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Maximal Violation of CHSH Inequality in a One-dimensional Quantum Harmonic Oscillator

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Abstract
It is shown that any eigenstate of quantum harmonic oscillator and the coherent state and the squeezed state displays quantum contextuality by the violation of a noncontextuality inequality.

1. Introduction

It has been shown by Gisin [1] that for any entangled state of two quantum systems it is possible to find pair of observables whose correlation's violate Bell / CHSH inequality. Different types of noncontextuality inequalities have been proposed and recent experiments have confirmed that quantum mechanics cannot be described by NCHV theories by violate of the inequalities [2]. Most of the investigators studied the system for discrete variables. Plastino and Cabello extended the study to continuous variables (CV) states using 18 variables based on position and momentum, motivated by the fact that CV states such as coherent and squeezed states have served as valuable resources in quantum optics and quantum information [3,4]. Braunstein et al. showed that for any Bell / CHSH inequality based on non commuting observables for both systems it is always possible to construct a state which will yield a violation, though not necessarily maximal [5]. G Kar showed that for any pair of observables chosen on both sides, there is violation of CHSH inequality and the maximal violation is achieved by maximally entangled states [6].

It is well known that the superposition of the eigenstates of quantum harmonic oscillator (QHO) may exhibit classical properties if co-efficients are appropriately selected and QHO is also exactly solvable and applied to many other systems [7]. The most commonly discussed Bell Inequality is the Clauser-Horne-Shimony- Holt inequality and the square of the bell operator is given by [8,9]. Taking the coherent state |α⟩, as an example, it can be seen as a poissonian superposition of photon number states, but its dynamics resembles a classical harmonic oscillator and it becomes more classical as |α⟩ is larger. Su et al. investigated quantum contextuality (QC) using two inequalities based on noncontextual-hidden-variable (NCHV) model [2,10]. The same approach has been exploited here to investigate the nonclassically of QHO and CV states including coherent state and squeezed state. The paper is organized as follows: We explore the QC of the eigenstates QHO in Sec.II and find that any eigenstate of QHO exhibits QC. In Sec.III, we study the coherent state as well as the squeezed state and show that they bear QC that can be demonstrated by measurement of four observables. Furthermore the state independent test of QC in CV states is also discussed in Sec.IV. We end with a summary and discussion in the last section.

2. QC of Eigenstates of QHO

It is known that the one dimensional quantum harmonic oscillator Hamiltonian gives

$$\mathrm{H} = \frac{p^2}{2m} + \frac{1}{2}\mathrm{m}\,\omega^2 x^2, \quad (1)$$

Where m is mass and ω is angular frequency and also the eigenenergy of eigenstate |n⟩ of H is En = ħ ω (n + 1).
We know by Roy-Singh the following non-contextuality inequality:

$$\mathrm{F} = \langle \mathrm{AB}\rangle + \langle \mathrm{BC}\rangle + \langle \mathrm{CD}\rangle - \langle \mathrm{DA}\rangle \le 2 \quad (2)$$

Where {A,B,C,D} is a set of observables taking values ± 1 classically and observable pairs (A,B),(B,C),(C,D),(D,A) are compatible. Relation (2), which is valid for NCHV model is formally related to the Clauser-Horne-Shimony-Holt inequality, but it is not limited to entangled states.

The square of the Bell operator is given by [11],

$$B^2_{CHSH} = 4I - [A,C][B,D]. \tag{3}$$

Now introduce four operators,

$$\begin{aligned} A &= \mathbb{1} \otimes \sigma_x \\ B &= \cos\beta\, \sigma_x \otimes \sigma_x \\ C &= -\sigma_y \otimes \sigma_y \\ D &= \cos\gamma\, \sigma_z \otimes \sigma_x \end{aligned} \tag{4}$$

where $\sigma_{(x,y,z)}$ are Pauli matrices and 1 is a 2 by 2 identity matrix and β,γ are parameters. These operators can be written in terms of number states,

$$A = |0\rangle\langle 1| + |1\rangle\langle 0| + |2\rangle\langle 3| + |3\rangle\langle 2|$$

$$B_1 = \frac{B}{\cos\beta} = |0\rangle\langle 3| + |1\rangle\langle 2| + |2\rangle\langle 1| + |3\rangle\langle 0|$$

$$C = |0\rangle\langle 3| - |1\rangle\langle 2| - |2\rangle\langle 1| + |3\rangle\langle 0| \tag{5}$$

$$D_1 = \frac{D}{\cos\gamma} = |0\rangle\langle 1| + |1\rangle\langle 0| - |2\rangle\langle 3| - |3\rangle\langle 2|.$$

$$\text{We set, } B_1 = \frac{B}{\cos\beta} \text{ and } D_1 = \frac{D}{\cos\gamma}. \tag{6}$$

Now the four observables A, B_1, C, D_1 are compatible operators i.e,

$$[A, B_1] = [B_1, C] = [C, D_1] = [D_1, A] = 0, \tag{7}$$

and $A^2 = B_1^2 = C^2 = D_1^2 = I$, where I is an identity matrix.

Here, $B^2_{CHSH} = 4I - [A,C]\left[\frac{B}{\cos\beta}, \frac{D}{\cos\gamma}\right]$.

$$\text{i.e, } B_{CHSH} = \sqrt{4I - [A,C]\left[\frac{B}{\cos\beta}, \frac{D}{\cos\gamma}\right]} \tag{8}$$

For different states |i⟩, (i=0,1,2,3), by choosing β,γ appropriately violation of BCHSH can be obtained. Specifically, we observe,

$$\langle 0| B_{CHSH} |0\rangle = 2\sqrt{2} > 2, \quad \text{when } \begin{array}{l} 1)\ \beta = 0,\ \gamma = \pi \\ 2)\ \beta = \pi,\ \gamma = 2\pi \\ 3)\ \beta = \pi,\ \gamma = 0 \\ 4)\ \beta = 2\pi,\ \gamma = \pi \end{array} \tag{9a}$$

$$\langle 1|\, B_{CHSH}|1\rangle = 2\sqrt{2} > 2\,, \quad \text{when } 1)\ \beta = 0,\ \gamma = 0 \quad 2)\ \beta = \pi,\ \gamma = \pi \quad 3)\ \beta = 2\pi,\ \gamma = 2\pi \tag{9b}$$

$$\langle 2|\, B_{CHSH}|2\rangle = 2\sqrt{2} > 2\,, \quad \text{when } 1)\ \beta = 0,\ \gamma = \pi \quad 2)\ \beta = \pi,\ \gamma = 0 \quad 3)\ \beta = 2\pi,\ \gamma = \pi \quad 4)\ \beta = \pi,\ \gamma = 2\pi \tag{9c}$$

$$\langle 3|\, B_{CHSH}|3\rangle = 2\sqrt{2} > 2\,, \quad \text{when } 1)\ \beta = 0,\ \gamma = 0 \quad 2)\ \beta = \pi,\ \gamma = \pi \quad 3)\ \beta = 2\pi,\ \gamma = 2\pi \tag{9d}$$

This tells that the four eigenstates { $|0\rangle,|1\rangle, |2\rangle, |3\rangle$ } shows QC and that the QC can be shown by violating the noncontextuality inequality (2) based on four observables with proper measurement settings as listed in Eq. (9).

We have according to Su et al. [2012], for every four eigenstates of QHO there exists a set of observables O capable of revealing the QC of the four states. For instance,

$O = \{ A, B_1, C, D_1 \}$ for $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle \}$ and

$O' = \{ A', B'_1, C', D'_1 \}$ for $\{|4\rangle, |5\rangle, |6\rangle, |7\rangle \}$.

Here A' is constructed by replacing the i-th eigenstate in A with the (i+4)-th eigenstate, i = { 0,1,2,3 }; and similarly for B'1,C',D'1. The direct sum of corresponding elements in O and O' makes a new set denoted by

$$O_2 = \{ A \oplus A', B \oplus B'_1, C \oplus C', D \oplus D'_1 \}$$

for an eight level system.
By repeating the procedure N-times, we finally obtain an observable set
ON = { A ⊕ A' ⊕... ⊕ A'', B ⊕ B'1 ⊕ ...⊕ B'', C ⊕ C' ⊕...⊕C'', D ⊕ D'1⊕...⊕ D''}
For 4N-level system $\{|4n\rangle, |4n+1\rangle, |4n+2\rangle, |4n+3\rangle\}$ (n=0,…N-1). When
N $\longrightarrow\infty$, the whole spectrum of QHO can be approximated by this 4N-level system. Thus the four observables for QHO can finally be written in a neat
form as

$$A_\infty = \sum_{n=0}^{\infty}(|4n\rangle\langle 4n+1| + |4n+1\rangle\langle 4n| + |4n+2\rangle\langle 4n+3| + |4n+3\rangle\langle 4n+2|)$$

$$B_{1\infty} = \sum_{n=0}^{\infty}(|4n\rangle\langle 4n+3| + |4n+1\rangle\langle 4n+2| + |4n+2\rangle\langle 4n+1| + |4n+3\rangle\langle 4n|)$$

$$C_\infty = \sum_{n=0}^{\infty}(|4n\rangle\langle 4n+3| - |4n+1\rangle\langle 4n+2| - |4n+2\rangle\langle 4n+1| + |4n+3\rangle\langle 4n|)$$

$$D_{1\infty} = \sum_{n=0}^{\infty}(|4n\rangle\langle 4n+1| + |4n+1\rangle\langle 4n| - |4n+2\rangle\langle 4n+3| - |4n+3\rangle\langle 4n+2|)\,. \tag{10}$$

Since X_∞ ($X = A, B_1, C, D_1$; $X^2 = 1$) are dichotomic observables, the classical upper bound of inequality (2) holds. Maximum violationof equation (2) can be obtained with the same parametric values of β and γ as in equation (9) by use of equation (10).

3. QC of Coherent and Squeezed States

It is known that the coherent states can be obtained by applying the unitary displacement operator Ɗ (α) on the vacuum state,

$$|\alpha\rangle = Ɗ\,(\alpha)\,|0\rangle = \; e^{\frac{-|\alpha|^2}{2}}\, \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|\mathrm{n}\,, \tag{10}$$

with $Ɗ\,(\alpha) = e^{\alpha a^{\dagger} - \,\alpha^{*} a}$, where α is the coherent parameter and $a^{\dagger}$,a are creation and annihilation operators.

We set the following set of observables by introducing the displacement operator Ɗ (α),

$$A^{\alpha} = Ɗ\,(\alpha)\;\mathrm{A}\;Ɗ^{\dagger}(\alpha)\,, \qquad B_1^{\alpha} = Ɗ\,(\alpha)\;\mathrm{B}_1\;Ɗ^{\dagger}(\alpha)$$

$$C^{\alpha} = Ɗ\,(\alpha)\;\mathrm{C}\;Ɗ^{\dagger}(\alpha), \qquad D_1^{\alpha} = \; = Ɗ\,(\alpha)\;\mathrm{D}_1\;Ɗ^{\dagger}(\alpha). \tag{11}$$

One can easily check that the violation of inequality (2) can be maximally violated for the coherent state with arbitrary coherent parameter α i.e,

$$\langle\alpha|\,\mathrm{B_{CHSH}}|\alpha\rangle = 2\sqrt{2} > 2\,, \qquad \text{when } \begin{array}{l} 1)\; \beta = 0,\; \gamma = \pi \\ 2)\; \beta = \pi,\; \gamma = 2\pi \\ 3)\; \beta = \pi,\; \gamma = 0 \\ 4)\; \beta = 2\pi,\; \gamma = \pi. \end{array} \tag{12}$$

It is found that coherent state |α⟩ always bears QC giving no importance of α.
A squeezed state is defined as,

$$|\xi\rangle = \mathrm{S}\,(\xi)\,|0\rangle = \; e^{\frac{-|\xi|^2}{2}}\, \sum_{n=0}^{\infty} \left(\frac{\xi}{2}\right)^{n} \frac{\sqrt{(2n)!}}{n!}\,|2\mathrm{n}\rangle, \tag{13}$$

with $\mathrm{S}(\xi) = \; e^{\frac{\xi\left(a^{\dagger}\right)^{2} - \xi^{*} a^{2}}{2\;,}}$, where ξ is the squeezed parameter.

Four observables are taken in this way,

$$A^{\xi} = \mathrm{S}\,(\xi)\;\mathrm{A}\;S^{\dagger}(\xi), \qquad B_1^{\xi} = \mathrm{S}\,(\xi)\;\mathrm{B}_1\;S^{\dagger}(\xi),$$

$$C^{\xi} = \mathrm{S}\,(\xi)\;\mathrm{C}\;S^{\dagger}(\xi), \qquad D_1^{\xi} = \mathrm{S}\,(\xi)\;\mathrm{D}_1\;S^{\dagger}(\xi) \tag{14}$$

can reveal the QC of the squeezed state, i.e.

$$\langle\xi|\,\mathrm{B_{CHSH}}|\xi\rangle = 2\sqrt{2} > 2\,, \qquad \text{when } \begin{array}{l} 1)\; \beta = 0,\; \gamma = \pi \\ 2)\; \beta = \pi,\; \gamma = 2\pi \\ 3)\; \beta = \pi,\; \gamma = 0 \\ 4)\; \beta = 2\pi,\; \gamma = \pi, \end{array} \tag{15}$$

regardless of ξ .Therefore QC exists also in the squeezed state also.

4. Summary

We have constructed four observables and have explored the contextual property of one-dimensional QHO. It is also shown that the eigen states of QHO bear QC. The coherent state and the squeezed state have also been investigated here.

The coherent state is considered because of its dynamical behavior, which is analogus to classical oscillators. It behaves classically in the case of large α, the coherent state is a quantum mechanical phenomenon. We construct suitable observables for both states and show that they also reveal contextuality of quantum phenomena.

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