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Research Article

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Is the Field of Real Numbers Really Complete?

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Abstract

The definition of a limit can only be applied to real numbers rather than infinity, and infinity is independent, necessary and important, so the definition of the limit is incomplete. Based on the incomplete definition of the limit, we can rigorously conclude that the field of real numbers is complete. However, from the point of view that the set of real numbers and the open interval (0, 1) are topologically equivalent but the (0, 1) interval is not a complete space, the completeness of the field of real numbers is inconsistent. We can give a complete definition of a limit through revision. According to the revised definitions, we can rigorously deduce that the field of real numbers is incomplete.

Keywords: Limit, Cauchy Sequence, Complete, Infinity, Incomplete

1. Introduction

When you open a math textbook, it says that the field of real numbers is complete [1]. A number field is complete if every Cauchy sequence in the field converges to an element of the field. A Cauchy sequence is a sequence whose terms become very close to each other as the sequence progresses. Formally, the sequence {Sn} is a Cauchy sequence if for every ε>0 there is an N>0 such that |Sn-Sm|<ε for n, m>N.

A sequence is said to be convergent if it approaches some limit. Formally, a sequence {Sn} converges to the limit S if for any ε>0 there exists an N such that |Sn-S|<ε for n>N. If {Sn} does not converge, it is said to diverge. Every bounded monotonic sequence converges. Every unbounded sequence diverges.

Every convergent sequence is a Cauchy sequence. The converse is not necessarily true. However, the converse is true for real numbers, and this means that the field of real numbers is complete.

The function f(x) = arctg(x)/π + 1/2 is an one-to-one correspondence from the set of real numbers to the open interval (0, 1), so the set of real numbers and the (0, 1) interval are topologically equivalent. However, the (0, 1) interval is not a complete space, because there is a sequence {1/n} in it, which is a Cauchy sequence, however does not converge in the (0, 1) interval — its 'limit', number 0, does not belong to the (0, 1) interval. So the conclusion that the field of real numbers is complete makes people feel very inconsistent.

2. The E-N Definition of a Limit Is Incomplete

The set of the sequence $\{e^{n}\}$ and the set of the sequence $\{e^{-n}\}$ are exactly the same when n is an integer. However, the monotonically decreasing sequence $\{e^{-n}\}$ is mathematically considered to converge to 0, while the monotonically increasing sequence $\{e^{n}\}$ tends to infinity and is mathematically considered to be divergent. The convergence of $\{e^{-n}\}$ and the divergence of $\{e^{n}\}$ are considered to be different, but the two sets formed by these two monotone sequences are exactly the same, which means that they seem to be the same again.

The definition of a limit in textbooks involves an arbitrarily small open neighborhood of the limit. The common perception is that infinity plus or minus any number is still infinity. Which means that an arbitrarily small open neighborhood of infinity is infinity according to the existing definition. However, each term in any sequence of numbers is a real number, so the definition of this limit can only be applied to real numbers rather than infinity.

The two endpoints, 0 and 1, are independent of the (0,1) interval, so it is incomplete to look at the two endpoints from the perspective of the (0,1) interval. Because the set of real numbers and the (0,1) interval are topologically equivalent, the two points 0 and 1 seem to correspond to infinity. Through the Pythagorean theorem, we can prove the existence of √2. From the perspective of decimal representation, √2 can be represented as an infinite series, and the existence of √2 also implies the existence of infinity. There are no points for plus or minus infinity on the real number line. Therefore, from the perspective of the set of real numbers, it is also incomplete to look at infinity.

The convergence process and the process that tends to infinity are both limiting processes, and there should be no essential difference. The convergence process and the process that tends to infinity can be regarded as the classification of limiting processes with definite limits. However, the definition of a limit can only be applied to real numbers rather than infinity, and infinity is independent, so the definition of a limit is incomplete.

It is easy to feel that time flows without end in one direction. From a time perspective, it is easy to find a time series that monotonically expands to infinity. Because of its unbounded nature, this time series does not conform to either the definition of a limit in textbooks or the definition of a Cauchy sequence. However, this time series reflects the necessity and importance of infinity, and the definition in textbooks omits these things. Therefore, the definition in textbooks does not reflect the connotation of completeness.

Based on the incomplete definition of a limit in textbooks, we can rigorously conclude that the field of real numbers is complete. However, from the point of view that the set of real numbers and the (0, 1) interval are topologically equivalent but the (0, 1) interval is not a complete space, the completeness of the field of real numbers is inconsistent. The problem is that the definition of a limit in textbooks is not complete, but we can give a complete definition of a limit through revision.

3. Revised Definitions

A sequence is said to be convergent if it approaches some limit. If a limit is a real number, then the definition of the limit is the ε-N definition of the limit in textbooks. The corresponding definition of a Cauchy sequence is the definition in textbooks. If a limit is unbounded, that is, infinity, then the limit is defined as follows:
the limit of a sequence $\{S_n\}$ tends to infinity (or negative infinity) if for any $\varepsilon>0$ there exists an N such that $S_n > \varepsilon$ (or $S_n < -\varepsilon$) for $n>N$.

The corresponding definition of a Cauchy sequence is defined as follows:

The sequence $\{S_n\}$ is a Cauchy sequence if for every $\varepsilon>0$ there is an $N>0$ such that $S_n > \varepsilon$ and $S_m > \varepsilon$ (or $S_n < -\varepsilon$ and $S_m < -\varepsilon$) for $n,m>N$.

A number field is complete if every Cauchy sequence in the field converges to an element of the field.

4. Results

According to the above revised definitions, we can rigorously deduce that a sequence with a definite limit is equivalent to a Cauchy sequence. However, for a sequence whose limit goes to infinity, although this sequence is also a Cauchy sequence, because infinity is not on the real number line, the field of real numbers is incomplete.

References

1. Tom M. Apostol, (1974). Mathematical Analysis (2nd edition), Published by Pearson January 1.