;a+n-1:n\in\mathbb{N}*\}$ taken p by p :

$$C_n^p=\frac{n!}{p!\times(n-p)!}$$

Examples:

$K_2^1(a) = (a) + (a+1)$; $\quad C_2^1 = \frac{2!}{1!\times(2-1)!} = 2$

$K_2^2(a) = a(a+1)$; $\quad C_2^2 = \frac{2!}{2!\times(2-2)!} = 1$

$K_3^2(a) = a(a+1) + a(a+2) + (a+1)(a+2)$; $\quad C_3^2 = \frac{3!}{2!\times(3-2)!} = 3$

$K_3^1(a) = (a) + (a+1) + (a+2)$; $\quad C_3^1 = \frac{3!}{1!\times(3-1)!} = 3$

$K_3^3(a) = a(a+1)(a+2)$; $\quad C_3^3 = \frac{3!}{3!\times(3-3)!} = 1$

The properties of this function include :

- $K_{2n}^{2p+1}(a) + (a+2n)K_{2n}^{2p}(a) = K_{2n+1}^{2p+1}(a)$
- $K_{2n}^{2p}(a) + (a+2n)K_{2n}^{2p-1}(a) = K_{2n+1}^{2p}(a)$
- $(a+2n)K_{2n}^{2n}(a) = K_{2n+1}^{2n+1}(a)$

Since we have for all, $n\in\mathbb{N}*, p\in\mathbb{N}, et\ n\geq p+1$:

$K_n^{p+1}(a) + (a+n)K_n^p(a) = K_{n+1}^{p+1}(a)$ and $(a+n)K_n^n(a) = K_{n+1}^{n+1}(a)$

Demonstration:

We know that :

$$C_n^{p+1} + C_n^p = \frac{n!}{(p+1)!\times(n-p-1)!} + \frac{n!}{p!\times(n-p)!}$$

$$= \frac{(n!)(n-p)+(n!)(p+1)}{(p+1)!\times(n-p)!}$$

$$= \frac{(n!)(n-p)+n!(p+1)}{(p+1)!\times(n-p)!}$$

$$= \frac{(n!)(n+1)}{(p+1)!\times(n-p)!}$$

$$= \frac{(n+1!)}{(p+1)!\times(n-p)!} = C_{n+1}^{p+1}(a)$$

And $C_n^n = C_{n+1}^{n+1}$

And since $(a+n)$ it is the (n+1) element of the set :

$\{a; a+1; a+2; \dots; a+n : n \in \mathbb{N}\}$

We therefore have the equivalence :

$$K_n^{p+1}(a) + (a+n)K_n^p(a) = K_{n+1}^{p+1}(a) \text{ And } (a+n)K_n^n(a) = K_{n+1}^{n+1}(a)$$

I find that the function $a \rightarrow K_n^p(a)$ also verifies the formula which can be stated as follows:

$$n \in \mathbb{N}^*, p \in \mathbb{N} \ : \ [K_n^n(a)]^{(p)} = p!\, K_n^{n-p}(a)$$

(p) Is the derivative of order p.

References

1. Bernoulli, J. (1713). Ars conjectandi, opus posthumum: accedit tractatus de seriebus infinitis, et epistola Gallice scripta de ludo pilæ reticularis. Impensis Thurnisiorum Fratrum.
2. Epistola gallice scripta de ludo pilae reticularis. (1713). Impensis Thurnisiorum.
3. Riemann, B. (1859). Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse. Ges. Math. Werke und Wissenschaftlicher Nachlaß, 2(145-155), 2.
4. https://fr.wikipedia.org/wiki/Valeurs_particuli%C3%A8res_de_la_fonction_z%C3%AAta_de
5. Reimann (consulted on 22/01/2024) https://www.bibmath.net/dico/index.php?action=affiche&quoi=./p/prolongementanalytique.html (consulted on 01/25/2024)