Research Article

Engineering: Open Access

ISSN: 2993-8643

Kernel Estimation of Entropy for the Weibull Distribution Using Generalized Progressive Hybrid Censored Data

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Submitted: 2026, Apr 08; **Accepted**: 2026, June 01; **Published**: 2026, June 30

Citation: Maswadah, M. (2026). Kernel Estimation of Entropy for the Weibull Distribution Using Generalized Progressive Hybrid Censored Data. *Eng OA, 4*(6), 01-16.

Abstract
In recent years, numerical methods have been widely and effectively applied in estimating parameters of lifetime distributions. Therefore, the primary objective of this paper is to introduce a novel numerical estimation approach namely, the kernel estimation method for estimating both the entropy and the parameters of the Weibull distribution, and to compare its performance with the commonly used Bayes estimation method in statistical inference. Through an extensive Monte Carlo simulation study, the entropy and distribution parameters are estimated using both techniques. The simulation results demonstrate that the kernel method generally outperforms the Bayes method, particularly when using informative gamma and kernel priors. In addition, real data applications are presented to illustrate the proposed methods and to compare their practical performance.

Keywords: Bayesian Estimation, Informative Gamma Prior, Kernel Prior, Kernel Estimation Method

1. Introduction

In statistical inference and reliability theory, numerical analysis has been widely employed to estimate distribution parameters through iterative techniques such as the Runge–Kutta, Picard, Adams, and integrable methods, see Maswadah [1-9]. The present study introduces a novel numerical approach based on kernel estimation for estimating the parameters and entropy of the Weibull distribution. This method is compared with the Bayesian approach utilizing informative gamma and kernel priors.

Entropy serves as a fundamental measure of uncertainty associated with a random variable. In information theory, it quantifies the expected information content of that variable. As such, entropy plays a crucial role in various fields including statistics, physics, chemistry, economics, insurance, financial analysis, and biology, where lower information in a sample corresponds to higher entropy and vice versa. One of the most widely adopted entropy measures was introduced by Shannon and is known as Shannon entropy [10]. This measure has proven effective across numerous applications. However, a notable limitation of Shannon's original formulation is that it can yield negative values for certain probability distributions, thereby undermining its interpretation as a measure of uncertainty [11]. To address this, refined the concept by defining differential entropy P(X) for a continuous random variable X with cumulative distribution function (cdf) F(x) and probability density function (pdf) f(x) as follows:

$$P(X) = P(f) = -\int_{-\infty}^{\infty} \log f(x) f(x) dx. \tag{1}$$

It is observed that a sharply peaked distribution corresponds to very low entropy, whereas a more spread out probability distribution yields substantially higher entropy. In this sense, entropy serves as a measure of uncertainty associated with a probability distribution

P(X). Moreover, as the entropy P(f) increases, the density function f(x) tends toward uniformity. Thus, entropy can be interpreted as a measure of the uniformity of a distribution, see [12]. Numerous authors have investigated the estimation of entropy for various lifetime distributions [13]. Derived the entropy of upper record values and established several upper and lower bounds using the hazard rate function [14]. Examined entropy estimation for the Weibull distribution under progressive censoring [15]. Proposed an efficient computational method for entropy based on progressively Type-II censored samples [16]. Derived estimators for the entropy function of a Rayleigh distribution under doubly generalized Type-II hybrid censoring, utilizing maximum likelihood estimators (MLEs), approximate MLEs, and Bayesian approaches [17]. Developed entropy estimators for the Weibull distribution under a generalized progressive hybrid censoring scheme (GPHCS) [18]. Provided entropy estimators for the generalized exponential distribution based on record values [19]. Derived entropy estimators for the double exponential distribution using multiply Type-II censored samples [20]. Obtained entropy estimators under hybrid censoring schemes.

To illustrate the proposed methodologies, we apply them to the Weibull distribution—one of the most widely used models in lifetime and reliability analysis. Its flexibility in modeling both constant and non-constant hazard rates makes it particularly valuable for analyzing lifetime data in medical, biological, and engineering contexts. Additionally, the Weibull distribution includes two well-known special cases: the exponential and Rayleigh distributions. Its parameters have been extensively studied under various prior assumptions in the literature.

Several researchers have employed informative priors for the parameters of the Weibull model [21]. Derived an informative conjugate prior by assuming that each parameter follows a gamma distribution [22,23]. Proposed priors based on prior knowledge of reliability levels or hazard rates at specific times, transforming such information into priors on the model parameters [24]. Obtained parameter estimates using both classical and Bayesian approaches [25]. Conducted reliability and quantile analyses for the Weibull distribution [26]. Applied three methods to estimate Weibull parameters derived MLEs for the Weibull parameters under progressively Type-II censored samples [27,28]. Studied parameter estimation based on progressively censored samples, as derived empirical Bayes estimates for the Weibull parameters, and later obtained parameter estimates using numerical methods such as Picard and Adams techniques under the GPHCS [9,29].

As a continuation of these efforts, the present paper aims to derive point estimates of the parameters based on generalized progressive hybrid censored samples using the kernel method, and to compare these estimates with those obtained via Bayesian methods, assuming the underlying distribution is the two-parameter Weibull distribution. The probability density function (pdf) and cumulative distribution function (cdf) of the Weibull distribution are given, respectively, by:

$$\mathbf{f(x) = \alpha\beta x^{\alpha-1}\exp(-\beta x^{\alpha})}, \qquad \mathbf{x > 0}. \tag{2}$$

$$\mathbf{F(x) = 1 - \exp(-\beta x^{\alpha})}, \qquad \mathbf{x > 0}. \tag{3}$$

$\boldsymbol{\alpha}, \boldsymbol{\beta} > \mathbf{0}$ are the shape and scale parameters, respectively.
For the pdf (2), the entropy P can be derived in simple form as follows:

$$\mathbf{P(\alpha,\beta) = (\gamma + \log(\beta))\left(1 - \frac{1}{\alpha}\right) - \log(\alpha\beta) + 1}$$

where $\boldsymbol{\gamma} = \mathbf{0.5772}$ is the Euler–Mascheroni constant, see Appendix A.

There are many cases in life testing experiments in which units are lost or removed from the test before failure. Therefore, progressive censoring is one of the familiar schemes in both industrial life testing applications and clinical trials that allows the removal of surviving experimental units before the termination of the test, despite the time of the experiment can be very long if the units are highly reliable [17]. Therefore, recently proposed a censoring scheme called the Type-II progressively hybrid censoring scheme, with the disadvantage that very few failures may occur, before the time point T. To provide a guarantee of the number of failures observed as well as the time to complete the test, see proposed the generalized progressive hybrid-censoring scheme, which modifies the progressive hybrid censoring scheme [17,30]. It allows the experiment to continue beyond time T to observe at least k failures if the number of failures are less than m. Thus, we have three cases for terminating the test: The first one if the specified number of failures m are less than T. The second case if T is less than m, where D is the number of observed units at the time T . The third case if T is less than k. The GPHCS algorithm can be seen in [2,3].

Thus, given a generalized progressive hybrid censored scheme, the likelihood function for the three different cases can be written in a unified form as follows:

$$L(\underline{x};\theta) = C\prod_{i=1}^{n} f(x_{i,m,n})\,[1-F(x_{i,m,n})]^{R_i}[1-F(T)]^{\delta R_T^*}. \tag{4}$$

$$n = \begin{cases} m, & \delta = 0, \quad \text{if} \quad X_{k:m:n} \le X_{m:m:n} < T \\ k, & \delta = 0, \quad \text{if} \quad T < X_{K:m:n} \le X_{m:m:n}, \\ D, & \delta = 1, \quad \text{if} \quad X_{k:m:n} < T < X_{m:m:n} \end{cases}$$

where R_T* is the number of surviving units that are removed at the stopping time

$T^* = \max\ \{X_{k:m:n}, \min\{X_{m:m:n}\},T\}\}$.

The GPHCS has been applied for some distributions such as the Weibull distribution, see and inverse Weibull distribution, see. Exponential distribution, see, Rayleigh distribution, see, shape-scale family, see [25], and the generalized Weibull distribution, see [26]. Some other distributions, see [1-9].

2. Estimation Methods

2.1 Kernel Method

We propose a simple and tractable algorithm for estimating the distribution parameters based on the kernel density estimate as the following:

The kernel estimates for the function $g(\alpha, \beta, P)$ can be derived by using the trivariate kernel density estimator for the unknown probability density function with support on [0,∞), which is defined as follows:

$$\hat{g}(\alpha,\beta,P) = \frac{1}{nh_1h_2h_3}\sum_{i=1}^{n} K\left(\frac{\alpha-\hat{\alpha}_i}{h_1},\frac{\beta-\hat{\beta}_i}{h_2},\frac{P-\hat{P}_i}{h_3}\right), \tag{5}$$

h_i, i = 1,2,3 are called the bandwidths or smoothing parameters, which are chosen such that $h_i \to 0$ and $nh_i \to \infty$ as $n \to \infty$, where n is the sample size. The influence of the smoothing parameter h is critical because it determines the amount of smoothing. However, the optimal choice for h_i which minimizes the mean squared errors is $h_i = 1.06\, S_i\, n^{-0.2}$, S_i the sample standard deviations. The optimal choice for the kernel function $K(.,.)$ can be used as the trivariate standard normal distribution for the parameters α , β and P.

1. Generate a random sample $X = (x_1, x_2,..., x_n)$ from the Weibull distribution.
2. Bootstrapping with replacement n samples from the random sample in (1) as follows:
$X_1 = (X_{11},..., X_{1n}), X_2 = (X_{21},..., X_{2n}), ..., X_n = (X_{n1},..., X_{nn})$.
3. For each sample in step 2, find the maximum likelihood estimator (MLE) for the parameters α, β, and P. Thus, we get the following random variables:
$A = (\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$, $B = (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n)$., Thus, we can derive the MLE for the entropy P as follows: $\hat{P} = (\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n)$.
Integrate the density function (5) with respect to θ from θ_0 to θ_1 for $\theta = (\alpha, \beta, P)$ as follows:

$$\int_{\theta_0}^{\theta_1} g(\theta)d\theta = \frac{1}{nh}\sum_{i=1}^{n}\int_{\theta_0}^{\theta_1} K\left(\frac{\theta-\theta_i}{h}\right)d\theta$$

Thus,

$$G(\theta_1) - G(\theta_0) = \frac{1}{nh}\sum_{i=1}^{n}\int_{\theta_0}^{\theta_1} K\left(\frac{\theta-\theta_i}{h}\right)d\theta$$

$$G(\theta_1) - G(\theta_0) = [\gamma - \frac{1}{nh}\sum_{i=1}^{n}\int_{0}^{\theta_0} K\left(\frac{\theta-\theta_i}{h}\right)d\theta] = [n\gamma - \sum_{i=1}^{n}\int_{\frac{-\theta_i}{h}}^{\frac{\theta_0-\theta_i}{h}} K(y)\,dy]/n$$

where $\gamma \in U(0,1)$.

Let $W(\frac{-\theta_i}{h}, \frac{\theta_0-\theta_i}{h}) = \int_{\frac{-\theta_i}{h}}^{\frac{\theta_0-\theta_i}{h}} K(y)\, dy$.

Thus,

$$G(\theta_1) - G(\theta_0) = \left[n\gamma - \sum_{i=1}^{n} W\left(\frac{-\theta_i}{h}, \frac{\theta_0-\theta_i}{h}\right)\right]/n. \tag{6}$$

It is known that the second order approximation of the first derivative, which is defined by the centered differencing, can be written as follows:

$$\frac{dG(\theta_m)}{d\theta} = \frac{G(\theta_1)-G(\theta_0)}{\theta_1-\theta_0} = g(\theta_m), \qquad \text{for} \quad \theta_0 < \theta_m < \theta_1. \tag{7}$$

From (6) and (7) we get the integral equation

$$\tilde{\theta}_1 = \tilde{\theta}_0 + C\left[n\gamma - \sum_{i=1}^{n} W\left(\frac{-\theta_i}{h}, \frac{\theta_0-\theta_i}{h}\right)\right],$$

Thus, the iterative process for the kernel method can be derived as follows:

$$\tilde{\theta}_{n+1} = \tilde{\theta}_n + C\left[n\gamma - \sum_{i=1}^{n} W\left(\frac{-\theta_i}{h}, \frac{\theta_n-\theta_i}{h}\right)\right], \text{ for n =1,2,3,...} \tag{8}$$

where $0 < C \le \frac{2h}{nL_1}$, and $L_1 = K(0)$.

The convergence of (8) is continued until two consecutive numerical solutions are almost the same, that is if $\left|\hat{\theta}_{n+1} - \hat{\theta}_n\right| < 10^{-5}$.

2.2 Bayes Method

In this section, the Bayes estimations will be derived based on the squared error loss function (SLF), $L\left(g(\theta), \hat{g}(\theta)\right) = (g(\theta) - \hat{g}(\theta))^2$, which is classified as a symmetric loss function and that penalizes overestimation and underestimation equally on (-∞,∞). For this loss function the Bayes estimator that minimizes the risk function is given by: $\hat{g}(\theta) = E\left(g(\theta)|\underline{X}\right)$.

2.2.1 Informative Gamma Prior

We consider the unknown parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have independent gamma prior distributions with joint probability density function defined as the following:

$$g(\alpha, \beta) \propto \alpha^{a-1}\beta^{c-1}e^{-b\alpha-d\beta}. \tag{9}$$

The hyper-parameter *a,b,c* and *d* are assumed to be known and positives to reflect the prior belief about the unknown parameters.

2.2.2 Informative Kernel Prior

For deriving the kernel prior, we introduce the bivariate kernel density estimator for the unknown probability density function g(α,γ) with support on (0,∞), which is defined as follows:

$$\hat{g}(\alpha,\beta) = \frac{1}{nh_1h_2}\sum_{i=1}^{n} K\left(\frac{\alpha-\hat{\alpha}_i}{h_1},\frac{\beta-\hat{\beta}_i}{h_2}\right), \tag{10}$$

h_i, i=1,2 are called the bandwidths or smoothing parameters, which are chosen such that $h_i \to 0$ and $nh_i \to \infty$ as $n \to \infty$, where n is the sample size. The influence of the smoothing parameter h is critical because it determines the amount of smoothing. However, the optimal choice for h_i, which minimizes the mean squared errors is given by $h_i = 1.06 S_i\, n^{-0.2}$, where S_i is the sample standard deviation. The optimal choice for the kernel function $K(.,.)$ can be used as the bivariate standard normal distribution for the parameters $\boldsymbol{\alpha}$ and β. Based on the properties of the MLEs of the parameters, which are converging in probability with the original parameters, the kernel prior estimate can be derived. It is worthwhile to mention that this kernel prior has been used for some distributions, see Ahsanullah et al., and Maswadah [31,32]. Thus, the priors (9) and (10) can be written as the general prior density as follows:

$$Q(\alpha,\beta)=h(\alpha,\beta)g(\alpha,\beta)$$

$$= \alpha^{a-1}\beta^{c-1}\hat{g}_1^{p_1}(\alpha)\,\hat{g}_2^{p_2}(\beta)\, exp(-b\alpha - d\beta). \tag{11}$$

with the following cases:
i) For the informative prior (9): $p_1 = p_2 = 0$.
ii) For the kernel prior (10): $p_1 = p_2 = 1$, and a=c=1 , b=d=0.
Thus, using the general prior (11) with the likelihood function of the GPHCS (4) the posterior density for the unknown parameters α and β can be written in a unified form as follows:

$$f(\alpha,\beta|\underline{X}) = K\hat{g}^{p_1}(\alpha)\hat{g}^{p_2}(\beta)\alpha^{n+a-1}\beta^{n+c-1}\, exp[-\alpha b + (\alpha-1)\sum_{i=1}^{n} ln(x_i)]$$

$$\times\, exp[-\beta(d + \textstyle\sum_{i=1}^{n}(R_i+1)x_i^{\alpha} + \delta R_T^{*} T^{\alpha}). \tag{12}$$

Based on (12) we can use the Tierney and Kadane approximation method to approximate all the Bayes estimators for the unknown parameters. Tierney and Kadane introduced an easily computable approximation for the posterior mean of a non-negative parameter or more generally, of a smooth function of the parameter that is non-zero on the interior of the parameter space [33]. For detail, let q(α,β) be a smooth, positive function in the parameter space. The posterior expectation of q(α,γ) can be obtained as:

$$q^{*} = E(q(\alpha,\gamma)|x) = \frac{\int_0^{\infty}\int_0^{\infty} e^{nH^{*}(\alpha,\gamma)}d\alpha d\gamma}{\int_0^{\infty}\int_0^{\infty} e^{nH(\alpha,\gamma)}d\alpha d\gamma}, \tag{13}$$

where $H(\alpha,\gamma) = \ln f(\alpha,\beta|x)/n$, and $H^{*}(\alpha,\beta)=H(\alpha,\beta)+\ln q(\alpha,\beta)/n$.
For (α,β) the Bayes estimator using Tierney and Kadane approximation for q(α,β) cd as follows:

$$q^{*} = \sqrt{|\Sigma^{*}|/|\Sigma|}\exp[n[H^{*}(\alpha,\beta) - H(\alpha,\beta)],$$

where $(\hat{\alpha},\hat{\beta})$ and $(\hat{\alpha}^{*},\hat{\beta}^{*})$ maximize the $H(\hat{\alpha},\hat{\beta})$ and $H^{*}(\hat{\alpha}^{*},\hat{\beta}^{*})$, respectively.

$$|\Sigma| = \begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix}^{-1} \text{ and } \quad |\Sigma*| = \begin{vmatrix} H_{11}^{*} & H_{12}^{*} \\ H_{21}^{*} & H_{22}^{*} \end{vmatrix}^{-1}$$

denote the minus of inverse of Hessians of $H(\alpha,\beta)$ and $H^{*}(\alpha,\beta)$ at $(\hat{\alpha},\hat{\beta})$ and $(\hat{\alpha}^{*},\hat{\beta}^{*})$ respectively. The derivatives of $H(\alpha,\beta)$ and $H^{*}(\alpha,\beta)$ have been derived in Appendix B.

3. Simulation Study

In this section, we evaluate the performance of kernel and Bayesian estimation methods through an extensive simulation study based on two criteria: the average bias (AVB) and the mean squared error (MSE), defined as follows:

$$\mathrm{AVB} = \frac{1}{L}\sum_{i=1}^{L} |\hat{\theta}_i - \theta|,\ \ \mathrm{MSE} = \frac{1}{L}\sum_{i=1}^{L}\left(\hat{\theta}_i - \theta\right)^2$$

$\hat{\theta}$ is the estimate of θ and L is the number of replications.

In our simulation study, we choose values for the hyperparameters of α and β say: (a,b,c,d)=(5,3,5,3) and two values for the parameter α=(1,2), and two values for the parameter β=(2,3) respectively.

Using the above parameter values for generating different samples from the Weibull distribution with sizes n = 20, 40 and 60 to represent small, moderate and large sizes. To assess the performance of the parameter and the entropy estimates using the kernel and Bayes methods based on the informative gamma and kernel priors. The AVB and MSE for each one were calculated using 1000 replications.

The simulation results are presented in Tables 3, 4, and 5. The main findings are summarized as follows:
From the simulation results in Tables, some of the points are quite clear based on these estimates and the others have been summarized in the following main points [13,25,34]:
1. It is clear that, in general, the estimated AVB and MSE values for the parameters and the entropy based on the kernel method has the smallest estimated values compared to the counterparts based on the Bayes' method.
2. In general, the parameters and the entropy have the estimated MSE values based on the kernel prior are often less than the counterparts estimates based on the informative gamma prior.
3. The estimated AVB and MSE values increase as the value of α increase and decrease as the value of β increase.
4. The estimated AVB and MSE values for the parameters decrease as the sample sizes and the termination time of the experiment T increase as expected.
5. Overall, the kernel method demonstrates competitive performance and often outperforms the Bayesian method under both informative and kernel priors. Furthermore, the entropy estimates based on the kernel method are consistently lower than those obtained from the Bayesian approach, suggesting that the information contained in the samples increases effectively with sample size.

In conclusion, the kernel method appears to be competitive with and even outperforms the Bayesian method based on informative and kernel priors. The entropy estimates obtained by the kernel method are lower than those obtained by the Bayesian method, indicating that the information content of the samples effectively increases as the sample size grows.

4 Real Data Applications

In this section, we analyze two real datasets to evaluate the performance of the proposed methods on the Weibull model, which is one of the most desirable and widely used lifetime distributions. This distribution has been applied in numerous fields, including emerging areas such as biomedical science and survival analysis, to model lifetimes characterized by specific mortality and failure rates. Therefore, we fit these datasets and assess the goodness of fit using the Kolmogorov–Smirnov (K-S), Anderson–Darling (A-D), and Chi-square (χ^2) tests, with a significance level of 0.05.

4.1 Ball Bearing Data

Consider the results of endurance tests on 25 deep-groove ball bearings. The original source of the data is given by, which reports the results of manufacturers' tests on 213 batches of ball bearings [35]. This dataset has been used in a series of papers as a complete sample of size 23; the misquotation may have started with, from which Lawless seems to have copied the data for use in subsequent works, such as [19] and his book [36,37]. The test results, in millions of revolutions before failure, are:
17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 67.80, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

These data have been used by several authors to fit a Weibull distribution, often treating the maximum likelihood estimates (MLEs) as the true parameter values after scaling the data by dividing each value by 100. The MLEs obtained are: shape parameter α = 2.1732, and scale parameter β = 1.5628, and an additional quantity P(X) = 0.3299.

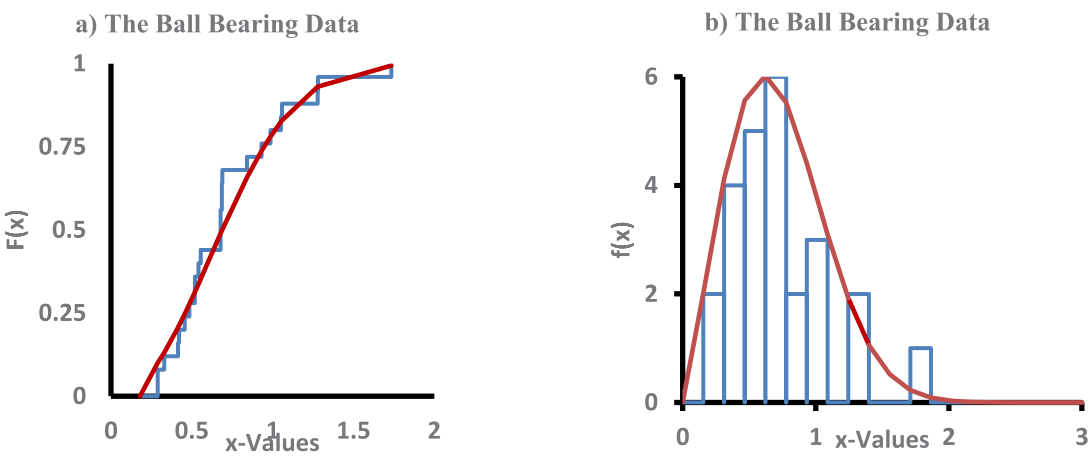

Figure 1: a) The Empirical CDF and the fitted CDF. b) The Histogram and the fitted PDF.

The Weibull model provides a good fit for this dataset, as shown in Table 1 and Figure (1a). In analyzing the failure behavior of the ball bearings based on this dataset, we obtain estimates for the scale parameter (α) and shape parameter (β), which characterize the failure distribution. These estimates are used to determine the mean time to failure.

The kernel and Bayes estimates for α are both approximately 2.0, suggesting that the failure distribution is moderately bell-shaped. This implies that the average number of failures remains relatively stable as the number of revolutions increases, as illustrated in Figure (1b). Similarly, the kernel and Bayes estimates for β are close to 1.5, further supporting the bell-shaped nature of the distribution and indicating that the failure rate stabilizes after a certain number of millions of revolutions.

Additionally, the estimated entropy values obtained from the kernel method are lower than those from the Bayes method, indicating that the sample contains a relatively high amount of information. This finding is consistent with the goodness-of-fit demonstrated in Figure (1a), further validating the appropriateness of the Weibull distribution for modeling this dataset.

4.2 Vinyl Chloride Data

Given that vinyl chloride is a known human carcinogen, exposure should be minimized to the greatest extent practicable, and concentrations should be maintained as low as technically feasible. It has been estimated that a concentration of 0.5 mg/L of vinyl chloride in drinking water is associated with an excess risk of liver and brain tumors for exposure beginning in adulthood, and this level would double the lifetime cancer risk for continuous exposure from birth.
In this context, we analyze a dataset originally presented in [5], consisting of 34 measurements (in mg/L) of vinyl chloride concentrations obtained from clean-up monitoring wells. The data are as follows:
5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

The Weibull distribution was found to provide a good fit to the dataset, as illustrated in Table 1 and Figure 2(a). To examine the concentration of vinyl chloride in groundwater from these wells, we estimated the parameters of the Weibull model, specifically the scale and shape parameters based on the dataset. The average concentration in the water was subsequently derived.

We noticed that the kernel and Bayes estimates for α are close to 1.0, indicating that the dataset is moderately right-skewed. This suggests that vinyl chloride concentrations tend to decrease over time, as shown in Figure (2b). Similarly, the kernel and Bayes estimates of the scale parameter β are close to 0.5, further confirming the right-skewed nature of the distribution and supporting the trend of decreasing concentration over time. For the entropy P(X), the kernel estimates are less than the Bayes estimates that indicate the information contained in the sample are very large to ensure this data is a good fit for the Weibull distribution as indicated in Figure (2a). Therefore, monitoring these wells is very significant.

The results in Table 1indicate that the Weibull model is a good fit for these datsets where the power of the tests is greater than the significance level of the tests, as shown in Figures (1a), and (2a). Also, the results in Table 2, indicate that the estimated values for the entropy and parameters based on the kernel method are smaller than those based on the Bayes' method for large values of T when considering the MLE values as the true values of the parameters. Thus, the results of the kernel method indicated that the information contained in the datasets gurantee the goodness of fit for the Weibull model, which ensure the simulation results.

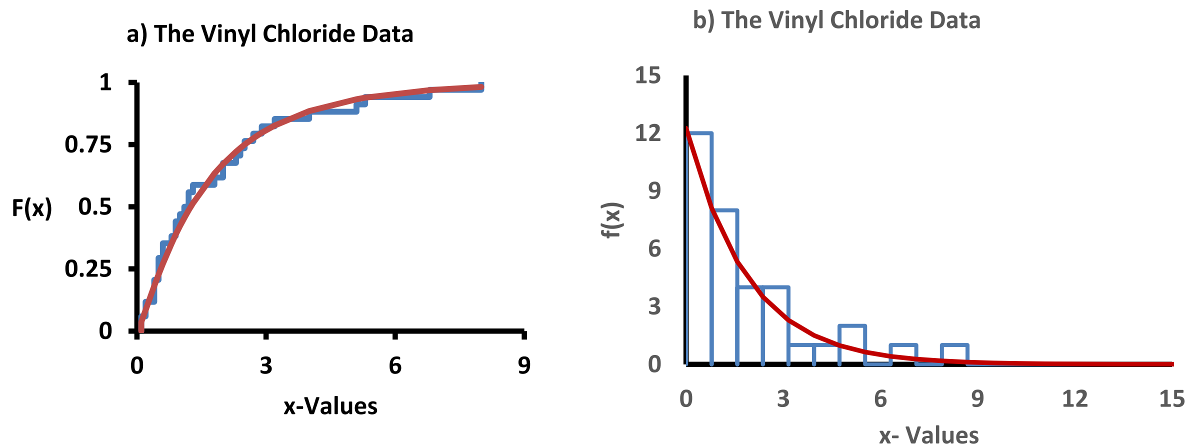

Figure 2: a) The Empirical CDF and the Fitted CDF. b) The Histogram and the Fitted PDF.

Data	The Tests	Critical value	Calculated value	The P-values	$\hat{\alpha}$	$\hat{\beta}$	$\hat{P}$
Ball Bearing Data. n=25	K-S	0.8546	0.8951	0.0485	2.1732	1.5628	0.3299
	A-D	0.7470	0.4029	0.4244			
	CH2	13.2377	2.0767	0.7044			
Vinyl Chloride Data. n=34	K-S	0.8624	0.5355	0.6525	1.0102	0.5263	1.6312
	A-D	0.7504	0.2826	0.6708			
	CH2	15.428	4.9912	0.4474			

Table 1: The Critical and Calculated Values for the K-S, A-D and CH2 Tests and their Powers (p- values).

Data	T	Par	Kernel Estamite		Bayes Estimate			
					Gamma Prior		Kernel prior	
			Estimate	MSE	Estimate	MSE	Estimate	MSE
Ball Bearing Data N=25	0.05	α	1.9581	0.0462	2.2933	0.0144	2.3581	0.0342
		β	1.3653	0.0393	1.7185	0.0242	1.6916	0.0166
		P	0.2947	0.0012	0.3233	0.0055	0.2961	0.0012
	1.5	α	1.9704	0.0411	2.3876	0.0459	2.3659	0.0372
		β	1.3725	0.0362	1.9528	0.1521	1.8291	0.0709
		P	0.3810	0.0026	0.2317	0.0097	0.2403	0.0080
The vinyl Chloride Data N=34	0.75	α	0.9131	0.0094	1.3731	0.1317	1.2793	0.0724
		β	0.4914	0.0012	0.3670	0.0254	0.2897	0.0559
		P	1.4987	0.0175	1.6462	0.0023	1.9582	0.1069
	3.5	α	0.9675	0.0018	1.0229	0.0016	1.0829	0.0053
		β	0.5003	0.0067	0.3049	0.0490	0.2491	0.0768
		P	1.5638	0.0045	2.1970	0.3202	2.3059	0.4554

Table 2: The Estimate and the mean Squared Errors (MSEs) for the Parameters α, β, and P based on the kernel and Bayesian Methods using Gamma and Kernel Priors for m=n/2, and k=m/2 with a=5, b=3, c=5, d=3.

5. Conclusion

The Bayesian estimation method, particularly when based on an informative prior, is generally more efficient than many traditional estimation techniques in statistical inference. In this study, we introduce a novel kernel-based estimation method for estimating both the entropy and the parameters of the Weibull distribution. We compare its performance with Bayesian estimates obtained under a generalized progressive hybrid censoring scheme. Our findings indicate that the proposed kernel estimators for the Weibull parameters and entropy outperform their Bayesian counterparts in terms of efficiency. Furthermore, the kernel entropy estimates are consistently lower than the Bayesian estimates, suggesting that the information contained in the samples is sufficient to support a good fit of the Weibull model to the data. This is illustrated in Figures 1a and 2a for the respective datasets [15,21].

Source of Funding

No funding was received for conducting this study.

Conflict of interest

The author has no conflicts of interest to declare that are relevant to the content of this article.

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Appendix A:

$$\ln f\,(x;\alpha,\beta) = \ln(\alpha\beta) + (\alpha-1)\ln x - \beta x^{\alpha}.$$

The entropy can be derived using (1) as follows:

$$\mathrm{P} = -\ln(\alpha\beta) - (\alpha-1)\mathrm{E}(\ln x) + \beta\mathrm{E}(x^{\alpha})$$

$$E(x^r) = \int_0^{\infty} x^r f(x)dx = \alpha\beta\int_0^{\infty} x^{r+\alpha-1}\exp(-\beta x^{\alpha})dx$$

Let $u = \beta x^{\alpha} \Rightarrow du = \alpha\beta x^{\alpha-1}dx, and \quad x = \beta^{-1/\alpha}u^{1/\alpha},$

Thus, $E(x^r) = \beta^{-\frac{r}{\alpha}}\int_0^{\infty} u^{\frac{r}{\alpha}}e^{-u}du = \beta^{-\frac{r}{\alpha}}\Gamma\left(\frac{r}{\alpha}+1\right).$

Let $r = \alpha$ we get $E[x^{\alpha}] = 1/\beta$.

Differentiating $E(x^r)$ with respect to r we get.

$$E(x^r \ln x) = -\frac{\beta^{-r/\alpha}}{\alpha}[\ln\beta\,\Gamma(r/\alpha+1) - \Gamma'(r/\alpha+1)]$$

Let $r = 0$ we get

$$E(\ln x) = -\frac{1}{\alpha}[\ln\beta\,\Gamma(1) - \Gamma'(1)] = \frac{\Psi(1)-\ln\beta}{\alpha}.$$

Thus, the entropy can be derived as follows:

$$\mathrm{P} = -\ln(\alpha\beta) - (\alpha-1)\frac{\Psi(1)-\ln\beta}{\alpha} + 1,$$

where $-\gamma = \Gamma'(1) = \Psi(1) = -0.5772$.

Appendix B:

$$f(\alpha,\beta|\underline{X}) = K\hat{g}^{p_1}(\alpha)\hat{g}^{p_2}(\beta)\alpha^{n+a-1}\beta^{n+c-1} \times \exp[-\alpha b + (\alpha-1)\sum_{i=1}^{n}\ln(x_i)]$$
$$\times \exp[-\beta(d + \textstyle\sum_{i=1}^{n}(R_i+1)x_i^{\alpha} + \delta R_T^{*}T^{\alpha}).$$

$$\mathrm{H}(\alpha,\beta) = \ln f(\alpha,\beta|\underline{X}) = [p_1\ln\hat{g}_1(\alpha) + p_2\hat{g}_2(\beta) + (n+a-1)\ln\alpha + (n+c-1)\ln\beta - \alpha b$$
$$+(\alpha-1)\textstyle\sum_{i=1}^{n}\ln x_i - \beta[d + \sum_{i=1}^{n}(R_i+1)x_i^{\alpha} + \delta R_T^{*}T^{\alpha}]]/n.$$

$$H_1 = \frac{\partial H}{\partial\alpha} = [p_1\frac{\hat{g}_1'}{\hat{g}_1(\alpha)} + (n+a-1)/\alpha - b + \sum_{i=1}^{n}\ln x_i - \beta[\sum_{i=1}^{n}(1+R_i)x_i^{\alpha}\ln x_i + \delta R_T^{*}T^{\alpha}\ln T]$$

$$H_{11}=\frac{\partial^2 H}{\partial\alpha^2}=[p_1\frac{\hat{g}_1(\alpha)\hat{g}_1^{//}(\alpha)-\hat{g}'^2_1(\alpha)}{\hat{g}_1^2(\alpha)}-\frac{n+a-1}{\alpha^2}$$

$$-\beta[\sum_{i=1}^{n}(1+R_i)\,{x_i}^{\alpha}(ln\,x_i)^2++\delta R_T^* T^{\alpha}(ln\,T)^2]]/n$$

$$H_2=\frac{\partial H}{\partial\beta}=[p_2\frac{\hat{g}_2'(\beta)}{\hat{g}_2(\beta)}+\frac{n+c-1}{\beta}-[d+\sum_{i=1}^{n}(R_i+1)x_i^{\alpha}+\delta R_T^* T^{\alpha}]]/n$$

$$H_{22}=\frac{\partial^2 H}{\partial\beta^2}=[p_2\frac{\hat{g}_2(\beta)g_2^{//}(\beta)-\hat{g}_2'^{\,2}(\beta)}{{\hat{g}_2}^2(\beta)}-(n+c-1)/\beta^2]/n.$$

$$H_{12}=\frac{\partial^2 H}{\partial\alpha\partial\beta}=[-\sum_{i=1}^{n}(1+R_i){x_i}^{\alpha}\,ln\,x_i-\delta R_T^* T^{\alpha}\,ln\,T]/n$$

The r^{th} derivative of the kernel density estimation can be defined as follows:

$$\frac{d^r\hat{g}(\alpha)}{d\alpha^r}=\hat{g}^r(\alpha)=\frac{1}{nh^{r+1}}\sum_{i=1}^{n}K^r\left(\frac{\alpha-\hat{\alpha}_i}{h}\right),\ (14)$$

where r = 0,1,2,3,-----.
Using the Gaussian kernel and (14), we have

$$\hat{g}_1(\alpha)=\frac{1}{n\sqrt{2\pi}}\sum_{i=1}^{n}e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2},$$

$$\hat{g}_1'(\alpha)=-\frac{1}{nh_1^2\sqrt{2\pi}}\sum_{i=1}^{n}\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right)e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{h_1})^2},$$

$$\hat{g}_1''(\alpha)=\frac{1}{nh_1^3\sqrt{2\pi}}\sum_{i=1}^{n}[(\frac{\alpha-\alpha_i}{h_1})^2-1]\,e^{-0.5(\frac{\alpha-\alpha_i}{h_1})^2}.$$

Similarly, for the kernel priors $\hat{g}_2(\beta)$.
we can define

$$H^*=H^*(\alpha,\beta)=H+ln\,h\,(\alpha,\beta)/n,$$

with $h(\alpha,\beta)=\alpha$.

Thus, $H^* = H + ln\,\alpha\,/\,n$,

$$\frac{\partial H^*}{\partial\alpha}=\frac{\partial H}{\partial\alpha}+1/n\alpha,\quad \text{and}\quad \frac{\partial^2 H^*}{\partial\alpha^2}=\frac{\partial^2 H}{\partial\alpha^2}-1/\alpha^2 n$$

$$\frac{\partial H^*}{\partial\beta}=\frac{\partial H}{\partial\beta},\qquad \frac{\partial^2 H^*}{\partial\gamma^2}=\frac{\partial^2 H}{\partial\gamma^2}.$$ Similarly,

let $H^*=H^*(\alpha,\beta)=H+ln\,h\,(\alpha,\beta)/n$

with $h(\alpha, \beta) = \beta$.

$$\frac{\partial H^*}{\partial \alpha} = \frac{\partial H}{\partial \alpha}, \quad \frac{\partial^2 H^*}{\partial \alpha^2} = \frac{\partial^2 H}{\partial \alpha^2},$$

and

$$\frac{\partial H^*}{\partial \beta} = \frac{\partial H}{\partial \beta} + \frac{1}{n\beta}, \quad \frac{\partial^2 H^*}{\partial \beta^2} = \frac{\partial^2 H}{\partial \beta^2} - \frac{1}{n\beta^2}.$$

$$\boldsymbol{H^* = H^*(\alpha, \beta) = H + \ln h(\alpha, \beta)/n,}$$

where $h(\alpha, \beta)$ is the Entropy P.

$$\boldsymbol{P = P(\alpha, \beta) = \gamma\left(1 - \frac{1}{\alpha}\right) - \log(\alpha) - \log(\beta)/\alpha + 1}$$

$$\boldsymbol{H^* = H^*(\alpha, \beta) = H + \ln P/n,}$$

$$\boldsymbol{\frac{\partial H^*}{\partial \alpha} = \frac{\partial H}{\partial \alpha} + \left[\frac{\frac{\gamma}{\alpha^2} - \frac{1}{\alpha} + \ln\beta/\alpha^2}{P}\right]/n \quad \text{and} \quad \frac{\partial H^*}{\partial \beta} = \frac{\partial H}{\partial \beta} - \left[\frac{1/\alpha\beta}{P}\right]/n\,.}$$

$$\boldsymbol{\frac{\partial^2 H^*}{\partial \alpha^2} = \frac{\partial^2 H}{\partial \alpha^2} + \left[\frac{P\left(-\frac{2\gamma}{\alpha^3} + 1/\alpha^2 - 2\ln\beta/\alpha^3\right) - \left(\frac{\gamma}{\alpha^2} - 1/\alpha + \ln\beta/\alpha^2\right)^2}{nP^2}\right]}$$

$$\boldsymbol{\frac{\partial^2 H^*}{\partial \beta^2} = \frac{\partial^2 H}{\partial \beta^2} + \left[\frac{P/\alpha\beta^2 - \left(\frac{1}{\alpha\beta}\right)^2}{P^2}\right]/n.}$$

$$\boldsymbol{\frac{\partial^2 H^*}{\partial \beta \partial \alpha} = \frac{\partial^2 H}{\partial \beta \partial \alpha} + \left[\frac{-P/\beta\alpha^2 - (1/\alpha\beta)\left(\frac{\gamma}{\alpha^2} - \frac{1}{\alpha} + \ln\beta/\alpha^2\right)}{P^2}\right]/n.}$$

n	m	k	α	β	T=0.75			T=1.5		
					Kernel estimations	Bayes Estimations		Kernel estimations	Bayes estimations	
						Gamma Prior	Kernel Prior		Gamma Prior	Kernel Prior
20	10	5	1	2	0.1036(0.0124)	0.1521(0.0235)	0.1470(0.0220)	0.0999(0.0114)	0.1478(0.0223)	0.1447(0.0213)
				3	0.1019(0.0120)	0.1577(0.0275)	0.1378(0.0193)	0.0984(0.0112)	0.1469(0.0220)	0.1376(0.0192)
			2	2	0.1692(0.0304)	0.3217(0.1038)	0.3253(0.1060)	0.1621(0.0278)	0.3171(0.1007)	0.3162(0.1001)
				3	0.1667(0.0297)	0.3182(0.1037)	0.3091(0.0957)	0.1623(0.0279)	0.3125(0.0978)	0.3062(0.0938)
		8	1	2	0.0989(0.0114)	0.1516(0.0233)	0.1476(0.0221)	0.0992(0.0114)	0.1492(0.0226)	0.1460(0.0216)
				3	0.1018(0.0121)	0.1572(0.0291)	0.1376(0.0193)	0.0995(0.0113)	0.1460(0.0217)	0.1367(0.0190)
			2	2	0.1705(0.0307)	0.3192(0.1020)	0.3203(0.1027)	0.1657(0.0290)	0.3175(0.1009)	0.3165(0.1002)
				3	0.1666(0.0293)	0.3147(0.0992)	0.3082(0.0951)	0.1609(0.0274)	0.3123(0.0976)	0.3061(0.0938)
	15	8	1	2	0.0979(0.0112)	0.1498(0.0229)	0.1460(0.0217)	0.1002(0.0114)	0.1442(0.0211)	0.1431(0.0207)
				3	0.0962(0.0108)	0.1488(0.0227)	0.1365(0.0189)	0.0966(0.0107)	0.1419(0.0204)	0.1373(0.0191)
			2	2	0.1656(0.0289)	0.3180(0.1012)	0.3179(0.1011)	0.1541(0.0252)	0.3140(0.0987)	0.3121(0.0974)
				3	0.1645(0.0286)	0.3135(0.0984)	0.3071(0.0944)	0.1535(0.0249)	0.3087(0.0954)	0.3039(0.0924)
		11	1	2	0.0951(0.0105)	0.1475(0.0221)	0.1452(0.0214)	0.0978(0.0109)	0.1444(0.0212)	0.1433(0.0208)
				3	0.0966(0.0108)	0.1433(0.0209)	0.1360(0.0188)	0.0988(0.0111)	0.1405(0.0200)	0.1360(0.0187)
			2	2	0.1566(0.0260)	0.3157(0.0998)	0.3143(0.0989)	0.1552(0.0255)	0.3134(0.0983)	0.3116(0.0972)
				3	0.1583(0.0265)	0.3113(0.0970)	0.3056(0.0935)	0.1584(0.0264)	0.3090(0.0955)	0.3043(0.0926)

40	20	10	1	2	0.0945(0.0102)	0.1388(0.0195)	0.1351(0.0185)	0.0960(0.0103)	0.1352(0.0185)	0.1340(0.0181)
				3	0.0940(0.0101)	0.1402(0.0203)	0.1287(0.0167)	0.0947(0.0101)	0.1323(0.0177)	0.1280(0.0165)
			2	2	0.1544(0.0252)	0.3064(0.0940)	0.3044(0.0927)	0.1493(0.0235)	0.3009(0.0906)	0.2994(0.0897)
				3	0.1560(0.0258)	0.3114(0.1057)	0.2963(0.0879)	0.1489(0.0233)	0.2980(0.0888)	0.2931(0.0859)
		15	1	2	0.0962(0.0104)	0.1367(0.0189)	0.1347(0.0183)	0.0961(0.0103)	0.1347(0.0184)	0.1335(0.0180)
				3	0.0948(0.0102)	0.1348(0.0184)	0.1286(0.0167)	0.0949(0.0102)	0.1326(0.0178)	0.1282(0.0166)
			2	2	0.1546(0.0251)	0.3040(0.0925)	0.3020(0.0912)	0.1507(0.0238)	0.3007(0.0904)	0.2992(0.0896)
				3	0.1540(0.0249)	0.3019(0.0912)	0.2946(0.0869)	0.1499(0.0236)	0.2982(0.0890)	0.2932(0.0860)
	30	15	1	2	0.0993(0.0110)	0.1370(0.0190)	0.1345(0.0183)	0.0929(0.0097)	0.1335(0.0180)	0.1327(0.0178)
				3	0.0972(0.0107)	0.1363(0.0188)	0.1287(0.0167)	0.0936(0.0098)	0.1313(0.0174)	0.1277(0.0164)
			2	2	0.1523(0.0245)	0.3050(0.0931)	0.3031(0.0920)	0.1448(0.0221)	0.3000(0.0901)	0.2987(0.0893)
				3	0.1522(0.0244)	0.3027(0.0918)	0.2951(0.0871)	0.1453(0.0223)	0.2973(0.0885)	0.2929(0.0858)
		23	1	2	0.0974(0.0105)	0.1337(0.0181)	0.1333(0.0179)	0.0969(0.0104)	0.1331(0.0179)	0.1331(0.0179)
				3	0.0989(0.0109)	0.1309(0.0173)	0.1283(0.0166)	0.0962(0.0103)	0.1304(0.0172)	0.1279(0.0165)
			2	2	0.1495(0.0234)	0.2987(0.0893)	0.2978(0.0887)	0.1478(0.0229)	0.2991(0.0895)	0.2982(0.0889)
				3	0.1481(0.0230)	0.2959(0.0876)	0.2925(0.0856)	0.1468(0.0227)	0.2956(0.0875)	0.2922(0.0854)
	30	15	1	2	0.0971(0.0105)	0.1320(0.0176)	0.1304(0.0171)	0.0977(0.0106)	0.1303(0.0171)	0.1297(0.0169)
				3	0.0964(0.0103)	0.1304(0.0172)	0.1252(0.0158)	0.0953(0.0100)	0.1280(0.0165)	0.1253(0.0158)
			2	2	0.1485(0.0231)	0.2977(0.0886)	0.2958(0.0875)	0.1459(0.0223)	0.2936(0.0863)	0.2929(0.0858)
				3	0.1480(0.0230)	0.2973(0.0885)	0.2898(0.0840)	0.1462(0.0223)	0.2923(0.0855)	0.2886(0.0833)
		23	1	2	0.0984(0.0107)	0.1319(0.0176)	0.1304(0.0171)	0.0987(0.0107)	0.1307(0.0172)	0.1301(0.0170)
				3	0.0960(0.0102)	0.1314(0.0174)	0.1258(0.0159)	0.0967(0.0103)	0.1286(0.0167)	0.1258(0.0159)
			2	2	0.1468(0.0227)	0.2971(0.0883)	0.2953(0.0873)	0.1450(0.0220)	0.2935(0.0862)	0.2927(0.0857)
				3	0.1500(0.0236)	0.2964(0.0879)	0.2895(0.0838)	0.1446(0.0219)	0.2922(0.0854)	0.2884(0.0832)
60			1	2	0.0973(0.0105)	0.1327(0.0178)	0.1310(0.0173)	0.0965(0.0103)	0.1295(0.0169)	0.1296(0.0169)
	45	23		3	0.0980(0.0107)	0.1313(0.0174)	0.1258(0.0159)	0.0952(0.0100)	0.1264(0.0161)	0.1251(0.0157)
			2	2	0.1488(0.0232)	0.2965(0.0879)	0.2948(0.0869)	0.1406(0.0207)	0.2919(0.0853)	0.2919(0.0852)
				3	0.1480(0.0230)	0.2963(0.0878)	0.2895(0.0838)	0.1422(0.0211)	0.2900(0.0841)	0.2879(0.0829)
		34	1	2	0.0956(0.0100)	0.1301(0.0170)	0.1300(0.0170)	0.0938(0.0097)	0.1301(0.0170)	0.1303(0.0171)
				3	0.0950(0.0099)	0.1263(0.0161)	0.1248(0.0157)	0.0922(0.0094)	0.1265(0.0161)	0.1254(0.0158)
			2	2	0.1405(0.0206)	0.2923(0.0855)	0.2921(0.0853)	0.1400(0.0205)	0.2916(0.0851)	0.2916(0.0851)
				3	0.1407(0.0207)	0.2901(0.0842)	0.2878(0.0828)	0.1393(0.0203)	0.2891(0.0836)	0.2874(0.0826)

Table 3: The Average Bias (AVB) and Mean Squared Errors (MSEs) in parentheses for the Weibull distribution parameter α using the kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4), et T=0.75 and T=1.5.

n	m	k	α	β	T=0.75			T=1.5		
					Kernel Estimations	Bayes estimations		Kernel estimations	Bayes estimations	
						Gamma Prior	Kernel Prior		Gamma Prior	Kernel Prior
20	10	5	1	2	0.1816(0.0347)	0.4647(0.2173)	0.4045(0.1637)	0.1724(0.0313)	0.4306(0.1856)	0.3834(0.1470)
				3	0.2284(0.0538)	0.8128(0.7067)	0.5646(0.3188)	0.2139(0.0472)	0.7147(0.5132)	0.5467(0.2989)
			2	2	0.1810(0.0346)	0.4049(0.1670)	0.4226(0.1790)	0.1716(0.0308)	0.3834(0.1474)	0.3783(0.1431)
				3	0.2274(0.0535)	0.6430(0.4368)	0.5685(0.3234)	0.2076(0.0446)	0.6025(0.3649)	0.5362(0.2876)
		8	1	2	0.1794(0.0338)	0.4461(0.2001)	0.3938(0.1553)	0.1745(0.0319)	0.4299(0.1850)	0.3835(0.1471)
				3	0.2314(0.0551)	0.8033(0.6800)	0.5644(0.3186)	0.2145(0.0476)	0.7135(0.5118)	0.5466(0.2987)
			2	2	0.1779(0.0332)	0.3921(0.1546)	0.3964(0.1572)	0.1723(0.0311)	0.3837(0.1476)	0.3786(0.1433)
				3	0.2220(0.0510)	0.6161(0.3847)	0.5503(0.3029)	0.2085(0.0449)	0.6020(0.3642)	0.5362(0.2875)
	15	8	1	2	0.1747(0.0320)	0.4452(0.1988)	0.3931(0.1546)	0.1657(0.0288)	0.3938(0.1551)	0.3622(0.1312)
				3	0.2238(0.0516)	0.7552(0.5777)	0.5548(0.3078)	0.2095(0.0453)	0.6387(0.4081)	0.5277(0.2784)
			2	2	0.1741(0.0318)	0.3878(0.1510)	0.3863(0.1493)	0.1668(0.0291)	0.3713(0.1379)	0.3608(0.1302)
				3	0.2193(0.0496)	0.6098(0.3749)	0.5427(0.2946)	0.2084(0.0448)	0.5793(0.3359)	0.5208(0.2712)
		11	1	2	0.1675(0.0295)	0.4155(0.1728)	0.3754(0.1410)	0.1648(0.0285)	0.3943(0.1555)	0.3627(0.1315)
				3	0.2139(0.0471)	0.6853(0.4705)	0.5394(0.2910)	0.2085(0.0448)	0.6379(0.4070)	0.5276(0.2783)
			2	2	0.1677(0.0296)	0.3792(0.1440)	0.3715(0.1380)	0.1645(0.0283)	0.3712(0.1379)	0.3608(0.1302)
				3	0.2104(0.0456)	0.5939(0.3537)	0.5305(0.2814)	0.2079(0.0446)	0.5784(0.3348)	0.5209(0.2714)
		10	1	2	0.1683(0.0296)	0.4317(0.1866)	0.3845(0.1479)	0.1612(0.0271)	0.3888(0.1512)	0.3609(0.1303)
				3	0.2139(0.0470)	0.7493(0.5746)	0.5562(0.3094)	0.2044(0.0429)	0.6379(0.4072)	0.5329(0.2840)
			2	2	0.1648(0.0285)	0.3944(0.1564)	0.3851(0.1486)	0.1615(0.0272)	0.3681(0.1355)	0.3564(0.1270)
				3	0.2126(0.0467)	0.6649(0.4614)	0.5576(0.3111)	0.1982(0.0404)	0.5789(0.3354)	0.5201(0.2705)

40	20	15	1	2	0.1603(0.0268)	0.4026(0.1621)	0.3686(0.1359)	0.1605(0.0269)	0.3889(0.1513)	0.3610(0.1303)
				3	0.2055(0.0434)	0.6716(0.4521)	0.5411(0.2928)	0.2031(0.0424)	0.6380(0.4073)	0.5329(0.2840)
			2	2	0.1652(0.0285)	0.3830(0.1469)	0.3713(0.1379)	0.1604(0.0269)	0.3678(0.1353)	0.3563(0.1270)
				3	0.2065(0.0439)	0.6079(0.3708)	0.5335(0.2847)	0.1978(0.0402)	0.5799(0.3365)	0.5202(0.2707)
	30	15	1	2	0.1664(0.0289)	0.4109(0.1689)	0.3730(0.1392)	0.1574(0.0259)	0.3807(0.1450)	0.3562(0.1269)
				3	0.2080(0.0445)	0.6924(0.4817)	0.5458(0.2979)	0.2000(0.0411)	0.6245(0.3902)	0.5293(0.2801)
			2	2	0.1643(0.0282)	0.3859(0.1491)	0.3760(0.1414)	0.1578(0.0260)	0.3650(0.1333)	0.3535(0.1250)
				3	0.2059(0.0436)	0.6149(0.3800)	0.5377(0.2892)	0.1959(0.0394)	0.5738(0.3294)	0.5175(0.2678)
		23	1	2	0.1599(0.0266)	0.3729(0.1391)	0.3517(0.1237)	0.1581(0.0260)	0.3662(0.1341)	0.3477(0.1209)
				3	0.1989(0.0406)	0.6024(0.3630)	0.5228(0.2734)	0.1963(0.0395)	0.6018(0.3622)	0.5228(0.2733)
			2	2	0.1575(0.0259)	0.3595(0.1293)	0.3486(0.1216)	0.1595(0.0265)	0.3595(0.1292)	0.3487(0.1216)
				3	0.1938(0.0386)	0.5630(0.3170)	0.5127(0.2629)	0.1944(0.0388)	0.5629(0.3169)	0.5125(0.2627)
	30	15	1	2	0.1607(0.0269)	0.3912(0.1531)	0.3637(0.1323)	0.1569(0.0256)	0.3720(0.1384)	0.3526(0.1243)
				3	0.2013(0.0415)	0.6568(0.4324)	0.5423(0.2941)	0.1976(0.0400)	0.6054(0.3666)	0.5280(0.2788)
			2	2	0.1603(0.0268)	0.3786(0.1435)	0.3665(0.1344)	0.1531(0.0244)	0.3584(0.1284)	0.3482(0.1213)
				3	0.2001(0.0411)	0.6091(0.3724)	0.5326(0.2837)	0.1910(0.0375)	0.5643(0.3185)	0.5150(0.2652)
		23	1	2	0.1604(0.0268)	0.3910(0.1529)	0.3636(0.1322)	0.1580(0.0259)	0.3719(0.1383)	0.3525(0.1243)
				3	0.2034(0.0424)	0.6602(0.4371)	0.5426(0.2944)	0.1963(0.0395)	0.6055(0.3667)	0.5281(0.2788)
			2	2	0.1588(0.0263)	0.3749(0.1406)	0.3629(0.1317)	0.1560(0.0253)	0.3582(0.1283)	0.3481(0.1212)
				3	0.1958(0.0394)	0.6020(0.3633)	0.5300(0.2809)	0.1909(0.0374)	0.5643(0.3186)	0.5148(0.2651)
60	45	23	1	2	0.1604(0.0267)	0.3931(0.1545)	0.3647(0.1330)	0.1504(0.0235)	0.3563(0.1270)	0.3427(0.1175)
				3	0.2025(0.0421)	0.6587(0.4356)	0.5424(0.2942)	0.1912(0.0375)	0.5727(0.3280)	0.5171(0.2674)
			2	2	0.1584(0.0261)	0.3734(0.1395)	0.3616(0.1307)	0.1514(0.0238)	0.3494(0.1221)	0.3409(0.1162)
				3	0.1978(0.0402)	0.6011(0.3622)	0.5298(0.2807)	0.1895(0.0368)	0.5456(0.2977)	0.5068(0.2569)
		34	1	2	0.1517(0.0239)	0.3599(0.1296)	0.3451(0.1191)	0.1507(0.0236)	0.3523(0.1241)	0.3401(0.1157)
				3	0.1950(0.0389)	0.5780(0.3341)	0.5191(0.2694)	0.1910(0.0374)	0.5679(0.3225)	0.5154(0.2656)
			2	2	0.1502(0.0235)	0.3510(0.1232)	0.3422(0.1171)	0.1492(0.0231)	0.3479(0.1210)	0.3397(0.1154)
				3	0.1924(0.0379)	0.5486(0.3010)	0.5082(0.2583)	0.1884(0.0364)	0.5419(0.2936)	0.5051(0.2552)

Table 4: The Average Bias (AVB) and Mean Squared Errors (MSEs) in Parentheses for the Weibull Distribution Parameter β using the Numerical and Bayes Methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4), at T=0.75 and T=1.5.

n	m	k	α	β	T=0.75			T=1.5		
					Kernel estimations	Bayes estimations		Kernel estimations	Bayes estimations	
						Gamma Prior	Kernel Prior		Gamma Prior	Kernel Prior
20	10	5	1	2	0.0350(0.0017)	0.0259(0.0010)	0.0457(0.0024)	0.0343(0.0016)	0.0351(0.0014)	0.0576(0.0035)
				3	0.0526(0.0042)	0.0982(0.0096)	0.0985(0.0097)	0.0523(0.0041)	0.0983(0.0097)	0.0985(0.0097)
			2	2	0.0403(0.0024)	0.1127(0.0130)	0.1079(0.0117)	0.0358(0.0019)	0.1181(0.0140)	0.1199(0.0144)
				3	0.0698(0.0067)	0.0543(0.0034)	0.0611(0.0037)	0.0615(0.0051)	0.0565(0.0033)	0.0697(0.0049)
		8	1	2	0.0365(0.0018)	0.0297(0.0012)	0.0509(0.0028)	0.0339(0.0016)	0.0345(0.0014)	0.0567(0.0034)
				3	0.0558(0.0045)	0.0981(0.0096)	0.0984(0.0097)	0.0539(0.0042)	0.0983(0.0097)	0.0985(0.0097)
			2	2	0.0407(0.0024)	0.1158(0.0135)	0.1148(0.0132)	0.0357(0.0019)	0.1180(0.0140)	0.1198(0.0144)
				3	0.0651(0.0058)	0.0552(0.0033)	0.0656(0.0043)	0.0591(0.0050)	0.0566(0.0033)	0.0697(0.0049)
	15	8	1	2	0.0350(0.0017)	0.0285(0.0011)	0.0516(0.0029)	0.0318(0.0014)	0.0566(0.0033)	0.0725(0.0053)
				3	0.0541(0.0043)	0.0982(0.0096)	0.0985(0.0097)	0.0465(0.0032)	0.0984(0.0097)	0.0985(0.0097)
			2	2	0.0368(0.0019)	0.1168(0.0137)	0.1175(0.0138)	0.0352(0.0018)	0.1227(0.0151)	0.1259(0.0159)
				3	0.0611(0.0052)	0.0556(0.0032)	0.0677(0.0046)	0.0594(0.0049)	0.0614(0.0038)	0.0752(0.0057)
		11	1	2	0.0331(0.0015)	0.0422(0.0019)	0.0619(0.0040)	0.0323(0.0014)	0.0559(0.0032)	0.0717(0.0052)
				3	0.0520(0.0039)	0.0983(0.0097)	0.0984(0.0097)	0.0475(0.0033)	0.0984(0.0097)	0.0985(0.0097)
			2	2	0.0349(0.0018)	0.1195(0.0143)	0.1220(0.0149)	0.0347(0.0017)	0.1225(0.0150)	0.1257(0.0158)
				3	0.0605(0.0050)	0.0580(0.0034)	0.0716(0.0051)	0.0571(0.0046)	0.0616(0.0038)	0.0751(0.0056)
	20	10	1	2	0.0318(0.0014)	0.0604(0.0040)	0.0826(0.0070)	0.0290(0.0011)	0.0825(0.0069)	0.0953(0.0091)
				3	0.0468(0.0032)	0.0977(0.0095)	0.0984(0.0097)	0.0448(0.0029)	0.0981(0.0096)	0.0985(0.0097)
			2	2	0.0352(0.0018)	0.1212(0.0148)	0.1239(0.0154)	0.0301(0.0012)	0.1282(0.0164)	0.1313(0.0172)
				3	0.0591(0.0049)	0.0552(0.0033)	0.0715(0.0051)	0.0524(0.0039)	0.0679(0.0046)	0.0822(0.0068)
		15	1	2	0.0294(0.0012)	0.0743(0.0057)	0.0905(0.0083)	0.0290(0.0011)	0.0825(0.0069)	0.0954(0.0092)
				3	0.0434(0.0028)	0.0980(0.0096)	0.0985(0.0097)	0.0446(0.0029)	0.0981(0.0096)	0.0985(0.0097)
			2	2	0.0303(0.0013)	0.1239(0.0154)	0.1272(0.0162)	0.0300(0.0012)	0.1283(0.0165)	0.1313(0.0172)

40				3	0.0560(0.0043)	0.0616(0.0039)	0.0779(0.0061)	0.0529(0.0039)	0.0677(0.0046)	0.0822(0.0068)
	30	15	1	2	0.0305(0.0013)	0.0704(0.0051)	0.0885(0.0079)	0.0287(0.0011)	0.0878(0.0078)	0.0987(0.0098)
				3	0.0437(0.0029)	0.0979(0.0096)	0.0984(0.0097)	0.0417(0.0026)	0.0981(0.0096)	0.0985(0.0097)
			2	2	0.0346(0.0017)	0.1234(0.0152)	0.1262(0.0159)	0.0304(0.0012)	0.1292(0.0167)	0.1321(0.0175)
				3	0.0569(0.0045)	0.0605(0.0038)	0.0767(0.0059)	0.0542(0.0040)	0.0692(0.0048)	0.0832(0.0069)
		23	1	2	0.0284(0.0011)	0.0921(0.0085)	0.1011(0.0103)	0.0274(0.0010)	0.0965(0.0094)	0.1040(0.0109)
				3	0.0433(0.0027)	0.0983(0.0097)	0.0985(0.0097)	0.0417(0.0026)	0.0983(0.0097)	0.0985(0.0097)
			2	2	0.0284(0.0011)	0.1309(0.0172)	0.1336(0.0178)	0.0294(0.0012)	0.1311(0.0172)	0.1337(0.0179)
				3	0.0526(0.0038)	0.0724(0.0052)	0.0851(0.0072)	0.0519(0.0038)	0.0724(0.0052)	0.0851(0.0072)
	30	15	1	2	0.0274(0.0010)	0.0926(0.0087)	0.1052(0.0111)	0.0269(0.0010)	0.1031(0.0107)	0.1109(0.0123)
				3	0.0419(0.0026)	0.0976(0.0095)	0.0984(0.0097)	0.0397(0.0023)	0.0980(0.0096)	0.0985(0.0097)
			2	2	0.0301(0.0012)	0.1285(0.0165)	0.1316(0.0173)	0.0281(0.0010)	0.1336(0.0179)	0.1359(0.0185)
				3	0.0526(0.0038)	0.0653(0.0043)	0.0828(0.0069)	0.0489(0.0034)	0.0757(0.0057)	0.0884(0.0078)
		23	1	2	0.0273(0.0010)	0.0930(0.0088)	0.1055(0.0112)	0.0269(0.0010)	0.1031(0.0107)	0.1109(0.0123)
				3	0.0408(0.0025)	0.0977(0.0095)	0.0984(0.0097)	0.0399(0.0023)	0.0981(0.0096)	0.0985(0.0097)
			2	2	0.0297(0.0012)	0.1294(0.0167)	0.1324(0.0175)	0.0286(0.0011)	0.1336(0.0179)	0.1359(0.0185)
				3	0.0521(0.0038)	0.0666(0.0045)	0.0835(0.0070)	0.0477(0.0032)	0.0757(0.0057)	0.0884(0.0078)
60	45	23	1	2	0.0282(0.0010)	0.0914(0.0085)	0.1045(0.0110)	0.0252(0.0009)	0.1121(0.0126)	0.1164(0.0136)
				3	0.0413(0.0025)	0.0977(0.0095)	0.0984(0.0097)	0.0393(0.0023)	0.0982(0.0096)	0.0985(0.0097)
			2	2	0.0293(0.0012)	0.1297(0.0168)	0.1327(0.0176)	0.0273(0.0010)	0.1361(0.0185)	0.1378(0.0190)
				3	0.0493(0.0035)	0.0669(0.0045)	0.0836(0.0070)	0.0475(0.0032)	0.0816(0.0067)	0.0914(0.0084)
		34	1	2	0.0261(0.0009)	0.1099(0.0121)	0.1150(0.0132)	0.0257(0.0009)	0.1139(0.0130)	0.1175(0.0138)
				3	0.0393(0.0023)	0.0981(0.0096)	0.0985(0.0097)	0.0395(0.0023)	0.0983(0.0097)	0.0985(0.0097)
			2	2	0.0268(0.0010)	0.1357(0.0184)	0.1375(0.0189)	0.0261(0.0009)	0.1366(0.0187)	0.1381(0.0191)
				3	0.0485(0.0032)	0.0806(0.0065)	0.0909(0.0083)	0.0473(0.0031)	0.0828(0.0069)	0.0920(0.0085)

Table 5: The Estimate (EST) and Mean Squared Errors (MSE) in Parentheses for the Entropy (P) of Weibull Distribution Using the kernel and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4), et T=0.75 and T=1.5.