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Mathematical Modeling of the Sliding Friction Coefficient Depending on Speed and Load

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Abstract
Friction is a complex of irreversible processes of mechanical energy transformation that occur during the motion of macroscopic bodies, particularly when a body slides over another one. This problem appears to be impossible to solve based on mechanical equations because it requires combining and solving a relatively large number of equations for all the particular elements of the system. This is also the reason why it appeals to modeling methods, which start from general cases to arrive at models having a determined structure based on hypotheses and peculiarities. This paper aims to determine the relative dependency of the dry friction coefficient on speed and normal pressure force through mathematical modeling, based on certain simplifying hypotheses.

Keywords: Friction coefficient, Dry sliding, Mathematical modeling, Hypotheses, Speed, Load: Contact area

1. Introduction

The dimensionless quantity known as the friction coefficient has been used for a long time in science and engineering. The friction coefficient is easy to define but not easy to understand on a fundamental level [1,2].

Conceptually defined as the ratio of two forces acting, respectively, perpendicular and parallel to an interface between two bodies under relative motion or impending relative motion, this dimensionless quantity turns out to be convenient for depicting the relative ease with which materials slide over one another under particular circumstances [1].

Even though the friction coefficient can be measured with little difficulty under laboratory conditions, the time, speed, and load-dependent characteristics and other conditions associated with both clean and dry and lubricated surfaces have proven exceedingly difficult to predict a priori from the first [1,2].

The materials involved in systems engineering are composed of different-sized elements. To study the influence of the sliding friction coefficient corresponding to different-sized elements Liu et al. used the discrete element method (DEM) on the simulation results, establishing a two-dimensional DEM model based on the experimental data for analysis [3].

Hentschke and Plagge investigated the coefficient of sliding friction on dry surfaces via scaling and dimensional analysis depending on sliding speed, temperature, and load [4]. They are finally obtaining a comparatively simple expression for the coefficient of friction, which allows a physically intuitive understanding that correlates with the experimental data for various speeds, temperatures, and pressures.

Friction may be studied as a complex of irreversible processes of mechanical energy transformation that occur during the motion of macroscopic bodies, in the exterior, and in a particular case, when a body slides over another one. It seems impossible to solve this problem based on mechanical equations, for example, because of the need to combine and solve a relatively large number of equations for all the particular elements of the system, which seem simple. This is also why scientists refer to modeling methods, which start from more general cases and pass to models having a determined structure based on peculiarities [4-8].

Therefore, the paper attempts to present a mathematical model of the friction coefficient during dry sliding of two elements in a given mechanical system, depending on two parameters: relative velocity/speed and load in the contact area.

2. Hypotheses

With the modeling trial shown here, we mean to settle a relatively steady dependence of the dry friction coefficient on the speed and normal pressure force by assuming the following hypotheses:- it is admitted that to achieve sliding friction, a certain point of closeness of the bodies sliding one over the other is necessary. The reciprocal normal force achieves this closeness, N, between the bodies in contact. The reciprocal normal force of the system of bodies sliding one over the other may have two components: the external one, N_e, generated from the system outside, and the internal one, N_i, caused by the reciprocal or rejection of the system bodies, thus:

$$N = N_e + N_i \tag{1}$$

It should be noted that N_e is easily determined, while Ni may be more difficult to establish;

- the relative motion speed of the sliding bodies is considered to be the main element that influences the dry friction phenomenon in a determined period;
- it is admitted theoretically that variation of the contact area geometrical dimensions does not play a decisive part, supposing that, because of the micro-roughness distortion of the surfaces in contact under the external forces, the actual contact surface is relatively constant [8-10];
- it is supposed that the friction force, F_f, depends on the normal force, N, with its external, N_e, and internal, N_i, components, and the sliding speed, v [2, 9, 11].

If we limit ourselves to these two elements, the process of actual bodies' friction is greatly simplified. This statement may be counteracted by admitting that, independently of the physical properties of the friction pair, the power dissipation takes always place and accompanies the friction process, changing single coupler bodies by other ones, the dependence structure between N, N_e, v, and F_f does not change, it is only the coefficients numerical value that change as the process may be considered to modify [2, 9-11].

3. Mathematical Model

The dependence

$$F_f = f(N, N_e, v), \tag{2}$$

it admits to modeling the friction process structure. From the proportion/ratio of the forces, it gets the equation:

$$\frac{F_f}{N} = f\left(\frac{N_e}{N}, v\right), \tag{3}$$

where, $\frac{F_f}{N}$ is an undetermined dimensionless function, while the dimensionless proportion/ratio, $\frac{N_e}{N} = k$ is considered as constant, the dependence (2) becomes:

$$F_f = N \cdot f(k, v), \tag{4}$$

which is verified, if $N = 0$, then $F_f = 0$, which means that, in the absence of the conditions for the tribological moving process, the friction force is too.

Equation (4) can also be written in vector form as:

$$\bar{F_f} = -\frac{\bar{v}}{v} \cdot N \cdot f(k, v), \tag{5}$$

where $\frac{\bar{v}}{v}$ is the speed vector versor.

Introducing the notion of a vector of friction coefficient $\bar{\mu}_a$ is obtained, the relation/equation:

$$\bar{\mu}_a = \frac{\bar{F_f}}{N} = -\frac{\bar{v}}{v} \cdot f(k, v), \tag{6}$$

that represents the mathematical modeling expression of the friction coefficient.

Function, $f(k, v)$ as a non-determined function can be decomposed into two parts (even and odd) according to the identity:

$$f(k, v) = \frac{f(k, v) + f(k, -v)}{2} + \frac{f(k, v) - f(k, -v)}{2}. \tag{7}$$

As the two parts (even and odd) of the function, $f(k, v)$, are analytical functions, they can also be written as:

$$f(k, v) = f_1(k, v^2) + v f_2(k.v^2). \tag{8}$$

This possibility of function decomposition, $f(k, v)$, results from the following function: $f(x) = \frac{x^5 + x^3 + 3x^2 + 6}{3(1 + x^2)}$ as a determined function in an interval.

Function $f(x)$ can be decomposed according to (7), thus: $f(-x) = \frac{-x^5 - x^3 + 3x^2 + 6}{3(1 + x^2)}$ being determined in the same interval as $f(x)$.

Noting the even part with $p(x)$ and the odd with $q(x)$, it is obtained [12, 13]:

$p(x) = \frac{f(x) + f(-x)}{2} = \frac{x^2 + 2}{1 + x^2} = 1 + \frac{1}{1 + x^2}$, respectively $q(x) = \frac{f(x) - f(-x)}{2} = \frac{x^3}{2}$ and finally it results: $f(x) = 1 + \frac{1}{1 + x^2} + \frac{x^3}{3}$.

The even part, $p(x)$, decreases by x, from 1 (one), and tends asymptotically to 0 (zero), when $x \to \infty$, and the odd part, $q(x)$, increases by x, from 1 (one), and conventionally to $+\infty$.

Coming back to relation/equation (6) and taking (8) into account, it is obtained:

$$\bar{\mu}_a = -\frac{\bar{v}}{v} \cdot f_1(k, v^2) + \bar{v} \cdot f_2(k, v^2). \quad (9)$$

The two terms in the relation (9) correspond to the even and odd parts of the function $f(k, v)$. Each of them may have a physical direction.

Taking into account this direction makes it necessary that in the relation (9) the function, $f_1(k, v^2)$ representing the even part of $\bar{\mu}_a$ decreases from the maximum and to tend asymptotically to 0 (zero), and the function, $f_2(k, v)$, i.e. the odd part increases and tend asymptotically to ∞. Thus, it is only the even term, $f_1(k, v^2)$ that represents the real solution because, by increasing the speed, the friction coefficient decreases and odd part term, $f_2(k, v)$ is not a real possibility, so the relation (9) becomes:

$$\bar{\mu}_a = -\frac{\bar{v}}{v} \cdot f_1(k, v^2). \quad (10)$$

Between the friction coefficient, $\bar{\mu}_a$, and its first and second derivate depending on v^2, i.e., between:

$$\bar{\mu}_a(v^2);\ \mu_a'(v^2);\ \mu_a''(v^2), \quad (11)$$

is can establish a relation, written as [7, 14]:

$$g(\bar{\mu}_a,\ \mu_a',\ \mu_a'') = 0, \quad (12)$$

which is a differential, linear, homogeneous equation of the second order, with constant coefficients.

To make calculations easier, it can be stated that the friction direction (of $\bar{\mu}_a$) is the reverse of the speed direction ($\bar{v}$), and the minus sign (-) can be understood.

If in the equation (12) it is substitute $\mu_a = -f_1(k, v^2)$, it becomes [7]:

$$g(f_1, f_1', f_1'') = 0, \quad (13)$$

For this purpose, we should solve the differential equation (12) having the following form [7]:

$$af_1'' + bf_1' + cf_1 = 0, \quad (14)$$

with a, b, c – constant coefficients of the linear homogeneous differential equation (14).

4. Discussion

Studying the solutions of these equations (14), it follows that the equation admits particular solutions of the form [7, 14, 15]:

$$f_1 = e^{rv^2}, \quad (15)$$

where: r – constant and v – variable.

Differentiating the equation (15) twice, we obtain: $f_1' = re^{rv^2}$ and $f_1'' = r^2e^{rv^2}$.

Substituting into equation (14) we have: $ar^2e^{rv^2} + b\,re^{rv^2} + ce^{rv^2} = 0$, simplifying with the exponential, e^{rv^2} and substituting the derivatives of f_1 by r and r^2 we get the characteristic equation:

$$ar^2 + br + c = 0, \quad (16)$$

having the roots: $r_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$. Noting $\frac{\sqrt{b^2-4ac}}{2a} = \omega$ or $\frac{b^2-4ac}{4a^2} = \omega^2$ the following solutions may be admitted:

1) if the roots r_1 and r_2 are real and distinct, thus $r_1 = \frac{-b}{2a} + \omega$ and $r_2 = \frac{-b}{2a} - \omega$, and the solution of the differential equation (14) is:

$$f_1 = C_1e^{r_1v^2} + C_2e^{r_2v^2} = e^{-\frac{b}{2a}v^2}\left(C_1e^{\omega v^2} + C_2e^{\omega v^2}\right), \quad (17)$$

where: C_1, C_2 - are integration constants.

By means of the hyperbolic trigonometry this solution may be written in a different form by definition we have: sinh $\omega v^2 = \frac{e^{\omega v^2} - e^{-\omega v^2}}{2}$ and cosh $= \frac{e^{\omega v^2} + e^{-\omega v^2}}{2}$, from where: sinh ωv^2 + cosh $\omega v^2 = e^{\omega v^2}$ and cosh ωv^2 – sinh $\omega v^2 = e^{-\omega v^2}$.

So, $C_1e^{\omega v^2} + C_2e^{\omega v^2} = (C_1 - C_2)\sinh \omega v^2 + (C_1 + C_2)\cosh \omega v^2$.

Noting: $C_1 + C_2 = A$, and $C_1 - C_2 = B$, the equation solution, f_1 becomes:

$$f_1 = e^{-\frac{b}{2a}v^2}(A\cosh \omega v^2 + B\sinh \omega v^2) \quad (18)$$

Additionally, if noting $\sqrt{A^2 - B^2} = \alpha$, sinh $\beta = \frac{B}{\alpha}$, cosh $\beta = \frac{A}{\alpha}$, and tanh $\beta = \frac{B}{A}$. Substituting in f_1, we get: $f_1 = e^{-\frac{b}{2a}v^2}$ (α cosh β·cosh ωv^2 + α sinh β·sinh ωv^2), from which it is get the final solution:

$$f_1 = e^{-\frac{b}{2a}v^2} \cdot \alpha \cosh(\omega v^2 + \beta). \quad (19)$$

2) if the roots r_1, r_2 are real and equal, respectively $r_1 = r_2 = r_0$ them $\omega = 0$. In this case equation admits the particular solution by the form: $y_1 = e^{r_0x}$ and also the second particular solution $y_2 = y_2 = xe^{r_0x}$, then solution of the differential equation (14) will be of form: $y = C_1e^{r_0x} + C_2xe^{r_0x}$ or $y = e^{r_0x}(C_1 + xC_2)$ and passing

to the original notations it follows that:

$$f_1 = e^{-\frac{b}{2a}v^2}(C_1 + xC_2). \qquad (20)$$

3) if the roots r_1 , $r_{,2}$ of the characteristic equation are imaginary, the equation solution, f_1 will be:

$$f_1 = e^{-\frac{b}{2a}v^2}\alpha_1 \cos(C_1' + v^2 C_2'), \qquad (21)$$

with $\alpha_1 \neq \alpha$, $C_1{}' \neq C_1$, and $C_2{}' \neq C_2$.

It is admitting that the first two forms/variants of the differential equation solution (14), model 1) and 2) are the most important characteristic of the dry friction coefficient dependence on speed for all the conventional limit value, with the third one 3), having complex roots can't have a physical interpretation.

Then, when the roots r_1 , $r_{,2}$ are real and distinct, the equation (10) may be written as:

$$\mu_a = \frac{1}{e^{-\frac{b}{2a}v^2}} \cdot \alpha \cosh(\omega v^2 + \beta), \qquad (22)$$

where: v – is relative speed in m/s; α – is constant; β = arc tanh B/A = arc tanh $\frac{C_1 - C_2}{C_1 + C_2}$, also a constant; $\omega = \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$ = constant.

When the roots $r_1 = r_{,2} = r_0$ are real and equal, the equation (10) may be written as:

$$\mu_a = \frac{1}{e^{-\frac{b}{2a}v^2}} \cdot (C_1 + v^2 C_2), \qquad (23)$$

with v in m/s, the values of C_1 and C_2 are in direct proportional ratio with μ_a; the coefficient a influences directly, and the b one inversely the friction coefficient μ_a.

5. Conclusions

These two solutions of the linear, homogenous differential equation of the second-order model are the most characteristic part of the functional dependence of the dry friction coefficient in comparison with the normal load and speed for all values.

The integration constants and the constant coefficients have physical directions that, if determined, give practical values to the mathematical modeling expressions.

To conclude, we must bear in mind that, even in the case of simplifying hypotheses of the friction process, the mathematical relations, although complicated, do not always offer general solutions.

References

1. Blau, P. J. (2001). The significance and use of the friction coefficient. *Tribology International, 34*(9), 585-591.
2. Pavelescu, D., & Tudor, A. (1987). The sliding friction coefficient—its evolution and usefulness. *Wear, 120*(3), 321-336.
3. Liu, P., Rui, Y., & Wang, Y. (2024). Study of the Sliding Friction Coefficient of Different-Size Elements in Discrete Element Method Based on an Experimental Method. *Applied Sciences, 14*(19), 8802.
4. Hentschke, R., & Plagge, J. (2021). Scaling theory of rubber sliding friction. *Scientific Reports, 11*(1), 18306.
5. Morgan, D. A. (2007). Differential and Integral Calculus, Ed. Cosimo Classics, ISBN 9781602063792.
6. Siretchi G. (1985). Differential and Integral Calculus, vol 2 (in Romanian), Ed. Scientific and Encyclopedic, Bucharest, Romania.
7. Lia, A., Morozan, T. (1978). Differential and Integral Calculus Problems (in Romanian), 3rd Edition Completed and Revised, Technical Publishing House, Bucharest, Romania.
8. Zhang, S., Li, D., & Liu, Y. (2022). Friction behavior of rough surfaces on the basis of contact mechanics: a review and prospects. *Micromachines, 13*(11), 1907.
9. Tudor, A. (1990). Real contact of the friction surfaces. Bucuresti, Editura Academiei Romane (in Rumanian).
10. Liang, X. M., Xing, Y. Z., Li, L. T., Yuan, W. K., & Wang, G. F. (2021). An experimental study on the relation between friction force and real contact area. *Scientific Reports, 11*(1), 20366.
11. Johnson, K.L. Contact Mechanics; Cambridge University Press: Cambridge, UK, 1989.
12. Debnath, P., Srivastava, H. M., Torres, D. F. M., & Cho, Y. J. (2025). Mathematical analysis. In Chapman and Hall/CRC eBooks.
13. Horwitz, A. L. A. N., & Ebrahimpour, A. R. Y. A. (2002). Engineering applications in differential and integral calculus. International Journal of Engineering Education, 18(1), 78-88.
14. Mahmudov, E. (2013). Single variable differential and integral calculus. Atlantis Press.
15. Mizraji, E. (2014). Differential and integral calculus for logical operations. A matrix–vector approach. *Journal of Logic and Computation, 25*(3), 613-638.