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Kinematic Model Yields Two Geo-Synchronous Orbits of E-M System and M-P-D System Validated by Total Energy Analysis

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Abstract
In Black Holes (BH) studies , a well-tested method of total energy analysis is used by which the radius of Innermost Stable Circular Orbit (ISCO) around the central BH is analyzed and the dimensionless spin parameter of BH is extracted. The same total energy analysis is utilized in case of E-M and M-P planet-satellite pairs and their extremum points are obtained. The energy maxima happens to be the inner unstable co-rotational orbit (aG1) and the energy minima happens to be outer stable corotational orbit(aG2) in case of E-M and M-P systems and their values from KM correspond to those obtained from Energy Analysis. This vindicates KM of binary pairs.

Keywords: Energy Maxima, Energy Minima, Geo-Synchronous Orbits, Innermost Stable Circular Orbit, Co-Rotational Orbit.

1. Introduction

Total Energy Analysis of the spinning BH at the centre of galaxies is a standard procedure for determining the rim of the accretion disk. Through X-Ray Telescopes (ESA's XMM-Newton Observatory and NASA's Nuclear Spectroscopic Telescope Array (NuSTAR)), Risaliti et.al.(2013) measured the high energy light emitted by iron atoms from the centre of the Galaxy. This enabled them to measure the radius of rim of accretion disk. From this rim radius the spin parameter of NGC1365 was extracted which turned out to be a*=0.85 (85% of the maximum spin SMBH , Super Massive Black Hole, can achieve). For this analysis Innermost Marginally Stable Circular Orbit needs to be obtained. This can be obtained by looking at the extremum of total energy profile.

Effective Potential in Schwarzschild Metric. [Chapter 9, lecture on Schwarzschild Space Time, http://eagle.phys.utk.edu/guidry/astro421/lectures/lecture490_ch9.pdf]

$$V(r) = \frac{\epsilon \times c^2}{2} - \frac{\epsilon \times c^2(\frac{GM}{c^2})}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \qquad 1$$

In (1),

$$Constant\ term = \frac{\epsilon}{2}; Newtonian\ Potential = -\frac{\epsilon GM}{r};$$

$$Contribution\ of\ Angular\ Momentum = \frac{L^2}{2r^2}; GR\ contribution = \frac{GML^2}{r^3};$$

Let 2GM = gravitational radius/Schwarzschild radius = r_G assuming 'c'=1.
For massive particles, ε = 1 and 2GM is replaced by gravitational radius/Schwarzschild radius = rG and let radial parameter be normalized as x = r/ rG.
Hence (1) becomes:

$$V(x) = \frac{1}{2} - \frac{1}{2x} + \frac{(\frac{L}{r_G})^2}{2x^2} - \frac{(\frac{L}{r_G})^2}{2x^3} \qquad 2$$

Differentiate (2) with respect to ‘x’ and we get:

$$\frac{\delta V}{\delta x} = \frac{3k}{2x^4} - \frac{k}{x^3} + \frac{1}{2x^2} \qquad where\ k = (\frac{L}{r_G})^2 \qquad 3$$

Equating the first derivative of V(x) to zero we get the roots where extremum of V(x) occur.
These roots are:

$$x_{max} = k + \sqrt{-3k + k^2}\ \ and\ x_{min} = k - \sqrt{-3k + k^2} \qquad 4$$

In Table 1. the roots of extremum are tabulated for different values of k.

Table 1: Energy Maxima and Energy Minima Roots calculated from Eq.3

k	Inner Energy maxima root	Outer Energy maxima root	Comments
1	none	none	No extremum
2	none	none	No extremum
3	3	3	Inner Marginally Stable CO at repeated roots (or coincident roots.
4	2(Unstable CO)	6(Stable CO)	Roots diverge
5	1.83(Unstable CO)	8.16(Stable CO)	Roots diverge
6	1.75(Unstable CO)	10.24(Stable CO)	Roots diverge
7	1.7(Unstable CO)	12.29(Stable CO)	Roots diverge
8	1.675(Unstable CO)	14.32(Stable CO)	Roots diverge
9	1.65(Unstable CO)	14.32(Stable CO)	Roots diverge
10	1.63(Unstable CO)	18.366(Stable CO)	Roots diverge

The roots for different k (a parameter of L) are graphically illustrated in Figure 1

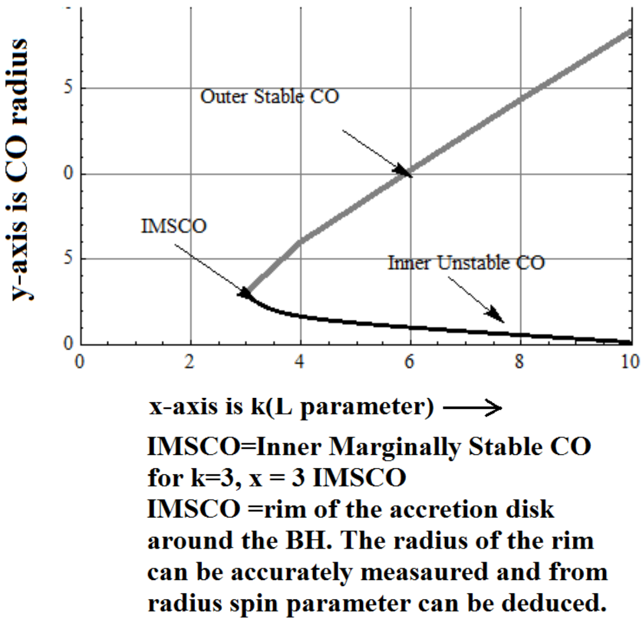

Figure 1: Graphical Illustration of coincident roots at IMSCO and diverging roots as a function of 'k'.

Risalliti et.al.(2013) have accurately determined the dimensionless spin parameter of the Black Hole at the center of Galaxy NGC1365 by accurately measuring the Innermost Stable Circular Orbit (ISCO). The radius of ISCO happens to be the energy minima point. At larger L(angular momentum) the extremum roots diverge and the energy maxima is the unstable inner circular orbit and energy minima is stable outer circular orbit. These are analogous to the two geo-synchronous orbits of E-M and M-P systems.

In Figure 2 and Figure 3 total energy profile is illustrated for k=3 and for k=4.

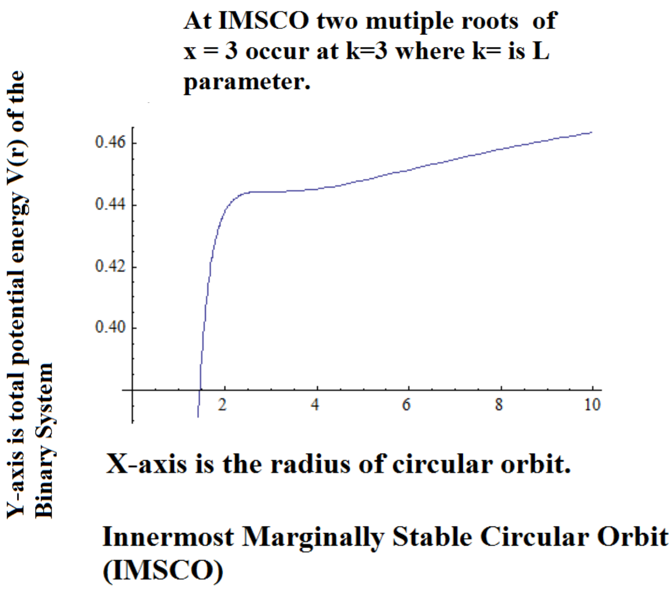

Figure 2: The total energy profile for IMSCO.

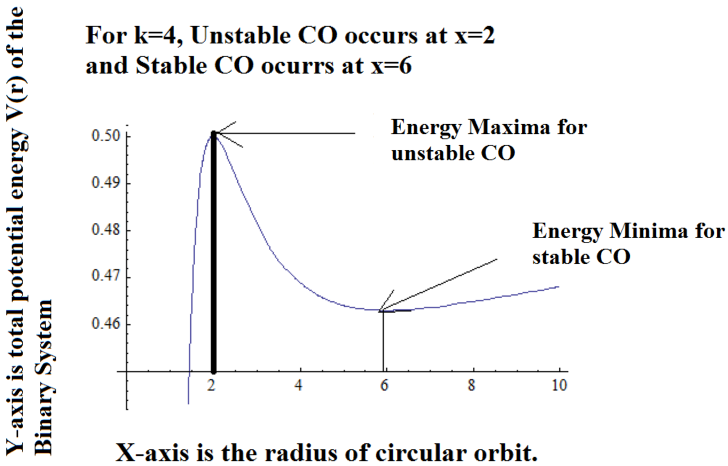

Figure 3: The total energy profile for ISCO (inner stable circular orbit) and OSCO (outer stable circular orbit).

2. To determine the extremum points from Total Energy Profile of E-M binary system.

Total Energy of Earth-Moon System = Rotational Kinetic Energy + Potential Energy + Translational Kinetic Energy.

Translational Kinetic Energy of the order of 1×108Joules due to recession of Moon for all practical purposes is negligible as compared to Rotational Kinetic Energy of the order of 1×1030Joules . Hence Translational Kinetic Energy is neglected in future analysis.

Moon is trapped in potential well created by the Earth.

Moon's potential energy = - $GM_{Earth}M_{Moon}/a$

G =Gravitational Constant=6.673×10^{-11} N-m^2/Kg^2;

M_{Earth} = mass of the Earth = 5.9742×10^{24} Kg;

M_{Moon} = mass of the Moon = E/81 = 7.348×10^{22} Kg;

a= semi-major axis of Moon's orbit around the Earth= 3.844×10^{8} m;

Rotational Kinetic Energy of Earth-Moon System = Spin Energy of the Earth + Orbital Energy of the Earth-Moon System + Spin Energy of the Moon =

$$\frac{1}{2}C\omega^2 + \frac{1}{2}\left(\frac{M_{Moon}}{1+\frac{M_{Moon}}{M_{Earth}}}\right)a^2 \times \Omega^2_{orbital} + \frac{1}{2} \times (0.4M_{Moon}R^2_{Moon})\Omega^2_{spin} \qquad 5$$

Where C = moment of inertia around polar axis = $0.3308M_{Earth}R_{Earth}^2 = 8.02\times10^{37}$ Kg-m^2;

Equatorial Radius of Earth=6.37814×10^{6} m;

Equatorial Radius of Moon=1.738×10^{6} m;

Earth angular spin velocity = $\omega = 2\pi/T_E$ = [2π/(86400)]radians/sec;

In this analysis we will consider all rates of rotation to be in Solar Days. We will consider one solar day as the present spin-period of Earth. Similarly while calculating Earth-Moon orbital angular momentum we will use present sidereal month expressed in 27.3 solar days.

Earth-Moon Orbital Angular Velocity = Ω = [2π/(27.3×86400)]radians/sec where sidereal month =27.3 d;

Since Moon is in synchronous orbit i.e. it is tidally locked with the Earth hence we see the same face of Moon and Moon's Orbital Angular Velocity = Moon's Spin Angular Velocity = Ω;

Therefore total rotational Kinetic Energy Equation 2 reduces to:

$$\frac{1}{2}C\omega^2 + \frac{1}{2}\left(\frac{M_{Moon}}{1+\frac{M_{Moon}}{M_{Earth}}}\right)a^2 \times \Omega^2 + \frac{1}{2} \times (0.4M_{Moon}R_{Moon}^2)\Omega^2 \qquad 6$$

Similarly total angular momentum of Earth-Moon System is as follows:

$$J_T = C\omega + \left(\frac{M_{Moon}}{1+\frac{M_{Moon}}{M_{Earth}}}\right)a^2 \times \Omega + (0.4M_{Moon}R_{Moon}^2)\Omega \qquad 7$$

Substituting the numerical values in Eq.7 we obtain:
$J_T = 3.44026\times10^{34}$ Kg-m^2/sec ;
From Sharma (2011), we have the following relation between Length of Sidereal Month and Length of Sidereal Day:

$$\frac{LOM}{LOD} = \frac{\omega}{\Omega} = E \times a^{1.5} - F \times a^2 \qquad 8$$

$$Where\ E = \frac{J_T}{BC}\ and\ F = \left(\frac{M_{Moon}}{C(1+\frac{M_{Moon}}{M_{Earth}})}\right)$$

$$Here\ B = \sqrt{G(E+m)}$$

Substituting the numerical values we get:
B=20.08884482×10^6 m$^{3/2}$/s;
E=2.13531×10^{-11} m$^{-3/2}$;
F = 9.05036×10^{-16} m^{-2} ;

If the numerical values of E and F are substituted in Eq.8. and the present value of 'a' is substituted we get LOM/LOD = 27.2 whereas we should get 27.3. This is because Eq.8 has been derived based on Keplarian Approximation. If LOM/LOD was derived from exact analysis we would get LOM/LOD in the present epoch as 27.3.

$$\frac{LOM}{LOD} = \frac{\omega}{\Omega} = E \times a^{1.5} - F \times a^2 = 1 \qquad 9$$

Rewriting total rotational Kinetic Energy expression from Eq.6 we get:

$$KE = \frac{1}{2}C\omega^2 + \frac{1}{2}\left(\frac{M_{Moon}}{1+\frac{M_{Moon}}{M_{Earth}}}\right)a^2 \times \Omega^2 + \frac{1}{2} \times (0.4M_{Moon}R_{Moon}^2)\Omega^2$$

Reshuffling the angular velocity terms we get:

$$KE = \frac{1}{2}\Omega^2\left[C\left(\frac{\omega}{\Omega}\right)^2 + \left(\frac{M_{Moon}}{1+\frac{M_{Moon}}{M_{Earth}}}\right)a^2 + (0.4M_{Moon}R_{Moon}^2)\right]; \qquad 10$$

Substituting Eq.9 in Eq.10 we get:

$$KE = \frac{1}{2}\Omega^2 \left[C(E \times a^{1.5} - F \times a^2)^2 + \left(\frac{M_{Moon}}{1 + \frac{M_{Moon}}{M_{Earth}}} \right) a^2 + (0.4 M_{Moon} R_{Moon}^2) \right]; \qquad 11$$

According to Kepler's 3rd Law:

$$a^3 \Omega^2 = G(M_{Earth} + M_{Moon}) \qquad 12$$

Substituting Eq.12 in Eq.11 we obtain:

$$KE = \frac{1}{2} \times \frac{G(M_{Earth} + M_{Moon})}{a^3} \left[C(E \times a^{1.5} - F \times a^2)^2 + \left(\frac{M_{Moon}}{1 + \frac{M_{Moon}}{M_{Earth}}} \right) a^2 + (0.4 M_{Moon} R_{Moon}^2) \right]; \qquad 13$$

Therefore total energy of the E-M System is:

$$TE = KE + PE$$

$$TE = \frac{1}{2} \times \frac{G(M_{Earth} + M_{Moon})}{a^3} \left[C(E \times a^{1.5} - F \times a^2)^2 + \left(\frac{M_{Moon}}{1 + \frac{M_{Moon}}{M_{Earth}}} \right) a^2 + (0.4 M_{Moon} R_{Moon}^2) \right] - \frac{G M_{Earth} M_{Moon}}{a} \qquad 14$$

To determine the stable and unstable equilibrium points in non-keplerian journey of Moon we must examine the Plot of Eq.14 from a = 8×10^6 m; to 'a' = 6×10^8 m;

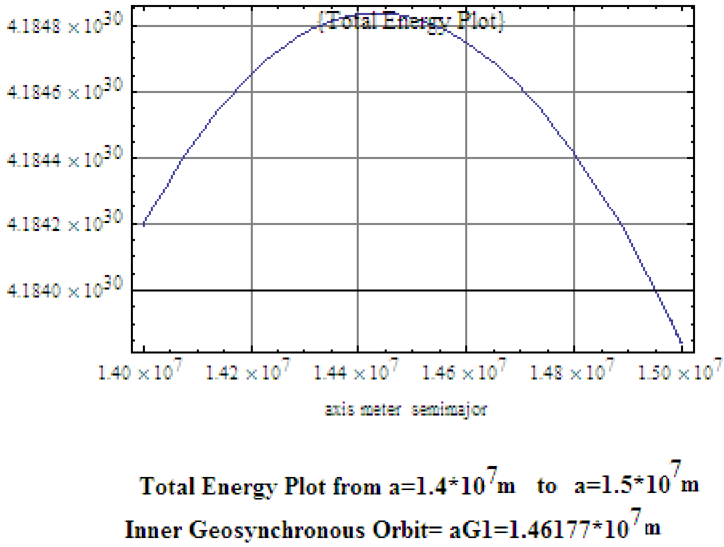

Figure 4: Plot of total energy in the range 1.4×10^7m and 1.5×10^7m around the inner geo-synchronous orbit of a = 1.46×10^7m.

We find an energy Maxima at inner geo-synchronous orbit (aG1=1.46×10^7m) hence it is unstable equilibrium point. When Moon is at inner-geosynchronous orbit, any perturbation launches Moon on either a sub-synchronous orbit or on extra-synchronous(or super-synchronous orbit). If it is launched on sub-synchronous orbit then it rapidly spirals in towards the primary body and if it is launched on extra-synchronous orbit then it spirals out from inner to outer geosynchronous orbit. In our case, Moon is fully formed beyond Roches' Limit which is 18,000Km [Ida et.al. 1997] just beyond inner Clarke's orbit or inner Geo-synchronous Orbit hence Moon is launched on expanding spiral orbit towards outer Clarke's Orbit or outer Geo-synchronous Orbit.

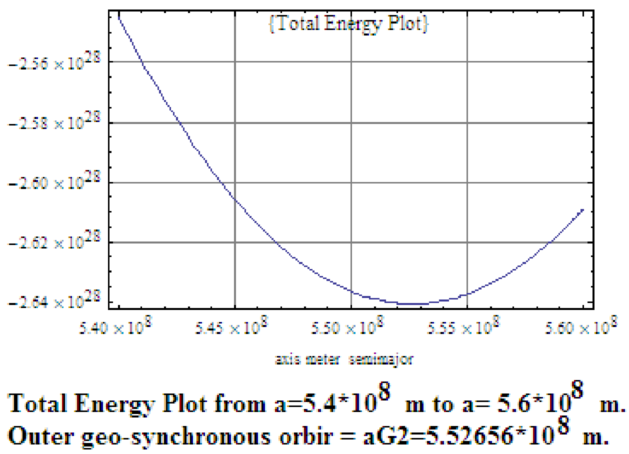

Figure 5: Plot of total energy in the range 5.4×10^8m and 5.6×10^8m around the outer geo-synchronous orbit of $a_{G2} = 5.527\times10^8$m.

At outer geosynchronous orbit ($\mathbf{a_{G2} = 5.527\times10^8}$**m**.)there is energy minima hence it is stable equilibrium point. Secondary body can never move beyond this orbit. Either it is stay-put in that orbit or it gets deflected back into a contracting spiral orbit.

The outer Geosynchronous Orbit defines the sphere of gravitational influence of Earth in much the same way as Hill Radius does for Earth in presence of Sun.

$$\boldsymbol{Hill\ Radius = R_H = R \times \left(\frac{M_+}{3M_\odot}\right)^{\frac{1}{3}}} \qquad \mathbf{15}$$

R = 1AU=1.49598×10^{11}m.

Substituting the mass of Earth and Sun, Hill Radius is 1.49×10^{9}m whereas $a_{G2}=5.527\times10^{8}$m.
Thus the results of KM are validated by an alternate method namely total energy profile analysis method commonly used in pinning down the spin parameter of Black Holes from the study of ISCO.

Total Energy Formalism and its extremum points for Mars-Phobos System.

Table 2: Globe and Orbit Parameters of Mars-Phobos-Deimos

parameters	Mars	Phobos	Deimos	Source
Mass(Kg)	0.64174×10^{24}	10.7046×10^{15}	2.24888×10^{15}	Ref 1,2
GM(Km^3/s^2)	0.042828382 $\times10^{6}$	(7.14±0.19) $\times10^{-4}$	(1.5±0.11) $\times10^{-4}$	Ref 2
Volumetric Mean Radius Or Median Radius($\times10^{3}$ m)	3389.5	11.2	6.1	Ref.1
Flattening	0.00589	irregular	irregular	Ref 1
Mean Density(Kg/m^3)	3933	1900	1750	Ref 1
Moment of Inertia($I/(MR^2)$)	0.366	0.4	0.4	Ref 1
Sidereal Spin period	24.6229h	0.31891d	1.26244d	Ref 1
Sidereal Orbital period(d)	-	0.31891d	1.26244d	Ref 1
a*(semi-major axis)($\times10^{6}$m)	-	9.378	23.459	Ref 1
Orbital eccentricity	-	0.0151	0.0005	Ref 1
Orbital inclination w.r.t. The equatorial plane of Mars(deg)	-	1.08	1.79	Ref 1

*Mean Orbital Distance from the center of Mars.

Reference 1. http://nssdc.gsfc.nasa.gov/planetary/factsheet/marsfact.html
Reference 2. Bills, Bruce G.; Neumann,Gregory A.; Smith,David E. and Zuber, Maria T. "Improved estimate of tidal dissipation within Mars from MOLA observations of the shadow of Phobos", *JOURNAL OF GEOPHYSICAL RESEARCH*, **110**, E07004, doi:10.1029/2004JE002376, 2005
Inspection of the Table clearly establishes that Phobos and Deimos are tidally locked with Mars. They present the same face to Mars all the time. The two satellites are moving in nearly circular orbits and are in nearly coplanar orbital plane. The orbital plane of the natural satellites are coplanar with the equatorial plane of Mars.
In this section we will study the energy profile of Phobos and Deimos during its tidally evolving trajectories.

Total Angular Momentum is::

$$J_T = C\omega + \left(m^* a_{present}^2 + I\right)\Omega \qquad 16$$

At Triple-synchrony where ω=Ω at a_{G1} and at a_{G2} we get the following relations:

$$J_T = C\omega + \left(m^* a_{present}^2 + I\right)\Omega \; = \left[C + \left(m^* a_{present}^2 + I\right)\right]\Omega = \left[1 + (\theta_2' \times a_{G1}^2 + \theta_1)\right]\frac{CB}{a_{G1}^{3/2}} \qquad 17$$

$$In\ Eq.17, \quad \theta_1 = \frac{I}{C}\ and\ \theta_2' = \frac{m^*}{C}\ ;$$

Solution of (17) gives the two Triple Synchrony Orbits defined as Clarke's Orbits or co-rotational orbits:

$$Inner\ Clarke's\ Orbit = \ a_{G1}\ and\ Outer\ Clarke's\ Orbit = a_{G2}$$

Rewriting (16) we get:

$$\frac{J_T}{C\Omega} = \left[\frac{\omega}{\Omega} + \left(\left(\frac{m^*}{C}\right) a_{present}^2 + \frac{I}{C}\right)\right] = \left[\frac{\omega}{\Omega} + \ \theta_2' a_{present}^2 + \theta_1\right] \qquad 18$$

Substituting Kepler's third law in (18) we get :

$$\left(\frac{J_T}{CB}\right) a^{\frac{3}{2}} = \left[\frac{\omega}{\Omega} + \ \theta_2' a_{present}^2 + \theta_1\right] \qquad 19$$

Rearranging the terms of (19) we get:

$$\frac{\omega}{\Omega} = \left(\frac{J_T}{CB}\right) a^{\frac{3}{2}} - (\theta_2' a^2 + \theta_1) = E a^{\frac{3}{2}} - F a^2 \qquad 20$$

$$where\ E = \frac{J_T}{BC}\ and\ F = \left(\theta_2' + \frac{\theta_1}{a^2}\right)$$

At a_{G2},

$$\omega = \Omega = \frac{B}{a_{G2}^{3/2}} \qquad 21$$

In (21) we could as well have taken a_{G1} in place of a_{G2} .
Substituting (21) in (20) we get:

$$\frac{\omega}{\Omega} = 1 = \left(\frac{J_T}{CB}\right) a_{G2}^{3/2} - (\theta_2' a_{G2}^2 + \theta_1) \qquad 22$$

Rearranging (22) we get:

$$E = \frac{J_T}{CB} = [1 + (\theta_2' a_{G2}^2 + \theta_1)] \frac{1}{a_{G2}^{\frac{3}{2}}} = [1 + (\theta_2 + \theta_1)] \frac{1}{a_{G2}^{\frac{3}{2}}}\ where\ \theta_2' a_{G2}^2 = \ \theta_2 \qquad 23$$

Therefore:

$$E = [1 + (\theta_2 + \theta_1)] \frac{1}{a_{G2}^{\frac{3}{2}}} = \frac{k_1}{a_{G2}^{\frac{3}{2}}}\ \ where\ k_1 = \ [1 + (\theta_2 + \theta_1)] \qquad 24$$

Now (20) can be rewritten as:

$$\frac{\omega}{\Omega} = \left(\frac{J_T}{CB}\right) a^{\frac{3}{2}} - \left(\theta_2' a_{present}^2 + \theta_1\right) = E a^{\frac{3}{2}} - F a^2 = \ k_1 x^{3/2} - \theta_2 x^2 - \theta_1\ \ where\ x = \frac{a}{a_{G2}};$$

Therefore:

$$\frac{\omega}{\Omega} = \ k_1 x^{3/2} - \theta_2 x^2 - \theta_1 \qquad 25$$

Now Total Energy (TE) = Kinetic Energy (KE)+Potential Energy(PE)

$$KE = rotational\ KE\ of\ the\ Primary + rotational\ KE\ of\ the\ Secondary + Orbital\ KE$$
$$= \frac{1}{2}C\omega^2 + \frac{1}{2}I\Omega^2 + \frac{1}{2}m^* a^2\Omega^2 = \frac{C\Omega^2}{2}[(\frac{\omega}{\Omega})^2 + \theta_1 + \theta_2' a^2] \qquad 26$$

Substituting (16) and (25) in (26) we get:

$$KE = \frac{CB^2}{2a^3}\left[(k_1 x^{\frac{3}{2}} - \theta_2 x^2 - \theta_1)^2 + \theta_1 + \theta_2' a^2\right] \qquad 27$$

Normalizing (27) with respect to a_{G2} we get:

$$KE = \frac{CB^2}{2{a_{G2}}^3} \times \frac{1}{x^3}\left[(k_1 x^{\frac{3}{2}} - \theta_2 x^2 - \theta_1)^2 + \theta_1 + \theta_2 x^2\right] \qquad 28$$

Let:

$$K = \frac{CB^2}{2{a_{G2}}^3} \quad and\ K_1 = \frac{GMm}{a_{G2}} \qquad 29$$

Substituting (29) and (28) in (26) we get:

$$TE = \frac{K}{x^3}\left[(k_1 x^{\frac{3}{2}} - \theta_2 x^2 - \theta_1)^2 + \theta_1 + \theta_2 x^2\right] - \frac{K_1}{x}\ where\ x\ normalized\ orbital\ radius \qquad 30$$

Differentiation and solving the Derivative = 0 will give the maxima Energy and minima Energy points.
The kinematic parameters (Sharma 2011) of Phobos and Deimos is given in Table 3.

Table 3: Kinematic parameters needed for Stability Analysis

parameters	Phobos	Deimos
J_T(total ang. mom.))($\times 10^{32}$Kg- m^2/s)	1.912715482	1.9127140479
C(moment of inertia of Mars) ($\times 10^{36}$Kg-m^2)	2.69843	2.69843
B($\sqrt{(G(M+m))}$) ($\times 10^6 m^{3/2}$/s)	6.54248	6.54248
Θ_1(Dimensionless)($\times 10^{-14}$)	19.9047	19.9047
Θ_2(Dimensionless)($\times 10^{-7}$)	16.5475	3.47639
k_1(Dimensionless)	1.0000016547487678	1.0000003476390125
K=(CB^2)($2a_{G2}^3$) ($\times 10^{27}$Joules)	6.77885	6.77885
K_1=(GMm)/a_{G2}($\times 10^{21}$Joules)	22.4346	4.71319
a_{G1}($\times 10^7$m)	2.04238	2.04238
a_{G2}($\times 10^{18}$ m)	7.4589	168.997
E($\times 10^{-11} m^{-3/2}$)	1.0834199115213353	1.0834190992037116
F($\times 10^{-22} m^{-2}$)	39.6697	8.33403

Setting up Equation (30) we get the Total Energy of the Binary-System as a function of x where x is the normalized orbital radius and normalization is with respect to a_{G1} in case of Phobos and Deimos because inner Clarke's Orbits are perceptible and outer Clarke's Orbit are inordinately large.

Total Energy Function Eq.(30) is differentiated with respect to 'x' and equated to Zero. This gives the extremum points. We obtain 3 extremum points as tabulated in Table 4.

Table 4: The Energy Extremum Points of Mars-Phobos and Mars-Deimos.

	Mars-Phobos	Mars -Deimos	Nature of Extremum	Comment
1^{st} Extremum	0.00060072	0.000327178	Minima	Towards the center
2^{nd} Extremum	1	1	Maxima	a_{G1}
3^{rd} Extremum	3.65205×10^{11}	8.27453×10^{12}	Minima	a_{G2}

3. Energy Profile around these three extremum points for Phobos.

For Mars-Phobos between 0.0005 and 0.00065:

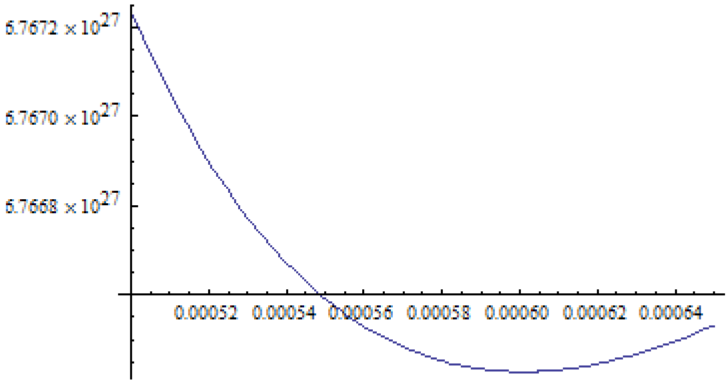

Figure 6: Energy Profile of Mars-Phobos between x = 0.0005 to 0.00065.

By inspection of Figure 6 we see that the first Energy minima occurs at x = 0.00060072 this corresponds to 12.269Km from the center of Mars. Hence it is a stable point. An orbit at 12.269Km is physically untenable because it will fall within body of Mars.

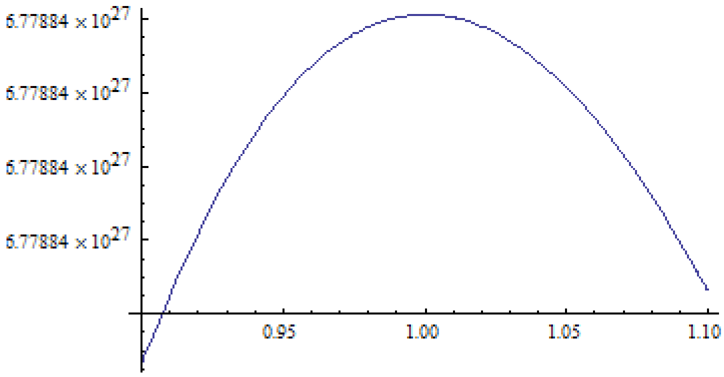

Figure 7: Energy Profile of Mars-Phobos between x = 0.9 to 1.1.

By inspection of Figure 7 we see that the Energy Maxima occurs at x = 1 which corresponds to a_{G1}= 2.04238×10^{7}m= 20,423Km. Hence inner Clarke's Orbit is an unstable point.

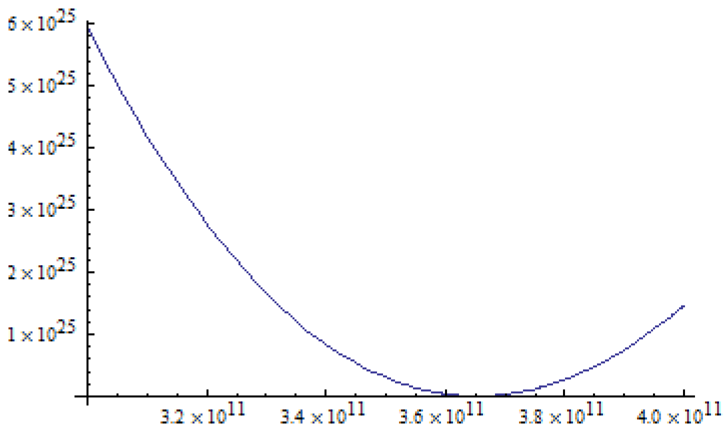

Figure 8: Energy Profile of Mars-Phobos between $x = 3\times10^{11}$ to 4×10^{11}.

By inspection of Figure 8, we see that Energy Minima occurs at $x = 3.65205\times10^{11}$ which corresponds to $a_{G2} = 7.4589\times10^{18}$m. Hence Outer Clarke's Orbit is a stable point.
Similar profiles are obtained for Mars-Deimos.

4. Discussion

This Energy Profile study clearly establishes that the secondary tumbles out of the Inner Clarke's Orbit at the slightest perturbation. If the secondary tumbles short of $a_{G1}= 2.04238\times10^{7}$m, it gets trapped in a death spiral and if it tumbles long of $a_{G1}= 2.04238\times10^{7}$m, it is launched on an outward expanding spiral path by gravitational sling shot effect (Sharma 2011) until it gets tidally locked into the Outer Clarke's Orbit. The time-constant of evolution is a strong function of 'q'=mass ratio. If q is vanishingly small, the time constant of evolution is practically infinite and the secondary hardly evolves out of its orbit of inception as is the case with our geo-stationary satellites. But as q exceeds 10-4, time constant of evolution becomes perceptible. At solar system or exo-solar system mass scale time scale of tidal evolution is scaled down from Gy to My to Ky to Y until beyond q = 0.2 up to q =1 in months and days the secondary component settles into Outer Clarke's Orbit configuration where it tends to get tidally interlocked with the primary.
This study has invoked Kinematic Model to study Earth-Moon Mars-Phobos and Mars-Deimos and correctly derived the two geo-synchronous orbits in case of Earth-Moon and two Clarke's orbits in case of Mars-Phobos and Mars-Deimos. By total energy analysis it has correctly arrived at the conclusion that the inner geosynchronous orbit or inner Clarke's orbit are unstable Circular Orbits and the outer geosynchronous orbit or outer Clarke's orbit are stable Circular Orbits just as the case is in the study of Black Holes.

These Clarke's Orbits or Geo-synchronous orbits correspond to the energy extremum of the system is direct validation of the Kinematic Model of the tidally interacting binary and it is a corroboration of the results arrived at.

5. Conclusion

Kinematic Model is a valid and reliable model is established by the fact that the derivation of geo-synchronous orbits and Clarke's orbits do correspond to the extremum energy points of the total energy of the system as is the case in Circular Orbits study around black Holes. Using Kerr metric we obtain inner unstable Circular Orbit and outer stable Circular Orbit in relativistic systems. In exact correspondence using K.M. the circular orbits are derived in non-relativistic systems as has been done in this study.

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Conflict of Interest

I have no conflict of interest whatsoever with anybody.

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