ISSN: 2690- 909X

Mini Review Article

Advances in Neurology and Neuroscience

Mathematical Modeling of Stress Using Fractal Geometry; The Power Laws and Fractal Complexity of Stress

Tahmineh Azizi

Florida State University

***Corresponding author**
Tahmineh Azizi, Department of Mechanical Engineering, Florida State University, USA .

Submitted: 12 Aug 2022; **Accepted**: 19 Aug 2022; **Published**: 25 Aug 2022

Citation: *Tahmineh Azizi (2022). Mathematical Modeling of Stress Using Fractal Geometry; The Power Laws and Fractal Complexity of Stress. Adv NeurNeur Sci. 5(3), 140-148.*

Abstract
In this study, we analyze the physiological data during real-world driving tasks to determine whether driver's relative stress is mono-fractal or multi-fractal. We use the PhysioNet database including long term ECG recordings from 15 healthy volunteers, taken while they were driving on a prescribed route including city streets and highways in and around Boston, Massachusetts. The vibration analysis such as power spectral densities (PSD) analysis has been performed to estimate the exponent from realizations of these pro- cesses and to find out if the signal of interest exhibits a power-law PSD. Multifractal dynamics of heartbeat interval signals have been assessed by multifractal spectrum analysis to explore the possibility that ECG recordings belong to class of multi-fractal process for which a large number of scaling exponents are re- quired to characterize their scaling structures. We apply Higuchi algorithm to find the fractal complexity of each cardiac rhythm for different time intervals. According to our analysis, we investigate that driver's ECG signals under relative stress follows fractal behavior unlike control healthy signals which are multi-fractal. Our findings provide a comprehensive framework for detect stress and differentiate people who experience stress with normal people without stress which is crucial in finding the best diagnostic and controlling strat- egy in fight against many health problems due to stress, such as high blood pressure, heart disease, obesity and diabetes. Moreover, being able to recognize stress can help us to manage it.

Keywords: Fractal geometry, Power Spectral Densities (PSD), Multifractal spectrum, Higuchi algorithm, Stress

Introduction

Although the well-known definition of stress is different from its scientific definition, but someone can sim- ply define it as a set of unspecific reactions that an organism demonstrates under environmental changes to maintain homeostasis. In general, these responses help the organism to adapt to the new conditions [1]. We call the stimuli that causes stress as stressors and the provoked situation as stressful [1, 2]. A large number of studies in neuroscience in the 20th century has been devoted to study stress [1–6]. According to some of them, stress may be the result of the physical action such as damage to the body, biochemical reasons such as reducing the blood glucose level, or biological factors such as infection by microorganisms agents [2–4]. The hypothalamic-pituitary-adrenal (HPA) axis which is a neuroendocrine system and is responsible for regulating numerous physiological processes can also causes many stress-related diseases such as post-traumatic stress disorder (PTSD) and major depressive disorder when its action is disrupted due to different reasons [3–6]. Recently, many efforts and therapies in stress management have been developed to decrease stress and promote health condition. Clearly, the combination of all these efforts in different fields will help in better understanding of the nature of stress and better controlling stress to prevent many different stress related diseases.

A fractal has been defined as a subset of Euclidean space with a dimension strictly higher than its topological dimension. For the first time, Mandelbort in 1983 [7] introduced these irregular geometric objects to the world. Fractals also can be defined as physical models for different phenomena which are distributed evenly in the embedding space. Fractals are well-known because of their unique property which is self-similarity in different scales. During recent decades, researchers in different field of sciences have developed varieties of interesting studies about the unique properties of fractals in our body [8–13]. Detecting the fractal pattern in electrocardiography or ECG signals is one of these important discoveries [8]. The complex heterogeneous, non-stationary and self-regulated processes with a wide range of characteristics in physiological signals cannot be easily recognized using the traditional techniques in signal analysis and require nonlinear tools to reveal their complex and irregular fluctuations. Vibration analysis as one of these techniques in studying the irregularity of the biomedical signals considers that the statistical properties of signals such as mean and standard deviation remain constant over time [15–17]. One the most widely used algorithms in vibration analysis of signals called power spectral density (PSD) which is useful to compare random vibrations of signals with different length. To compute the PSD, one may need to estimate the fast Fourier transform (FFT) of the signal and then multiplying its amplitude by its complex conjugate. Normalizing this output to the frequency bin width, makes PSD more reliable compared to FFT when we are studying the random signals [15–17].

Multifractal analysis which has been used frequently in signal analysis [11–13], helps to classify different classes of signals by approximating scaling exponents and defining the scale invariance characteristics of the given process. The approximated scaling exponents can be used to characterize the statistical properties of dif- ferent subsets of a signal. According to different studies in fractal geometry, we only need one global exponent to characterized the homogeneous structure of monofractal signals, however, the heterogeneous structure of multifractal signals require more exponents to get indexed [18].
‘
In this study, we apply different quantitative and non-linear techniques to find a rigorous mathematical and the- oretical framework in studying the complexity of stress. Our public databases which is called Stress database in PhysioNet includes 15 multiparameter recordings from healthy volunteers, taken while they were driving on a prescribed route including city streets and highways. We perform vibration analysis methods such as power spectral densities (PSD) to study the power law behaviors of this database. We estimate the power law scaling exponents for all these recordings and then we apply multifractal analysis to study the multifractal structure and complex dynamics of these signals. Finally, Higuchi fractal dimension analysis of the recordings will complement this research.

Materials, Methods and Results

Data

In this study, we use 15 recordings (see figure (1)) of the Stress database which is contributed to PhysioNet by its creator, Jennifer Healey. This database contains a collection of multiparameter recordings from healthy volunteers, taken while they were driving on a prescribed route including city streets and highways in and around Boston, Massachusetts. The objective of the study for which these data were collected was to investigate the feasibility of automated recognition of stress on the basis of the recorded signals, which include ECG, EMG (right trapezius), GSR (galvanic skin resistance) measured on the hand and foot, and respiration [22, 24].

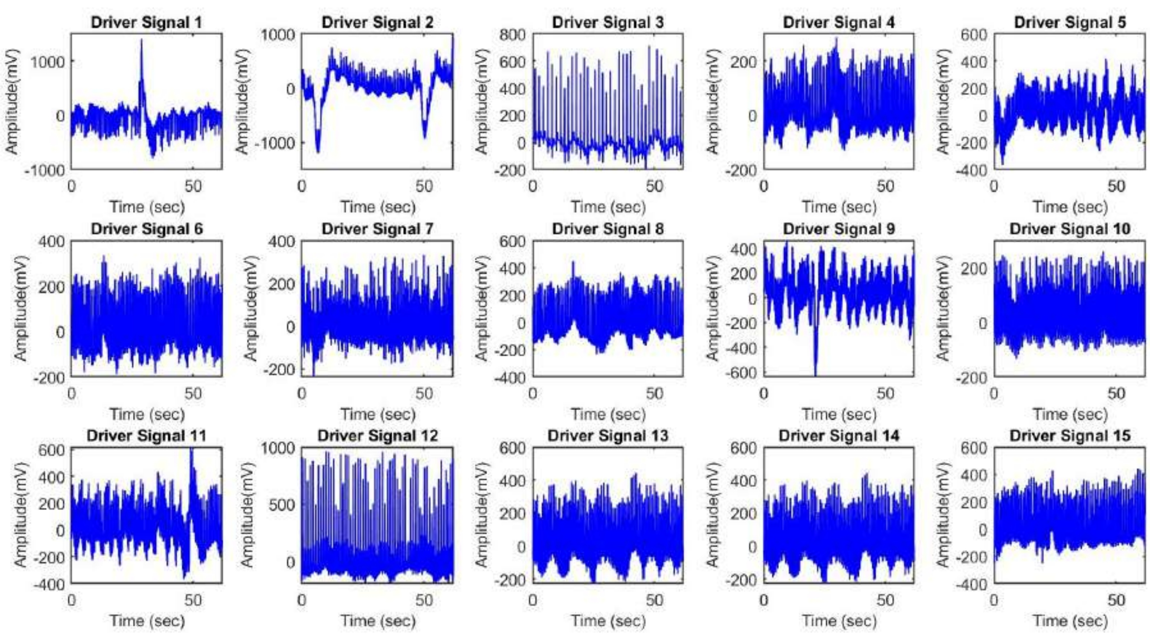

Figure 1: Recordings from 15 healthy subjects in Stress PhysioNet database.

Vibration frequency analysis; Power spectral densities (PSD) and scaling exponents

In this section, we apply power spectral densities (PSD) and exponent analysis on these ECG recordings to estimate the exponent from realizations of these processes and to find out if the signal of interest exhibits a power-law PSD. Here, we estimate the power spectral density using an averaging estimator technique called welch (PSD) method with overlapped segmentation.

Among different vibration frequency analysis algorithms, the fast Fourier transform (FFT) is one of the most widely used to compute discrete Fourier transform (DFT). However, it has some disadvantages. Basically, FFTs can only work well when there exist a finite number of dominant frequency components in the database. To overcome this problem, we are going to apply another vibration analysis method, called power spectral densities (PSD), which has been applied successfully to characterize random vibration in signals. To compute the power spectral densities, we need to multiply each frequency bin of fast Fourier transform by its complex conjugate to obtain a real spectrum and then normalize the results to frequency bin width.

Multifractal Analysis and Discrete Wavelet Transform (DWT)
In this section, we perform multi-fractal analysis to discover whether some type of power-law scaling exists for various statistical moments at different scales of these ECG signals. Here, we estimate the multifractal spectrum using a technique called (DWT) method.

According to recent studies about physiological signals, healthy signals reveal multifractal structure. In this section, using multifractal analysis, we test the three ECG databases we have to explore which one of them belongs to class of multifractal process, means that it requires larger number of scaling exponents to character- ize the scaling structures. We start with reviewing the general idea behind multifractal analysis using several different studies [25–46].

In general, fractal dimension determines the complexity of a fractal object by measuring the changes of cover- ings relative to the scaling factor. It also specifies the space filling capacity of a fractal object with respect to its scaling properties in the space. The relationship between scaling and covering is often hard to be characterized. The variation in the number of coverings, N(e), with respect to the scaling factor e, can be written as

$$N(\epsilon) \propto \epsilon^{-D} \tag{2.1}$$

where D is the fractal dimension. The relation (2.1) is called scaling law that has been used to demonstrate the size distribution of many objects in nature. The box counting formula which has been widely applied to approximate the fractal dimension of an irregular object is defined as

$$D_B = \lim_{a \to 0} \frac{\ln(N(a))}{\ln(1/a)} \tag{2.2}$$

However, this monofractal dimension is not able to fully characterize complex scaling behaviors of many irregular objects in the real world. That's why to study irregular objects like ECG signals one may need to apply the multifractal algorithm. The multifractal analysis used a spectrum of singularity exponents to provide a detailed and local description of complex scaling behaviors. In order to quantify local densities of the fractal set, we approximate the mass probability using the following formula

$$P_i(a) = \frac{N_i(a)}{N} \tag{2.3}$$

where Ni(a) is the number of mass in the ith subset of measure a, N is the total mass of the set. When we scale the mass probability Pi(a) with measure a of a multifractal set, it also demonstrates the power law behavior:

$$P_i(a) \propto a^{\alpha_i} \tag{2.4}$$

where αi is the singularity exponent characterizing the local scaling in the ith subset. The multifractal spectrum f (α) provides a statistical distribution of singularity exponents αi. In general, f (α) may be estimated using the Legendre transformation

$$f(\alpha) = q\alpha - \tau(q)$$
$$\alpha(q) = \frac{d\tau(q)}{dq}$$

where q is the moment and τ(q) is the mass exponent of the qth order moment. In addition, the multifractal measures may be specified by scaling of qth moments of Pi(a) as

$$\sum_{i=1}^{N(a)} P_i^{q(a)} \propto a^{\tau(q)} = a^{(q-1)D_q} \tag{2.5}$$

where Dq = τ(q) is the generalized fractal dimension. For q = 0 equation (2.3) becomes
$(q - 1)$
$N(a) \propto a^{-D0}$

which is similar to formula (2.1).
One of the most widely used techniques approximate multifractal spectrum of signals called wavelet analysis [37–42]. This method uses discrete wavelet analysis which is robust enough to characterize the distribution of scaling exponents and provides a good estimation to changes of regularity of a signal. To specify the spectrum of singularity of the pointwise regular function f , wavelet analysis associates the dimension of fractal sets to Hölder exponent *H(t)* [41]. The Hölder exponent of a fractal process *f (t)* can be defined as follows:

Definition 2.1 [42] *A fractal process f (t) satisfies a Hölder condition, when there exist H(t) > 0, such that*

$$|f(t') - f(t)| \simeq |t' - t|^{H(t)} \tag{2.6}$$

We can find H(t) for constant f from the coarse Hölder exponents as

$$h_\xi(t) = \frac{1}{\log \xi} \log \sup_{|t'-t|<\xi} |f(t') - f(t)| \tag{2.7}$$

The following sets may be defined to extract the geometry of a signal

$$E_h^{[d]} = \{t : H(t) = d\} \tag{2.8}$$

with varying d, these sets describe the local regularity of signal. We call the map

$$d \mapsto \dim(E^{[d]}) \tag{2.9}$$

which is a compact form of the singularity structure of the fractal process f , the multifractal spectrum of f [42]. In a global setting, to describe the complexity of a signal, we may need to count the intervals over which the fractal process f evolves with Hölder exponent H(t) and it gives an estimation of dim(E[d]). The grain exponent which is a discrete approximation to hξ (t) can be written as the following form [42].

$$h_k^{(n)} := -\frac{1}{n}\log_2 \sup\{|f(s)-f(t)| : (k-1) \tag{2.10}$$

$$2^{-n} \leq s \leq t \leq (k+2)2^{-n}\}$$

Therefore, the grain multifractal spectrum has the form [43–46].

$$F(d) = \lim_{\xi\to 0}\lim_{n\to\infty}\frac{\log N^n(d,\xi)}{n\log 2} \tag{2.11}$$

where

$$N^n(d,\xi) = \#\{k : |h_k^{(n)} - d| < \xi\} \tag{2.12}$$

Time-Frequency Analysis and Continuous Wavelet Transform (CWT)

The continuous wavelet transform (CWT) of a dataset h(t) is given by (Mallat, 1998) [14, 15]

$$\mathrm{CWT}(u,s) = \int_{-\infty}^{\infty} h(t)\frac{1}{|s|^{0.5}}\Phi^*\left(\frac{t-u}{s}\right)dt \tag{2.13}$$

where s is scale, u is displacement, Φ is the mother wavelet used, and ∗ means complex conjugate. The CWT is therefore a convolution of the data with scaled version of the mother wavelet. Of course, the time coordinate t in equation (2.13) could equally well be the spatial coordinate x if profile data were being analyzed.

Continuous Wavelet Transform (CWT) compute a linear time-frequency representation of non-stationary signals such as ECG signals called scalogram by breaking the ECG signals into scales by preserving time shifts and time scales. Therefore, the wavelet transform makes the analysis of the ECG signal in different frequency ranges easier and we can extract useful information from the time intervals between its consecutive waves of the physiological signals [15]. To compute the scalogram of a signal which is function of time and frequency, at first, we split the signal into overlapping segments, then we need to compute the absolute value of the continuous wavelet transform coefficients of each segment and finally, plot it.

Higuchi Fractal Dimension Algorithm

In this section, a quantitative analysis commonly known as the Fractal Dimension (FD) using the Higuchi algorithm has been carried out to illustrate the fractal complexity of input signals.

There are different methods to study the complexity of a fractal process. Using box counting method we can compute the dimension in two-dimensional space and also we can specify the complexity of two dimensional images [47]. However, this method does not provide us reliable information when we analyze ECG databases since it fails to recognize the sudden changes in the time series data set [48]. A variety of algorithms such as Higuchi algorithm, power spectrum analysis, and Katz algorithm have been developed to study the complexity of irregular signals such as physiological signals [49–51]. Here we use one of the most common used algorithms to estimate the fractal dimension of three groups of ECG data; Higuchi Algorithm.

Assume we have a finite time series $x_1, x_2, x_3, \ldots, x_N$. Then, we construct k new time series x^k

$$x_m, x_{m+k}, x_{m+2k}, \ldots, x_{[m+Ak]}$$

where $A = (N - m)/k$. For each time interval k and the initial time m such that $m = 1, 2, \ldots, k$, we compute the length of using

$$L_m^k = \frac{\sum_{i=1}^{[A]} |x_{m+ik} - x_{m+(i-1)k}|}{k} R$$

where $R = (N-1)/[A]k$ is the curve length normalization factor. To find the average of curve length for
each k, we calculate the mean of L^k for m = 1, 2, . . . , k and take the average for $k = 1, \ldots, k_{max}$. Next, we
plot log(L^k) versus log($1/k$) for different time interval k. Finally, we find the slope of regressed line which is obtained by the least-squares approximation as the Higuchi fractal dimension for time interval $k = 400$.

Discussion of results

Signals without characteristic in scale also called scale free signals (with fixed statistical properties like mean and variance after any stretching or shrinking factors) have a wide range of application in geophysics, finance and physiology, the importance of different approaches in nonlinear dynamics theory have been increased and motivated us to apply them in biomedical signal processing. In this section, we employ some of these methods including power spectral densities (PSD), Higuchi fractal dimension algorithm, scaling exponents and multi- fractal analysis.

In figure [2], we can see the fitted least squares approximation to the logarithm of power spectral density of all long-term signals.

As we can see from figure [2], PSD may not be useful to classify the data, however, the power-law exponent.

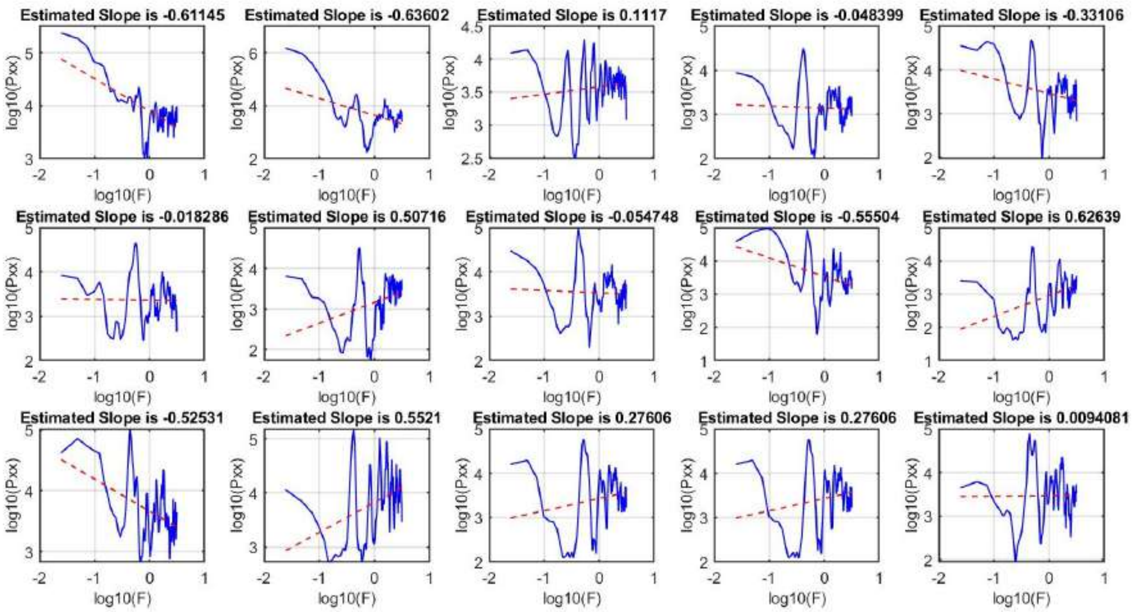

Figure 2: Fitted least squares approximation to the logarithm of power spectral density of Stress PhysioNet database obtained by wavelet techniques.

can be a good measure of complexity for the recordings and their power-law properties.
To differentiate the time series, we approximate the scaling exponents for stress database and plot them in figure [3].

By looking at figure [3], we can easily say that all signals are monofractal since we have a narrow range of scaling exponent.

We plot the multi-fractal spectra of this database to compare the width of the scaling exponent for each spec- trum. From multifractal analysis results of signals in the stress database (see figure (4)), we can easily see that we have a short range of support of exponents for all recorded signals, which indicates they have mono-fractal structure.

Therefore, the signals with stress show a clear loss of multifractality and are homogeneous and monofractal

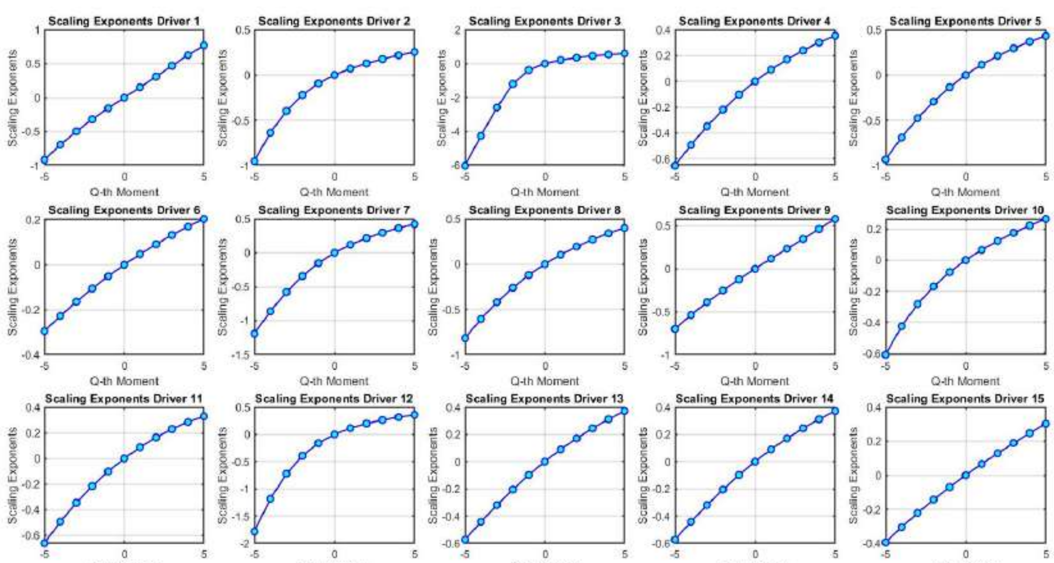

Figure 3: Scaling exponent of power spectral density of Stress PhysioNet database.

since their spectrum displays a narrow width of scaling exponent. Here, the recordings demonstrate similar scaling features throughout the signal and they can be characterized by only a single global exponent. In sum- mary, the multi-fractal analysis demonstrates different level of complexity and non-linear dynamics of signals and can be used to characterize them since it provides different range of exponents useful to classify data.

Another important outcome from our multifractal analysis is recognizing obvious changes in the shape of D(h) curves for signals which is crucial in finding the best strategies to better controlling the stress. We have displayed the scalogram plots of all signals for in figure [5].

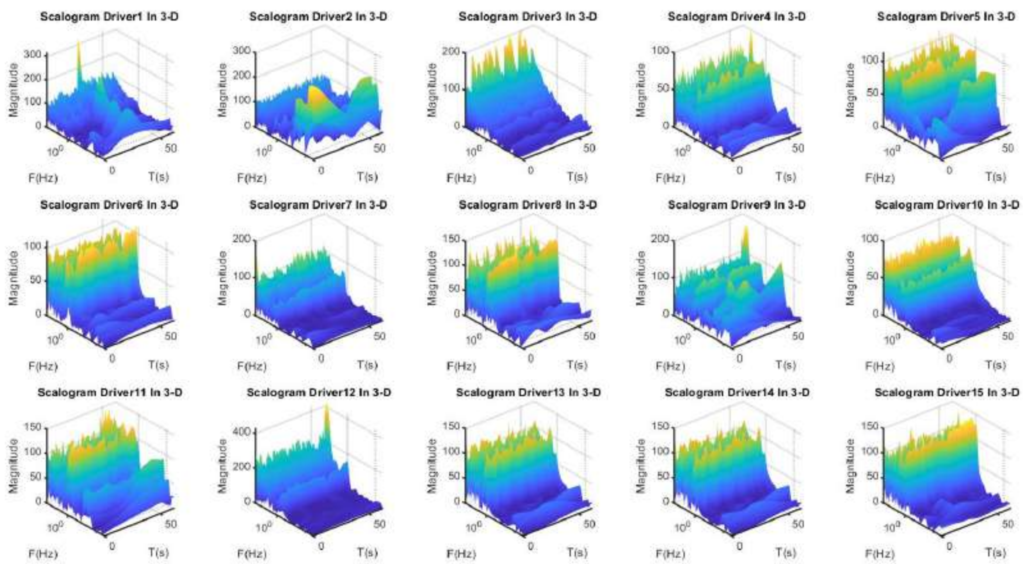

Figure 5: Time-frequency representations of Stress PhysioNet database.

Here, we can see that the monofractal characteristics and nonlinear features of data are encoded in the fre quency domain of the vibrations.

We have estimated the fractal dimension of data and plotted their regression models for each data in figure (6). We have determined the optimal value for k_{max} such that after this value, there is no change in fractal dimension and it is $k_{max} = 400$.

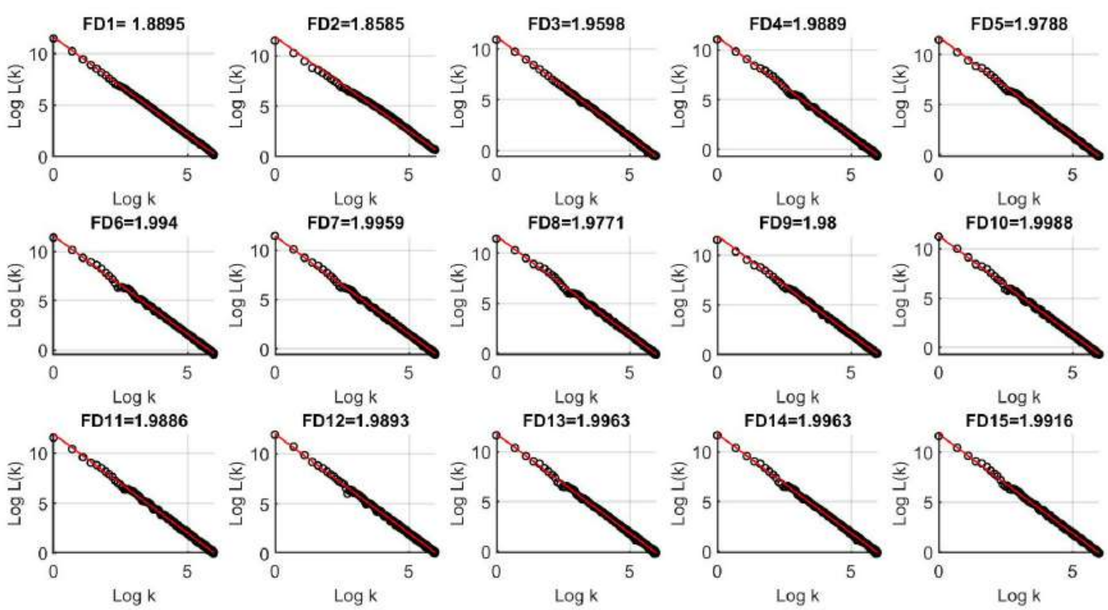

Figure 6: Plots of $\log(L^k)$ versus $\log(k)$ for time interval $k = 400$, the logarithmic scale and the corresponding slope of fitted regression line (the Higuchi fractal dimension) for Stress PhysioNet database.

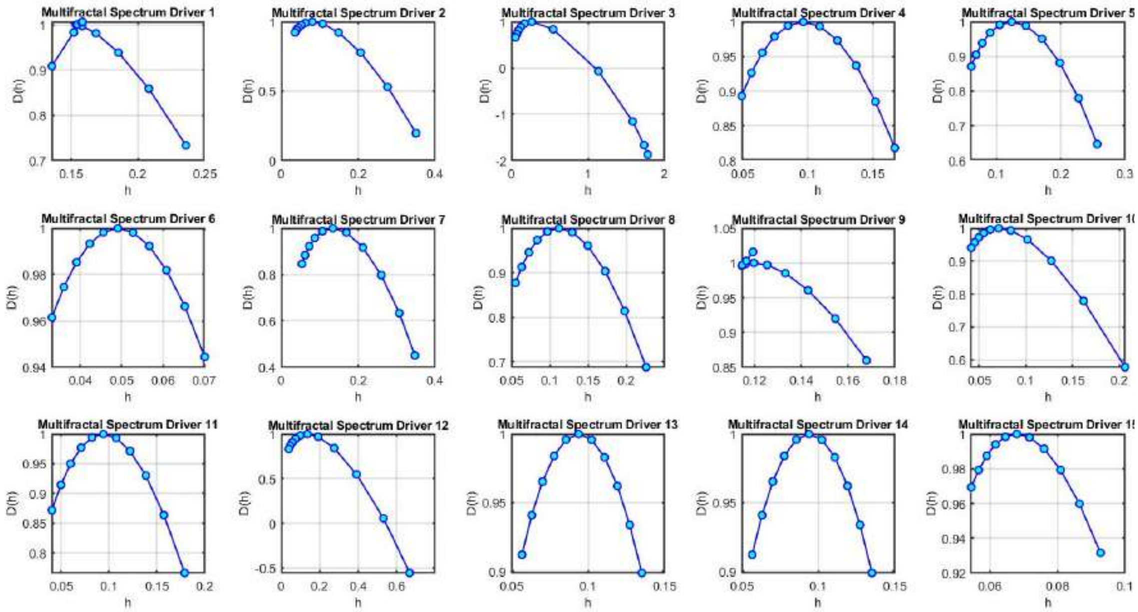

Figure 4: The multifractal spectrum of Stress PhysioNet database.

Conclusion

In this research, we have studied the fractal structure of Stress PhysioNet database. We have performed different nonlinear techniques for classifying the data such as vibration analysis and wavelet analysis. To specify the signals patterns and complexity of the stress database, we have estimated the power-law exponent and (PSD), the fractal dimension (FD) and we carried out the multifractal analysis which are reliable and well-known methods in time series data analysis. We have plotted the logarithm of power spectral densities (PSD) against the logarithm of frequency to estimate the exponent using the slope of linear regression of these processes. Moreover, we have estimated the scaling exponents of the signals and we noticed that the stress signals have narrow range of scaling law. We continued the analysis by looking at the fractal structures of data using Higuchi method. Higuchi algorithm approximated the fractal dimensions for all recordings for the optimal time interval k = 400. According to Higuchi algorithm, fractal dimension can be used to compare different individuals in database. However, fractal dimension cannot be used as a diagnosis tool for clinical purposes unless further analysis and studies need to be performed in this area. Finally, we have reported on evidences for monofractality in stress database using multifractal analysis. According to the multifractal analysis, we recognized a narrow range of scaling exponents for all recordings which revealed the loss of multifractality in stress database. These results suggest that the multifractal analysis and scaling exponents may be considered as two indicators to compare the complexity of stress signals. Likewise, the multifractal analysis can be used as a controlling and regulating mechanism of the stress and have the potential to be used as diagnostic tools in patient examinations. Furthermore, it can be considered as a computational framework to further analysis of physiological signals and clinical databases and fight against stress. In addition, these results of comparison between different subjects using variety of non-linear methods indicate that it requires ongoing researches and studies to develop a realistic and comprehensive model which helps to control and regulate the stress.

Despite the fact that there is still a big gap between theoretical and experimental research about stress, we hope that our framework offers a useful model for future investigations of the mechanism operating on the stress and the system related to that during any changes which cause stress. This approach should be considered only as a starting point in theoretical and mathematical framework in studying this complex problem, and we hope to develop it in interactions with empirical and experimental research.

References

1. Savić, D., Knežević, G., & Opačić, G. (2000). A mathematical model of stress reaction: Individual differences in threshold and duration. Psychobiology, 28(4), 581-592.
2. Yaribeygi, H., & Sahraei, H. (2018). Physiological/neurophysiological mechanisms involved in the formation of stress responses. Neurophysiology, 50(2), 131-139.
3. Lupien, S. J., Maheu, F., Tu, M., Fiocco, A., & Schramek, T. E. (2007). The effects of stress and stress hormones on human cognition: Implications for the field of brain and cognition. Brain and cognition, 65(3), 209-237.
4. Lupien, S. J., McEwen, B. S., Gunnar, M. R., & Heim, C. (2009). Effects of stress throughout the lifespan on the brain, behaviour and cognition. Nature reviews neuroscience, 10(6), 434-445.
5. Lupien, S. J., Juster, R. P., Raymond, C., & Marin, M. F. (2018). The effects of chronic stress on the human brain: From neurotoxicity, to vulnerability, to opportunity. Frontiers in neuroendocrinology, 49, 91-105.
6. Lautarescu, A., Craig, M. C., & Glover, V. (2020). Prenatal stress: Effects on fetal and child brain development. International review of neurobiology, 150, 17-40.
7. Mandelbrot B B. The fractal geometry of nature. WH freeman

New York, 1, 1982.
8. Lapidus, M. L., Van Frankenhuijsen, M., & Mandelbrot, B. B. (2004). Fractal geometry and applications: a jubilee of Benoît Mandelbrot. American Mathematical Soc..
9. Meyer, M., & Stiedl, O. (2003). Self-affine fractal variability of human heartbeat interval dynamics in health and disease. European Journal of Applied Physiology, 90(3), 305-316.
10. Acharya, R., Bhat, P. S., Kannathal, N., Rao, A., & Lim, C. M. (2005). Analysis of cardiac health using fractal dimension and wavelet transformation. Itbm-Rbm, 26(2), 133-139.
11. Azizi, T. (2022). Measuring fractal dynamics of FECG signals to determine the complexity of fetal heart rate. Chaos, Solitons & Fractals: X, 100083.
12. Azizi, T. (2022). On the fractal geometry of gait dynamics in different neuro-degenerative diseases. Physics in Medicine, 100050.
13. Azizi, T. (2022). On the fractal geometry of different Heart rhythms. Chaos, Solitons & Fractals: X, 100085.
14. Cooper, G. R. J., & Cowan, D. R. (2008). Comparing time series using wavelet-based semblance analysis. Computers & Geosciences, 34(2), 95-102.
15. Mallat, S. (1999). A wavelet tour of signal processing. Elsevier.
16. Rangayyan, R. M. (2015). Biomedical signal analysis. John Wiley & Sons.
17. Peng, C. K., Havlin, S., Stanley, H. E., & Goldberger, A. L. (1995). Quantification of scaling exponents and crossover phenomena in nonstationary heartbeat time series. Chaos: an interdisciplinary journal of nonlinear science, 5(1), 82-87.
18. Azizi, S., & Azizi, T. (2022). The Fractal Nature of Drought: Power Laws and Fractal Complexity of Arizona Drought. European Journal of Mathematical Analysis, 2, 17-17.
19. Ivanov, P. C., Amaral, L. A. N., Goldberger, A. L., Havlin, S., Rosenblum, M. G., Struzik, Z. R., & Stanley, H. E. (1999). Multifractality in human heartbeat dynamics. Nature, 399(6735), 461-465.
20. Sassi, R., Signorini, M. G., & Cerutti, S. (2009). Multifractality and heart rate variability. Chaos: An Interdisciplinary Journal of Nonlinear Science, 19(2), 028507.
21. Lin, D. C., & Sharif, A. (2010). Common multifractality in the heart rate variability and brain activity of healthy humans. Chaos: An Interdisciplinary Journal of Nonlinear Science, 20(2), 023121.
22. Healey, J. A., & Picard, R. W. (2005). Detecting stress during real-world driving tasks using physiological sensors. IEEE Transactions on intelligent transportation systems, 6(2), 156-166.
23. Goldberger, A. L., Amaral, L. A. N., Glass, L., Hausdorff, J. M., Ivanov, P. C., Mark, R. G., ... & Stanley, H. E. (2000). Components of a new research resource for complex physiologic signals. PhysioBank, PhysioToolkit, and Physionet.
24. Goldberger, A. L., Amaral, L. A., Glass, L., Hausdorff, J. M., Ivanov, P. C., Mark, R. G., ... & Stanley, H. E. (2000). PhysioBank, PhysioToolkit, and PhysioNet: components of a new research resource for complex physiologic signals. circulation, 101(23), e215-e220.
25. Ren, S. Z., Tombacz, E., & Rice, J. A. (1996). Dynamic light scattering from power-law polydisperse fractals: Application of dynamic scaling to humic acid. Physical Review E, 53(3), 2980.
26. Halley, J. M., Hartley, S., Kallimanis, A. S., Kunin, W. E., Lennon, J. J., & Sgardelis, S. P. (2004). Uses and abuses of fractal methodology in ecology. Ecology letters, 7(3), 254-271.
27. Bonnet, E., Bour, O., Odling, N. E., Davy, P., Main, I., Cowie, P., & Berkowitz, B. (2001). Scaling of fracture systems in geological media. Reviews of geophysics, 39(3), 347-383.
28. Li, C., Ma, T., & Shi, J. (2003). Application of a fractal method relating concentrations and distances for separation of geochemical anomalies from background. Journal of Geochemical exploration, 77(2-3), 167-175.
29. Foroutan-pour, K., Dutilleul, P., & Smith, D. L. (1999). Advances in the implementation of the box-counting method of fractal dimension estimation. Applied mathematics and computation, 105(2-3), 195-210.
30. Li, J., Du, Q., & Sun, C. (2009). An improved box-counting method for image fractal dimension estimation. Pattern recognition, 42(11), 2460-2469.
31. Posadas, A. N., Giménez, D., Quiroz, R., & Protz, R. (2003). Multifractal characterization of soil pore systems. Soil Science Society of America Journal, 67(5), 1361-1369.
32. Thomas, T. R., & Rosén, B. G. (2008). Implementation of Whitehouse's method for calculating properties of self-affine fractal profiles. Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, 222(8), 1547-1550.
33. Stanley, H. E., Amaral, L. N., Goldberger, A. L., Havlin, S., Ivanov, P. C., & Peng, C. K. (1999). Statistical physics and physiology: monofractal and multifractal approaches. Physica A: Statistical Mechanics and its Applications, 270(1-2), 309-324.
34. Ţălu, Ş. (2012). Mathematical methods used in monofractal and multifractal analysis for the processing of biological and medical data and images. ABAH Bioflux, 4(1), 1-4.
35. Weiss, B., Clemens, Z., Bódizs, R., Vágó, Z., & Halász, P. (2009). Spatio-temporal analysis of monofractal and multifractal properties of the human sleep EEG. Journal of neuroscience methods, 185(1), 116-124.
36. Wink, A. M., Bullmore, E., Barnes, A., Bernard, F., & Suckling, J. (2008). Monofractal and multifractal dynamics of low frequency endogenous brain oscillations in functional MRI. Human brain mapping, 29(7), 791-801.
37. Rodríguez-Liñares, L., Méndez, A. J., Lado, M. J., Olivieri, D. N., Vila, X. A., & Gómez-Conde, I. (2011). An open source tool for heart rate variability spectral analysis. Computer methods and programs in biomedicine, 103(1), 39-50.
38. Wendt, H., & Abry, P. (2007). Multifractality tests using bootstrapped wavelet leaders. IEEE Transactions on Signal Processing, 55(10), 4811-4820.
39. Wendt, H., Abry, P., & Jaffard, S. (2007). Bootstrap for empirical multifractal analysis. IEEE signal processing magazine, 24(4), 38-48.

40. Jaffard, S., Lashermes, B., & Abry, P. (2006). Wavelet leaders in multifractal analysis. In Wavelet analysis and applications (pp. 201-246). Birkhäuser Basel.
41. Jaffard, S. (2004). Wavelet techniques in multifractal analysis. PARIS UNIV (FRANCE).
42. Riedi, R. H. (1999). Multifractal processes. Rice Univ Houston Tx Dept Of Electrical And Computer Engineering.
43. Mandelbrot, B. B. (1972). Possible refinement of the lognormal hypothesis concerning the distribution of energy dissipation in intermittent turbulence. In Statistical models and turbulence (pp. 333-351). Springer, Berlin, Heidelberg.
44. Riedi, R. (1995). An improved multifractal formalism and self-similar measures. Journal of Mathematical Analysis and Applications, 189(2), 462-490.
45. Hentschel, H. G. E., & Procaccia, I. (1983). The infinite number of generalized dimensions of fractals and strange attractors. Physica D: Nonlinear Phenomena, 8(3), 435-444.
46. Halsey, T. C., Jensen, M. H., Kadanoff, L. P., Procaccia, I., & Shraiman, B. I. (1986). Fractal measures and their singularities: The characterization of strange sets. Physical review A, 33(2), 1141.
47. Panigrahy, C., Garcia-Pedrero, A., Seal, A., Rodríguez-Esparragón, D., Mahato, N. K., & Gonzalo-Martín, C. (2017). An approximated box height for differential-box-counting method to estimate fractal dimensions of gray-scale images. Entropy, 19(10), 534.
48. Öztürk, Y. (2019). Fractal Dimension as a Diagnostic Tool for Cardiac Diseases. Int. J. Current Eng. Technol, 9, 425-431.
49. Higuchi, T. (1988). Approach to an irregular time series on the basis of the fractal theory. Physica D: Nonlinear Phenomena, 31(2), 277-283.
50. Gomolka, R. S., Kampusch, S., Kaniusas, E., Thürk, F., Széles, J. C., & Klonowski, W. (2018). Higuchi fractal dimension of heart rate variability during percutaneous auricular vagus nerve stimulation in healthy and diabetic subjects. Frontiers in physiology, 9, 1162.
51. Katz, M. J. (1988). Fractals and the analysis of waveforms. Computers in biology and medicine, 18(3), 145-156.