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Joint Field Theoretical Description of The Electron and Neutrino Scattering Off Nuclei

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1. Prelude

Despite the unpleasant current situation, our institute continues to work in a remote regime, allowing us to do some explorations at a distance. Accepting this invitation to contribute to an extensive development of the theory of electroweak interactions with nuclei, let me remind of several pages from the 90s [1-4]. Then, in this century, we have extended our explorations by applying field theoretical methods. In particular, working in the late 90s on the LTP (Dubna) in the papers by M.Shirokov and me [5,6] we developed the notion of the so-called clothed particles, i.e., particles with physical properties, put forward in the QFT by Greenberg and Schweber [7,8].

2. Clothed Particle Representation (CPR) in Action

within its basic idea to remove from the total Hamiltonian H for a system of interacting fields, e.g., meson and nucleon ones, undesirable (bad) terms that prevent one-body states to be H eigenvectors, viz., in the case of the nucleon, for instance,

$$H|\vec{p};\mathrm{cloth}\rangle = E_{\vec{p}}|\vec{p};\mathrm{cloth}\rangle, \quad E_{\vec{p}} = \sqrt{m^2+\vec{p}^2} \tag{1}$$

for nucleon momentum $\vec{p}$ and mass m[1], instead of the bare particle representation (BPR), where bare one-particle states $|\vec{p}$; bare⟩ are not the H eigenstates.

In Refs. [5,6] we have seen how one can go from the division $H = H_0 + V$ to $H = K_F + K_I$ using the unitary clothing transformations (UCTs). By the way, it means that $K_I|\vec{p}$; cloth⟩ = 0.

An attractive feature of the UCT method is that it allows to build up both interaction operators responsible for physical processes between clothed particles (bosons and fermions)

$$\begin{aligned} H = K_F + K_I(ff\to ff) + K_I(\bar f\bar f\to \bar f\bar f) + K_I(f\bar f\to f\bar f) + K_I(bf\to bf) \\ + K_I(b\bar f\to b\bar f) + K_I(bb\to f\bar f) + K_I(f\bar f\to bb) + \cdots \end{aligned} \tag{2}$$

and opens a fresh look at finding the mass and charge shifts (key points in renormalization theories), see Refs. [9-12] and my talks at FB18 conference and at FB20 conference.

3. Links between in(out) and clothed particle states in QFT

As well-known, when evaluating the *S*-matrix in the Heisenberg picture,

$$S_{if} = \langle f; out \mid i; in\rangle \tag{3}$$

one has to deal with the *in*(*out*) states (see, e.g., [13]), in particular, one-particle state

$$|\vec{p};\ in\,(out)\rangle = a^{\dagger}_{in\,(out)}(\vec{p})\,|\Omega\rangle\,, \tag{4}$$

where $|\Omega\rangle$ is the physical vacuum. The creation (destruction) *in*(*out*) operators $a^{\dagger}_{in(out)}$ $\left(a_{in(out)}\right)$ meet canonical commutation relations for bosons and fermions. By definition, these states are the H eigenstates

$$H|\vec{p};in(out)\rangle = E_{\vec{p}}|\vec{p};in(out)\rangle. \tag{5}$$

Omitting important details (see also Sec. 4 of [14] and Sec. 1.4 of [15]), one can prove the relations between states in the CPR and *in*(*out*) formalism for the one particle

$$|\vec{p};in(out)\rangle \equiv a^{\dagger}_{in(out)}(\vec{p})|\Omega\rangle = a^{\dagger}_{c}(\vec{p})|\Omega\rangle, \tag{6}$$

and two particles

$$|\vec{p}_1\vec{p}_2;in\rangle \equiv a^{\dagger}_{in}(\vec{p}_1)a^{\dagger}_{in}(\vec{p}_2)|\Omega\rangle = \Omega^{(+)}_{c}a^{\dagger}_{c}(\vec{p}_1)a^{\dagger}_{c}(\vec{p}_2)|\Omega\rangle, \tag{7}$$

$$|\vec{p}_1\vec{p}_2;out\rangle \equiv a^{\dagger}_{out}(\vec{p}_1)a^{\dagger}_{out}(\vec{p}_2)|\Omega\rangle = \Omega^{(-)}_{c}a^{\dagger}_{c}(\vec{p}_1)a^{\dagger}_{c}(\vec{p}_2)|\Omega\rangle \tag{8}$$

with the Møller operators $\Omega^{(\pm)}_{c} \equiv \lim_{t\to\mp\infty} \exp(iHt)\exp(-iK_F t)$, that hold under the condition

$$\lim_{t\to\pm\infty} W_D(t) = 1, \tag{9}$$

where the UCT in the D picture $W_D(t) = e^{iH_F t}We^{-iH_F t}$ and the limit is implied in the strong sense. To some extent, relation (3.4) does not seem unexpected, since both one-particle clothed states and *in*(*out*) states, being equally normalized, are H eigenvectors. Of course, it does not mean that $a_{in(out)}(\vec{p}) = a_c(\vec{p})$!

1D. Some Deviation

For instance, in case of opposite-charged scalar particles, we have

$$\varphi_{in}\,(x) = \int d\vec{k}\left[A_{in}\left(\vec{k}\right) f_k\,(x) + B^{\dagger}_{in}\left(\vec{k}\right) f^{*}_{k}\,(x)\right]. \tag{10}$$

In fact, we can prove

$$|\vec{k};in(out)\rangle \equiv A^{\dagger}_{in(out)}(\vec{k})|\Omega\rangle = \lim_{t\to-\infty(+\infty)} A^{\dagger}_{int}(\vec{k},t)|\Omega\rangle = A^{\dagger}_{c}(\vec{k})|\Omega\rangle. \tag{11}$$

Eq. (11) helps us to do the next step when deriving Eqs. (7), (8)

$$A^{\dagger}_{in}A^{\dagger}_{in}|\Omega\rangle = A^{\dagger}_{in}A^{\dagger}_{c}|\Omega\rangle. \tag{12}$$

Of course, we remember that $A_{in} \neq A_c$. Following a common practice, we introduce the interpolating fields, these mediators between the in (incoming) ($t \to -\infty$) fields and the out (outgoing) ($t \to +\infty$) fields. Recall that for a given Heisenberg field $\varphi(x)$, the corresponding interpolating field is determined by

$$\varphi_{int}\,(x) = \int d\vec{k}\left[A_{int}\left(\vec{k},t\right) f_k\,(\vec{x}) + B^{\dagger}_{int}\left(\vec{k},t\right) f^{*}_{k}\,(\vec{x})\right], \tag{13}$$

where

$$A_{int}\left(\vec{k},t\right) = \left(f_k^*, \varphi_{int}\right), \;\; B_{int}^{\dagger}\left(\vec{k},t\right) = \left(\varphi_{int}, f_k\right) \quad \text{and} \quad f_k\left(\vec{x}\right) \equiv f_k\left(\vec{x}, t=0\right) \tag{14}$$

with respect to the definition

$$\left(f_1, f_2\right) \equiv i \int d\vec{x}\left[f_1(x)\partial_0 f_2(x) - f_2(x)\partial_0 f_1(x)\right] \tag{15}$$

and the completeness condition

$$\int d\vec{k}\, f_k^*\left(x\right) f_k\left(x'\right)|_{t=t'} = \frac{1}{2\omega_{\vec{k}}}\delta\left(\vec{x}-\vec{x}'\right), \tag{16}$$

$$f_k\left(x\right) = \left[\left(2\pi\right)^3 2\omega_{\vec{k}}\right]^{-1/2} e^{-ikx}, \quad k = (\omega_{\vec{k}}, \vec{k}), \quad \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m_b^2}, \tag{17}$$

for the so-called plane wave solutions of the Klein-Gordon equation

$$\left(\; k_x + m_b^2\right)\varphi_{in(out)}\left(x\right) = 0. \tag{18}$$

At the same time, in accordance with conventional relation between D-picture and H-picture operators

$$O\left(t\right) = \exp\left(iHt\right) O_D\left(t=0\right)\exp\left(-iHt\right), \tag{19}$$

we have

$$\varphi_D\left(\vec{x}\right) = \int d\vec{k}\left[A(\vec{k})f_k\left(\vec{x}\right) + B^{\dagger}(\vec{k})f_k^*\left(\vec{x}\right)\right], \tag{20}$$

$$\varphi\left(x\right) = \int d\vec{k}\left[A(\vec{k},t)f_k\left(\vec{x},t\right) + B^{\dagger}(\vec{k},t)f_k^*\left(\vec{x},t\right)\right]. \tag{21}$$

4. Two-Body Currents

Recall that by using the method of unitary clothing transformations, the total field Hamiltonian H and other operators of great physical meaning (e.g., the Lorentz boosts and current density operators) are expressed through commutators of generators R of UCTs $W = e^R$ with primary operators, e.g., when calculating the transition matrix elements $\langle f|J^{\mu}(0)|i\rangle$ between the initial $|i\rangle$ and final $|f\rangle$ states we will employ the Campbell–Hausdorff formula so

$$J^{\mu}(0) = e^R J_c^{\mu}(0)e^{-R} = J_c^{\mu}(0) + [R, J_c^{\mu}(0)] + \frac{1}{2}[R,[R, J_c^{\mu}(0)]] + \ldots, \tag{22}$$

with primary Noether current $J_c^{\mu}(0)^2$ in which "bare" operators $\{\alpha\}$ are replaced by the clothed partners $\{\alpha_c = W^{\dagger}\alpha W\}$.

We have proposed in [10] a recursive technique for evaluating multiple commutators that inevitably appear along our guideline. This technique has been realized in the case of interacting boson and fermion fields with Yukawa-type couplings.

At this point, let me address the field theoretical description of electron scattering on the deuteron. The corresponding amplitude is proportional to the matrix element of the current density operator sandwiched between the initial deuteron state $|i\rangle = |d\rangle$ and final two-nucleon states ($|d\rangle$ for elastic scattering and $|np\rangle$ for breakup). It is the case where our task is reduced to

$$\langle \text{two-body}|J_{\mu}(0)|d\rangle = \langle \text{two-body}|[J_{\mu}^{[1]} + J_{\mu}^{[2]}]|d\rangle \tag{23}$$

$$J^{[1]} = \sum\!\!\!\!\!\!\int d1'd1\, F(1',1)b_c^{\dagger}(1')b_c(1), \tag{24}$$

$$J^{[2]} = \sum\!\!\!\!\!\!\int d1'd2'd1d2\, F_{MEC}(1',2',1,2)b_c^{\dagger}(1')b_c^{\dagger}(2')b_c(1)b_c(2). \tag{25}$$

For brevity, the Lorentz label is omitted.

It is important to stress that the many-nucleon currents introduced in such a way do not depend on the choice of states between which we calculate the amplitude. In particular, it means that the one-nucleon matrix elements

$$\langle 1'|J^{[1]}|1\rangle = F(1',1), \tag{26}$$

where $|1\rangle = b_c^\dagger(1)|\Omega\rangle$ and the operator $J^{[1]}$ is the same as in Eq. (23), with

$$F^\alpha(p'\mu',p\mu) = e\,m\,\bar{u}(p'\mu')\left\{F_1[(p'-p)^2]\gamma^\alpha + i\sigma^{\alpha\beta}\frac{(p'-p)_\beta}{2m}F_2[(p'-p)^2]\right\}u(p\mu) \tag{27}$$

N.B. It is an attractive feature of our approach.
Schematically, this structure looks as

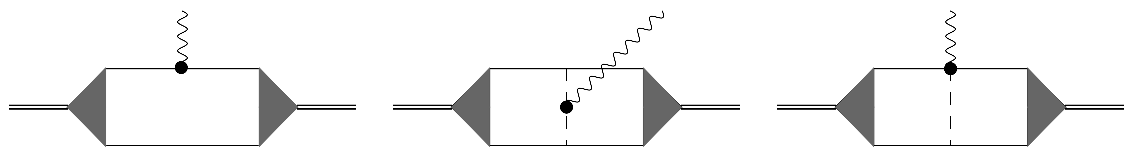


Figure 1: Different contributions to the current operator

Here, the c-functions are determined by

$$F^\mu(1',1) = e\,m\,\bar{u}(1')\left\{F_1[(p_1'-p_1)^2]\gamma^\mu + i\sigma^{\mu\nu}\frac{(p_1'-p_1)_\nu}{2m}F_2[(p_1'-p_1)^2]\right\}u(1), \tag{28}$$

with the Dirac (Pauli) form factor F_1 (F_2) and $F_{MEC} = F_{MCC} + F_{MNN}$, e.g., for π mesons, looks as

$$\begin{aligned} F^\mu_{\pi CC}(1',2',1,2) = i\frac{eg_\pi^2 m^2}{2(2\pi)^6}[\vec{\tau}_1\times\vec{\tau}_2]^z g_{11}^\pi(p_1'p_1)g_{11}^\pi(p_2'p_2)\frac{\bar{u}(1')\gamma_5 u(1)}{(p_1'-p_1)^2-m_\pi^2} \\ \times\frac{\bar{u}(2')\gamma_5 u(2)}{(p_2'-p_2)^2-m_\pi^2}(p_2'-p_1'+p_1-p_2)^\mu, \end{aligned} \tag{29}$$

$$\begin{aligned} F^\mu_{\pi NN}(1',2',1,2) = {} & \frac{eg_\pi^2 m^2}{2(2\pi)^6}g_{11}^\pi(p_2'p_2)\frac{\bar{u}(2')\gamma_5 u(2)}{(p_2'-p_2)^2-m_\pi^2} \\ & \times\Big[\tfrac{1}{2}(\vec{\tau}_1\cdot\vec{\tau}_2+\tau_2^z-i[\vec{\tau}_1\times\vec{\tau}_2]^z)g_{11}^\pi(p_1 s)\,\bar{u}(1')\gamma^\mu\Gamma(1',2',1,2)\gamma_5 u(1) \\ & +\tfrac{1}{2}(\vec{\tau}_1\cdot\vec{\tau}_2+\tau_2^z+i[\vec{\tau}_1\times\vec{\tau}_2]^z)g_{11}^\pi(p_1's')\,\bar{u}(1')\gamma_5\Gamma(1,2,1',2')\gamma^\mu u(1)\Big] \end{aligned} \tag{30}$$

$$\Gamma(1',2',1,2) = \frac{1}{2E_{\vec{s}}}\left[(\not{s}+m)\frac{E_{\vec{p}_1}-E_{\vec{p}_2}+E_{\vec{p}_2'}-E_{\vec{s}}}{(p_1-s)^2-m_\pi^2} + (\not{s}_- - m)\frac{E_{\vec{p}_1}-E_{\vec{p}_2}+E_{\vec{p}_2'}+E_{\vec{s}}}{(p_1+s_-)^2-m_\pi^2}\right], \tag{31}$$

where $s=(E_{\vec{s}},\vec{s})$, $s'=(E_{\vec{s}'},\vec{s}')$, $\vec{s}=\vec{p}_1+\vec{p}_2-\vec{p}_2'$, $\vec{s}'=\vec{p}_1'+\vec{p}_2'-\vec{p}_2$, g^π and $g_{11}^\pi(p'p)$ the corresponding cutoff factors. Henceforth, we accept the abbreviation $\not{s}=s^\mu\gamma_\mu$.

Our calculations with such currents are underway.

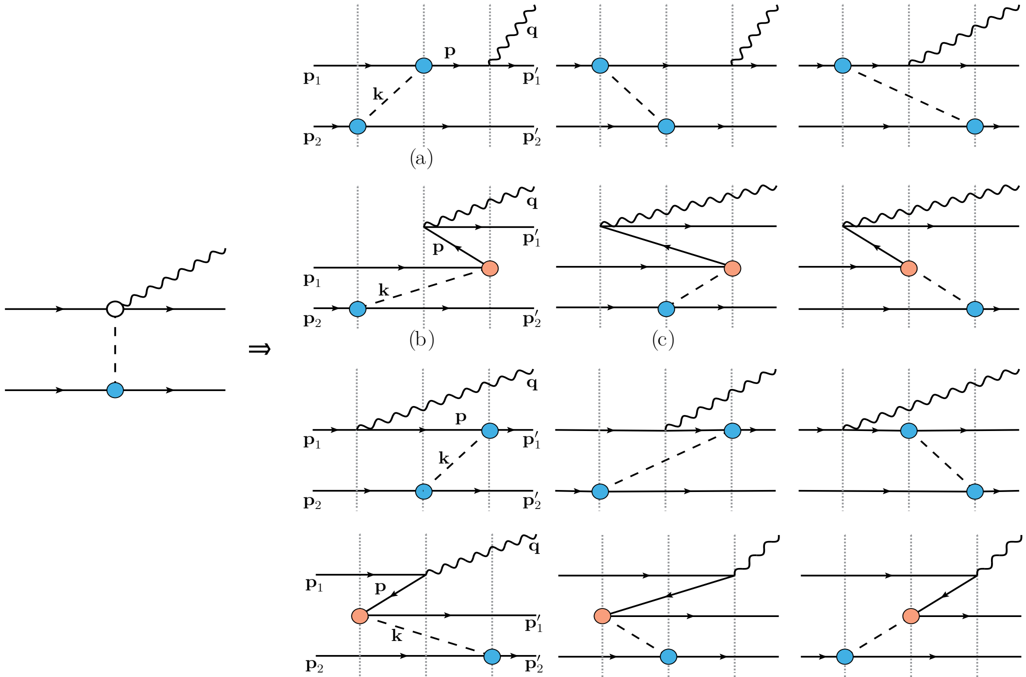


Figure 2: Illustration of the mechanisms that contribute to the seagull exchange current. Blue and orange circles correspond to the g_{11} and g_{12} cutoffs, respectively.

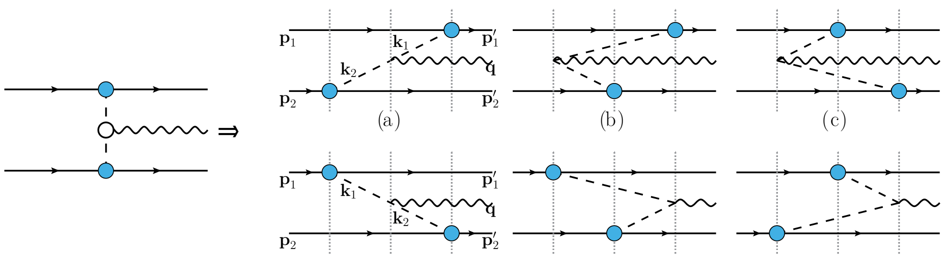


Figure 3: Illustration of the mechanisms that contribute to the mesonic meson exchange current.

Blue circles correspond to the g_{11} cutoff. These figures are taken from [15].

Many things under our consideration remain intact for describing the scattering of other leptons on few-nucleon systems.

5. Final State Interactions in Inclusive and Semi-Inclusive Processes

Special attention in our studies is paid to the effects due to interactions between reaction products in processes induced by leptons off nucleons and nuclei below and above the pion production threshold. In this context, let me recall our collaborative research on the pion photoproduction off the deuterium $d(\gamma, \pi^+)nn$ and pion electroproduction in the reaction $d(e, e'\pi^+)nn$ [16,17].

These inclusive (in final states, only pions are detected) and semi-inclusive (pions are observable with scattered electrons) reactions are typical to illustrate a general idea, viz., we rewrite the expression

$$d\sigma_{\gamma\pi d} = (2\pi)^4 \sum\!\!\!\!\!\!\int_{nn} \delta(E_\pi - E_\gamma + E_{nn} - E_d)|\Gamma_{nn}|^2 d\vec{p}_\pi, \tag{32}$$

$$\Gamma_{nn} = \langle \psi_{nn}^{(-)} | F_{\gamma\pi} | d \rangle, \tag{33}$$

where $F_{\gamma\pi}$ the corresponding transition operator, $\langle\psi_{nn}^{(-)}|$ is the H eigenvector that belongs to the energy $E_{nn} = E_{n1} + E_{n2}$ of the final nn-pair, in the form

$$d\sigma_{\gamma\pi d} = (2\pi)^4 \langle d|F_{\gamma\pi}^{\dagger}\delta(E - E_{\pi} - H)F_{\gamma\pi}|d\rangle d\vec{p}_{\pi}. \tag{34}$$

In its turn, the delta function $\delta(x - H) = -\pi^{-1}\mathrm{Im}(x + i0 - H)^{-1}$,

$$(x + i0 - H)^{-1} = g_0(x + i0) + g_0(x + i0)t_{nn}(x + i0)g_0(x + i0) \tag{35}$$

with the free resolvent $g_0(z) = (z - H_0)^{-1}$, so evaluation of our cross section reduces to plane wave contribution FSI contribution linear in t-matrix of nn-scattering.

See details in my talk at Gordon Conference on photonuclear reactions, Tilton, 1976.

6. Well-Forgotten Pages from Past to Future

First of all, we would like to recall that[3] the electroweak interaction is part of the Standard Model and based on a local $SU(2) \times U(1)$ gauge symmetry. After spontaneous symmetry breaking via the Higgs mechanism, we get for the interaction part of the Lagrangian [18]

$$\mathcal{L}_{int} = -\frac{g}{2\sqrt{2}}\left(\mathcal{J}_{\alpha}^{CC}W^{\alpha\dagger} + \text{ h.c. }\right) - \frac{g}{2\cos\theta_W}\mathcal{J}_{\alpha}^{NC}Z^{\alpha} - e\mathcal{J}_{\alpha}^{EM}A^{\alpha} \tag{36}$$

The weak charged current (CC) $\mathcal{J}_{\alpha}^{CC}$, the weak neutral current (NC) $\mathcal{J}_{\alpha}^{NC}$ and the electromagnetic current (EM) $\mathcal{J}_{\alpha}^{EM}$ couple to the charged W-boson field W^{α}, the neutral Z-boson field Z^{α} and the photon field A^{α}, respectively. The currents can be separated into a leptonic part, denoted by j_{α}, and a hadronic part J_{α}:

$$\mathcal{J}_{\alpha} = j_{\alpha} + J_{\alpha}. \tag{37}$$

Ibid. we encounter an essential simplification for further consideration where one uses the following transition from the point-like vertex to the vertex with composite particles Then, for example, the amplitude of the neutrino-nucleon scattering amplitude can be expressed in

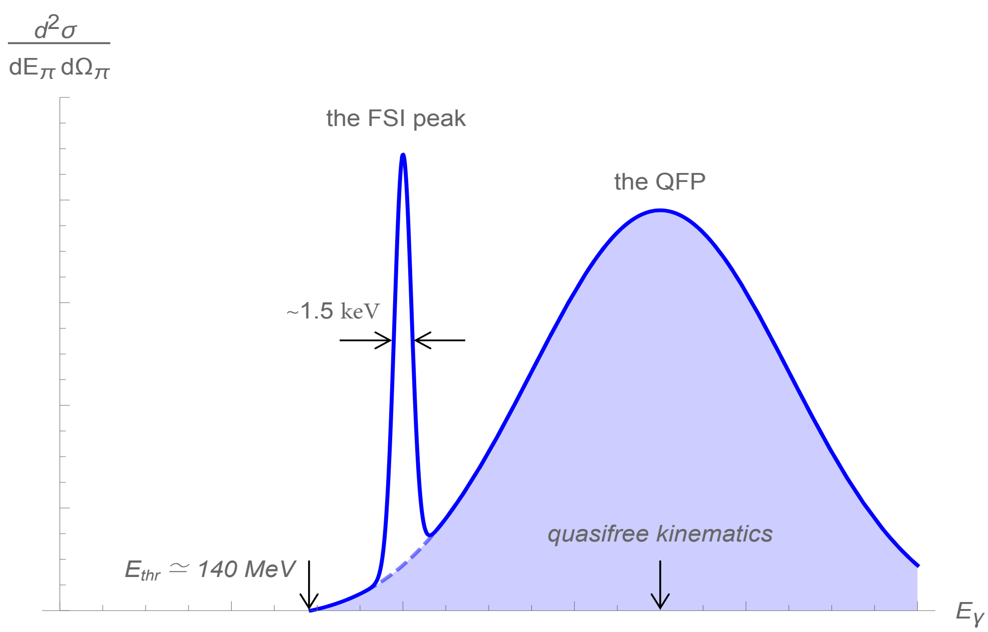
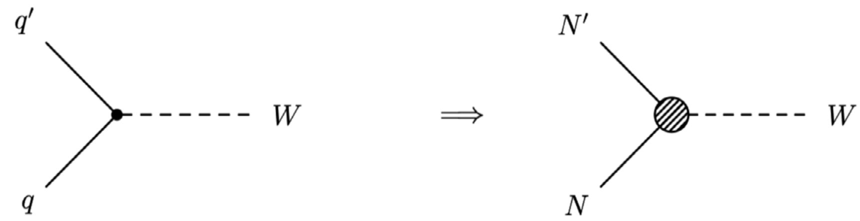


Figure 4: FSI – final state interaction peak (very sharp one separated from E_{thr} at the distance 0.075 MeV; its height is proportional to the square of the nn scattering length (a_{nn}) value and width ≈ 1.5 KeV). The distinctive feature of the quasifree peak (QFP) (in general, such a wide bump in inclusive energy spectra) is that it is centered near the energy of the reaction $\gamma + p \to \pi^{+} + n$ on free proton at rest.



terms of the matrix elements of the hadronic current density operator sandwiched between the nucleon initial state (*N*) and final system state (*X*),

$$\mathcal{M}=\left(\frac{g}{2\sqrt{2}}\right)^{2}\bar{u}_{l}\left(k'\right)\gamma_{\alpha}\left(1-\gamma_{5}\right)u_{\nu}(k)\frac{i}{q^{2}-M_{W}^{2}}\left(-g^{\alpha\beta}+\frac{q^{\alpha}q^{\beta}}{M_{W}^{2}}\right)\langle X\left(p'\right)|\,J_{\beta}(0)|N(p)\rangle. \quad (38)$$

In this context, we could address the Fock-Weyl criterion and its consequences [19]. See also the survey [5,20].

6.1. An Effective Way of Ensuring Gauge Independent Treatment of Single-Photon Processes on Nuclei. Extension of The Siegert Theorem [21]

As shown in [22] (cf. [23-25]), the photonuclear reaction amplitude of interest (to be more definite for the photon emission with energy E_γ and momentum $\vec{k}$), given in the conventional form:

$$T_{if}=\left[2(2\pi)^{3}E_{\gamma}\right]^{-1/2}\left\langle \vec{P}_{i}-\vec{k};f\left|\varepsilon^{\mu}\hat{J}_{\mu}(0)\right|\vec{P}_{i};i\right\rangle \quad (39)$$

can be expressed through the electric $(\vec{E}(\vec{k}))$ and magnetic $(\vec{H}(\vec{k}))$ field strengths:

$$\vec{E}(\vec{k})=i\left[2(2\pi)^{3}E_{\gamma}\right]^{-1/2}\left(E_{\gamma}\vec{\varepsilon}(\vec{k})-\vec{k}\varepsilon_{0}(\vec{k})\right),\quad \vec{H}(\vec{k})=i\left[2(2\pi)^{3}E_{\gamma}\right]^{-1/2}\vec{k}\times\vec{\varepsilon}(\vec{k}). \quad (40)$$

these manifestly gauge independent quantities, and the matrix elements $\vec{D}_{if}(\vec{k})$ and $\vec{M}_{if}(\vec{k})$ of the so-called generalized electric and magnetic dipole moments of the nucleus:

$$T_{if}=\vec{E}(\vec{k})\cdot\vec{D}_{if}(\vec{k})+\vec{H}(\vec{k})\cdot\vec{M}_{if}(\vec{k}). \quad (41)$$

Formulas for the matrix elements were first derived in [22] without separation of the center of mass (CM) motion, and thus they can be used in relativistic nuclear models or in problems, where such a separation becomes hardly feasible as for photomeson processes on nuclei (see [17] and Refs. therein).

6.2. Single-Photon Emission Amplitude in Terms of Electric and Magnetic Field Strengths

Using the nonrelativistic ansatz we will prove that

$$\left\langle \vec{P}_{i}+\vec{q};f\left|\varepsilon^{\mu}\hat{J}_{\mu}(0)\right|\vec{P}_{i};i\right\rangle=i\left[\vec{q}\varepsilon_{0}(q)-q_{0}\,\vec{\varepsilon}(q)\right]\vec{D}\left(q\right)-i\left[\vec{q}\times\vec{\varepsilon}(q)\right]\vec{M}(q) \quad (42)$$

with

$$\vec{D}(q)=-\frac{1}{q_{0}}\int_{0}^{1}\left(\vec{P}_{i}+\lambda\vec{q}\right|\hat{\vec{R}}[\hat{H},\hat{\rho}(0)]\left|\vec{P}_{i}\right)d\lambda, \quad (43)$$

$$\vec{M}(q)=-\int_{0}^{1}\left(\vec{P}_{i}+\lambda\vec{q}\right|\hat{\vec{R}}\times\hat{\vec{J}}\left(0\right)\left|\vec{P}_{i}\right)\lambda d\lambda. \quad (44)$$

Here, to comprise both the photon absorption and emission, the four-momentum transfer $q = (q_0, \vec{q})$ is determined with $q_0 = E_f - E_i$ $(\vec{q} = \vec{P}_f - \vec{P}_i)$ for the photoabsorption and $q_0 = E_i - E_f$ $(\vec{q} = \vec{P}_i - \vec{P}_f)$ for the photoemission.

It is the case where the l.h.s. of Eq. (42) may be written as

$$\left\langle \vec{P}_i + \vec{q}; f \left| \varepsilon^\mu \hat{J}_\mu(0) \right| \vec{P}_i; i \right\rangle = \varepsilon_0(q) \left\langle f \right| \left(\vec{P}_i + \vec{q} \right| \hat{\rho}(0) \left| \vec{P}_i \right) \left| i \right\rangle - \vec{\varepsilon}(q) \left\langle f \right| \left(\vec{P}_i + \vec{q} \right| \hat{\vec{J}}(0) \left| \vec{P}_i \right) \left| i \right\rangle , \tag{45}$$

and it is convenient to employ the representation

$$\vec{\varepsilon} e^{\vec{q} \cdot \hat{\vec{a}}} = \int_0^1 \left\{ \left[\hat{\vec{b}}, (\vec{\varepsilon} \cdot \hat{\vec{a}}) e^{\lambda \vec{q} \cdot \hat{\vec{a}}} \right] + \lambda \hat{\vec{a}} \times [\vec{\varepsilon} \times \vec{q}] \, e^{\lambda \vec{q} \cdot \hat{\vec{a}}} \right\} d\lambda \tag{46}$$

This equation with arbitrary c-vectors $\vec{\varepsilon}$ and $\vec{q}$ is valid for the two operators $\hat{\vec{a}} = (\hat{a}_1, ..., \hat{a}_n)$ and $\hat{\vec{b}} = \left(\hat{b}_1, ..., \hat{b}_n \right)$ that meet the commutation relations

$$\left[\hat{a}_j, \hat{b}_k \right] = \delta_{j,k} \qquad (j, k = 1, ..., n) \tag{47}$$

Of course, it is implied that each operator a_i and b_i is defined on an infinitely dimensional space.

With the help of (46) we get

$$\begin{aligned} \vec{\varepsilon} \cdot \left(\vec{P}_i + \vec{q} \right| \hat{\vec{J}}(0) \left| \vec{P}_i \right) = -i \int_0^1 d\lambda \left(\vec{P}_i + \lambda \vec{q} \right| \vec{\varepsilon} \cdot \hat{\vec{R}} [\hat{H}, \hat{\rho}(0)] \left| \vec{P}_i \right) \\ - i \, [\vec{q} \times \vec{\varepsilon}] \int_0^1 \lambda d\lambda \left(\vec{P}_i + \lambda \vec{q} \right| \hat{\vec{R}} \times \hat{\vec{J}}(0) \left| \vec{P}_i \right) \end{aligned} \tag{48}$$

In fact, one has

$$\left(\vec{P}_i + \vec{q} \right| = \left(\vec{P}_i \right| \exp \left(-i \vec{q} \hat{\vec{R}} \right) , \tag{49}$$

where $\hat{\vec{R}}$ is the total CM coordinate operator. Now, putting in $\hat{\vec{a}} = i \hat{\vec{R}}$, $\vec{q} = -\vec{q}$ and $\hat{\vec{b}} = -\hat{\vec{P}}$ we come to

$$\begin{aligned} \vec{\varepsilon} \cdot \left(\vec{P}_i + \vec{q} \right| \hat{\vec{J}}(0) \left| \vec{P}_i \right) = -i \int_0^1 d\lambda \left(\vec{P}_i \right| \vec{\varepsilon} \cdot \hat{\vec{R}} e^{-i\lambda \vec{q} \cdot \hat{\vec{R}}} [\hat{\vec{P}}, \hat{\vec{J}}(0)] \left| \vec{P}_i \right) \\ - i \, [\vec{q} \times \vec{\varepsilon}] \int_0^1 \lambda d\lambda \left(\vec{P}_i \right| e^{-i\lambda \vec{q} \cdot \hat{\vec{R}}} \hat{\vec{R}} \times \hat{\vec{J}}(0) \left| \vec{P}_i \right) . \end{aligned} \tag{50}$$

Formula (46) works owing to the canonical relations (cf. (47))

$$\left[\hat{R}_j, \hat{P}_k \right] = i \delta_{j,k} \qquad (j, k = 1, 2, 3) \tag{51}$$

In addition, we have accounted for the equality

$$(A|\left[\hat{A},\hat{B}\right]|A)=0 \tag{52}$$

with the matrix elements between the eigenvectors $|A)$ of a given operator $\hat{A}$ (in our case $\hat{A}=\vec{P}$). The conversion (50) is culminative in deriving Eq. (48) since it enables us to employ the continuity equation (CE) that results in (48). Further,

$$\langle f|\left(\vec{P}_i+\vec{q}\right|\hat{\rho}(0)\left|\vec{P}_i\right)|i\rangle=(E_f-E_i)^{-1}\langle f|\left(\vec{P}_i+\vec{q}\right|[\hat{H},\hat{\rho}(0)]\left|\vec{P}_i\right)|i\rangle \tag{53}$$

or

$$\langle f|\left(\vec{P}_i+\vec{q}\right|\hat{\rho}(0)\left|\vec{P}_i\right)|i\rangle=(E_f-E_i)^{-1}\langle f|\int\limits_0^1 d\lambda\frac{d}{d\lambda}\left(\vec{P}_i+\lambda\vec{q}\right|[\hat{H},\hat{\rho}(0)]\left|\vec{P}_i\right)|i\rangle\,. \tag{54}$$

Once more, the CE in combination with Eq. (52) helps us to see that

$$\left(\vec{P}_i\right|[\hat{H},\hat{\rho}(0)]\left|\vec{P}_i\right)=\left(\vec{P}_i\right|[\hat{\vec{P}},\,\hat{\vec{J}}(0)]\left|\vec{P}_i\right)=0 \tag{55}$$

At last, using the equation

$$\frac{d}{d\lambda}\left(\vec{P}_i+\lambda\vec{q}\right|=-i\vec{q}\left(\vec{P}_i+\lambda\vec{q}\right|\hat{\vec{R}} \tag{56}$$

we obtain

$$\langle f|\left(\vec{P}_i+\vec{q}\right|\hat{\rho}(0)\left|\vec{P}_i\right)|i\rangle=-i\frac{\vec{q}}{q_0}\langle f|\int\limits_0^1 d\lambda\left(\vec{P}_i+\lambda\vec{q}\right|\hat{\vec{R}}[\hat{H},\hat{\rho}(0)]\left|\vec{P}_i\right)|i\rangle \tag{57}$$

Substituting expressions (48) and (57) into the r.h.s. of Eq. (45), we arrive at Eq. (42). The representation (41) follows from (42) at $q_0=-E_\gamma$, $\vec{q}=-\vec{k}$.

6.3. Gauge Independent Expression for The Amplitude [17]

An incompleteness of the description may lead to results that are not gauge independent. To restore the gauge independence (GI) of the treatment, one often adds an extra term to the amplitude making the subtraction

$$J_\mu\to J_\mu-q_\mu\;q\cdot J/q^2. \tag{58}$$

Of course, this procedure cannot reflect the complexity of the reaction mechanisms such as, e.g., the two-body processes. Moreover, it does not affect the transverse components of the transition matrix and is not unambiguous admitting extra subtraction of an arbitrary vector X_μ such that $q \bullet X = 0$.

In our consideration, to provide the GI of calculations, we make use of the extension [22,23,25,26] of the Siegert theorem expressing the amplitude in an explicitly gauge independent way through the Fourier transforms of electric $(\vec{E}(\vec{q}))$ and magnetic $(\vec{H}(\vec{q}))$ field strengths,

$$T_{if}=\vec{E}(\vec{q})\vec{D}_{\mathrm{if}}+\vec{H}(\vec{q})\vec{M}_{\mathrm{if}} \tag{59}$$

$$\vec{E}(\vec{q})=i[2(2\pi)^3\omega]^{-\frac{1}{2}}\;(\omega\vec{\varepsilon}-\varepsilon_0\vec{q}), \tag{60}$$

$$\vec{H}(\vec{q})=i[2(2\pi)^3\omega]^{-\frac{1}{2}}\;[\vec{q}\times\vec{\varepsilon}], \tag{61}$$

with $\vec{D}_{\rm if}$ $\vec{M}_{\rm if}$ being matrix elements of generalized electric and magnetic dipole moments of the hadronic system containing the information on the nuclear dynamics.

To get representation (59), consider the expression

$$\delta(\vec{P}_i+\vec{q}-\vec{P}_f)\langle\vec{P}_f,f\mid J^\mu(0)\mid\vec{P}_i,i\rangle=(2\pi)^{-3}\int\exp(i\vec{q}\vec{s})\,j^\mu_{if}(\vec{s})\,{\rm d}\vec{s}\,, \tag{62}$$

$$j^\mu_{if}(\vec{s})\equiv\Big(\rho_{if}(\vec{s}),\vec{j}_{if}(\vec{s})\Big)=\langle\vec{P}_f,f\mid J^\mu(\vec{s})\mid\vec{P}_i,i\rangle=\langle\vec{P}_f,f\mid J^\mu(0)\mid\vec{P}_i,i\rangle\ {\rm e}^{-i(\vec{P}_f-\vec{P}_i)\vec{s}}\,. \tag{63}$$

Multiplying the space part of the matrix element (62) by an arbitrary vector $\vec{\varepsilon}(\vec{q})$ and applying the Foldy trick [26]

$$\vec{\varepsilon}{\rm e}^{i\vec{q}\vec{s}}=\int_0^1\{\nabla_{\vec{s}}(\vec{\varepsilon}\vec{s}{\rm e}^{i\lambda\vec{q}\vec{s}})\ -\ i\lambda\vec{s}\times[\vec{q}\times\vec{\varepsilon}]{\rm e}^{i\lambda\vec{q}\vec{s}}\}\ {\rm d}\lambda\,, \tag{64}$$

with help of the GI condition ${\rm div}\vec{j}_{if}(\vec{s})=-i(E_f-E_i)\rho_{if}(\vec{s})$, we get

$$\delta(\vec{P}_i+\vec{q}-\vec{P}_f)\langle\vec{P}_f,f\mid\vec{J}(0)\mid\vec{P}_i,i\rangle=i(E_f-E_i)\vec{d}_{if}(\vec{q})-i[\vec{q}\times\vec{m}_{if}(\vec{q})]\,, \tag{65}$$

$$\vec{d}_{if}(\vec{q})=(2\pi)^{-3}\int_0^1{\rm d}\lambda\int{\rm e}^{i\lambda\vec{q}\vec{s}}\vec{s}\rho_{if}(\vec{s})\ {\rm d}\vec{s}\,,\quad\vec{m}_{if}(\vec{q})=(2\pi)^{-3}\int_0^1\lambda{\rm d}\lambda\int{\rm e}^{i\lambda\vec{q}\vec{s}}\left[\vec{s}\times\vec{j}_{if}(\vec{s})\right]\ {\rm d}\vec{s}\,. \tag{66}$$

Then, due to charge conservation, one may write

$$\int i(E_f-E_i)\vec{q}\vec{d}_{if}(\vec{q})\ {\rm d}\vec{P}_i=\omega\langle\vec{P}_f,f\mid J^0(0)\mid\vec{P}_f-\vec{q},i\rangle \tag{67}$$

$$-(E_f-E_i(\vec{P}_f))\langle\vec{P}_f,f\mid J^0(0)\mid\vec{P}_f,i\rangle=\omega\langle\vec{P}_f,f\mid J^0(0)\mid\vec{P}_f-\vec{q},i\rangle \tag{68}$$

Integration of Eq. (65) over $\vec{P}_i$ with taking into account the relationship (68), results in representation (59), with quantities $\vec{D}_{if}$ and $\vec{M}_{if}$ being defined as

$$\vec{D}_{if}=-\int\frac{E_f-E_i}{\omega}\vec{d}_{if}(\vec{q})\ {\rm d}\vec{P}_i\,,\qquad\vec{M}_{if}=-\int\vec{m}_{if}(\vec{q})\ {\rm d}\vec{P}_i\,. \tag{69}$$

In case of reaction $e + d \rightarrow e' + \pi+ + n + n$, one has

$$\vec{D}_{if}=i\omega^{-1}\int_0^1\nabla_{\lambda\vec{q}}[\ (\ \omega+M_{\rm d}-\sqrt{M_{\rm d}^2+(1-\lambda)^2\vec{q}^2}\ )\ J^0_{if}(\lambda\vec{q})\ ]\ {\rm d}\lambda \tag{70}$$

$$\vec{M}_{if}=i\int_0^1\nabla_{\lambda\vec{q}}\times\vec{J}_{if}(\lambda\vec{q})\ \lambda\ {\rm d}\lambda \tag{71}$$

where $J^\nu_{if}(\lambda\vec{q})$ is obtained from $\langle\pi^+{\rm nn;\ out}\mid J^\nu(0)\mid{\rm d}\rangle$ replacing the deuteron momentum by $(1-\lambda)\vec{q}$.

Note whereas quantities $\vec{d}_{if}$ and $\vec{m}_{if}$ in Eq. (65) are singular and not proportional to the delta function expressing the momentum conservation, the representation (59) is free of singularities. Furthermore, it has been derived here (cf. [23]) without decomposition of the e.m. current into the part associated with the motion of the hadronic system as a whole and the intrinsic current and, therefore, can be employed in relativistic calculations.

This representation generates a correction term additional to the "canonical" expression, which restores the GI of the amplitude in calculations that fail to satisfy the requirement $q_\mu J^\mu_{if}(\vec{q}) = 0$. However, when this condition does hold, this correction is equal to zero automatically. In the long-wave limit, Eq. (59) provides the fulfillment of the Siegert theorem [27] for electric transitions in reactions with nonmeson channels [22,25]. For pion photoproduction on the free nucleon at threshold, it leads (as shown in [28]) to the Kroll-Ruderman result [29] emerging here as a particular case of the Siegert theorem.

7. To conclude

Accent on joint description of the electron and neutrino scattering off atomic nuclei for this exposition is not accidental since from the physics point of view they have much in common. In fact, for the both cases the so-called one-boson-exchange approximation (OBEA) works well with photons and W-Z bosons as interaction mediators, respectively. At the beginning, it opens a comparatively simple way to studying different properties of nuclear structure. However, many-particle reaction mechanisms (in particular, via the meson exchange currents) complicate such a consideration so understanding of their role gets some priority. We have shown a number of fresh analytical tools when constructing the e.m. and weak current density operators. We foresee good prospects for applications of our approach in the theory of electroweak interactions
with nuclei as a whole.

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