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Review Article

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How to Generate the Hyper Exponential Functions

Uchida Keitaroh

Applied Mathematics Department

***Corresponding author**
Uchida Keitaroh, Applied Mathematics Department, Japan, E-mail: keitaroh_uchida@eco.ocn.ne.jp

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The method using repetitive integrals

The Hyper exponential functions of n-order can be generated using repetitive integrals.

$$x \in R, y \in R$$

$$k_0(x;.,n,j) = seed(x;j) = \frac{x^j}{j!} \quad (j = 0, 1, 2, 3 \dots n-1)$$

$$k_{i+1}(x;f,n,j) = \int_0^x \int_0^x \dots \int_0^x f(x)\, k_i(x;f,n,j)\, dx^n$$

$$Exph_j^n(x;f(x)) = \sum_{i=0}^{\infty} k_i(x;f,n,j)$$

$$y = Exph_j^n(x;f(x))$$

$$\frac{d^n y}{dx^n} = f(x)y$$

Uniform convergence in the wider sence

The following is the proof of uniform convergence in the wider sence of the Hyper exponential functions of n-order:

$$k_0(x;.,n,j) = seed(x;j) = \frac{x^j}{j!} \quad (j = 0, 1, 2, 3 \dots n-1),$$

$$|k_1(x;f,n,j)| = \left| \int_0^x \int_0^x \dots \int_0^x f(x) k_0(x;.,n,j) dx^n \right| \le M \left| \frac{x^{n+j}}{(n+j)!} \right|.$$

Similary,

$$|k_2(x;f,n,j)| = \left| \int_0^x \int_0^x \dots \int_0^x f(x) k_1(x;f,n,j) dx^n \right| \le M^2 \left| \frac{x^{2n+j}}{(2n+j)!} \right|.$$

In general,

$$|k_i(x;f,n,j)| \le M^i \left| \frac{x^{in+j}}{(in+j)!} \right|.$$

We thus obtain the following results:

$$\sum_{i=0}^{\infty} |k_i(x;f,n,j)| \le \left| \frac{x^j}{j!} \right| + M \left| \frac{x^{n+j}}{(n+j)!} \right| + M^2 \left| \frac{x^{2n+j}}{(2n+j)!} \right| + \dots + M^i \left| \frac{x^{in+j}}{(in+j)!} \right| + \dots$$
$$= M^{-\frac{j}{n}} \left[M^{\frac{j}{n}} \left| \frac{x^j}{j!} \right| + M^{\frac{n+j}{n}} \left| \frac{x^{n+j}}{(n+j)!} \right| + M^{\frac{2n+j}{n}} \left| \frac{x^{2n+j}}{(2n+j)!} \right| + \dots + M^{\frac{in+j}{n}} \left| \frac{x^{in+j}}{(in+j)!} \right| + \dots \right]$$
$$\le M^{-\frac{j}{n}} Exp \left[M^{\frac{1}{n}} |x| \right].$$

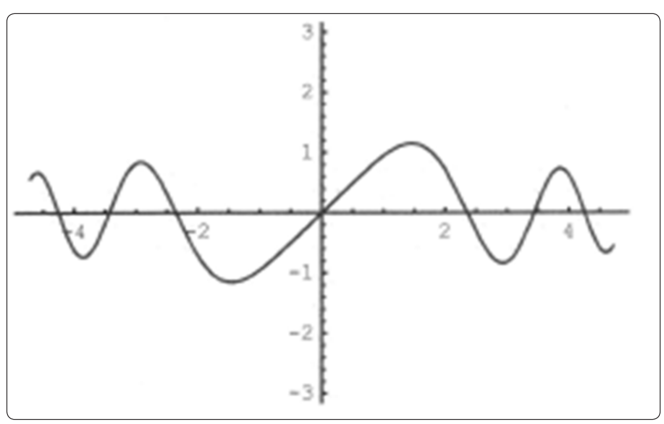
This proof is based on the majorant theorem of Weierstrass

The following is a sample of the program using mathematica

$$n = 14;$$
$$seed = k(0) = x;$$
$$f = -x^2;$$
$$\mathrm{Do}\left(k(i+1) = \int_0^x \int_0^x f\, k(i)\, dx\, dx,\ \{i, 0, n\}\right)$$
$$\mathrm{exph2}(x\_) = \sum_{i=0}^{n} k(i);$$
$$\mathrm{Print}(\mathrm{exph2}(x\_));$$

$$\mathrm{Plot}(\mathrm{exph2}(x), \{x, -1.5\pi, 1.5\pi\}, \mathrm{PlotRange} \to \{-\pi, \pi\})$$

$$x\_ - \frac{x\_^5}{20} + \frac{x\_^9}{1440} - \frac{x\_^{13}}{224\,640} + \frac{x\_^{17}}{61\,102\,080} - \frac{x\_^{21}}{25\,662\,873\,600} + \frac{x\_^{25}}{15\,397\,724\,160\,000} - \frac{x\_^{29}}{12\,502\,952\,017\,920\,000} + \frac{x\_^{33}}{13\,203\,117\,330\,923\,520\,000} - \frac{x\_^{37}}{17\,586\,552\,284\,790\,128\,640\,000} + \frac{x\_^{41}}{28\,841\,945\,747\,055\,810\,969\,600\,000} - \frac{x\_^{45}}{57\,107\,052\,579\,170\,505\,719\,808\,000\,000} + \frac{x\_^{49}}{134\,315\,787\,666\,209\,029\,452\,988\,416\,000\,000} - \frac{x\_^{53}}{370\,174\,310\,808\,072\,085\,172\,436\,074\,496\,000\,000} + \frac{x\_^{57}}{1\,181\,596\,400\,099\,366\,095\,870\,415\,949\,791\,232\,000\,000}$$



linear independence

The following is the proof that the Hyper exponential functions of n-order are linearly independent one another:

$$p_j(x) = Exph_j^n(x; f(x)) \ (j = 0, 1, 2 \ldots n-1),$$

$$H_n(x) = \begin{vmatrix} p_0 & p_1 & p_2 & \cdots & p_{n-1} \\ p_0' & p_1' & p_2' & \cdots & p_{n-1}' \\ p_0'' & p_1'' & p_2'' & \cdots & p_{n-1}'' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_0^{(n-1)} & p_1^{(n-1)} & p_2^{(n-1)} & & p_{n-1}^{(n-1)} \end{vmatrix}.$$

$$\frac{dH_n}{dx} = \begin{vmatrix} p_0' & p_1' & p_2' & \cdots & p_{n-1}' \\ p_0' & p_1' & p_2' & \cdots & p_{n-1}' \\ p_0'' & p_1'' & p_2'' & \cdots & p_{n-1}'' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_0^{(n-1)} & p_1^{(n-1)} & p_2^{(n-1)} & \cdots & p_{n-1}^{(n-1)} \end{vmatrix} + \begin{vmatrix} p_0 & p_1 & p_2 & \cdots & p_{n-1} \\ p_0'' & p_1'' & p_2'' & \cdots & p_{n-1}'' \\ p_0'' & p_1'' & p_2'' & \cdots & p_{n-1}'' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_0^{(n-1)} & p_1^{(n-1)} & p_2^{(n-1)} & \cdots & p_{n-1}^{(n-1)} \end{vmatrix} + \cdots + f(x) \begin{vmatrix} p_0 & p_1 & p_2 & \cdots & p_{n-1} \\ p_0' & p_1' & p_2' & \cdots & p_{n-1}' \\ p_0'' & p_1'' & p_2'' & \cdots & p_{n-1}'' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_0 & p_1 & p_2 & \cdots & p_{n-1} \end{vmatrix} \equiv 0.$$

$\therefore$

$H_n(x)$ is constant

The first term of all functions of the diagonal element of the matrix H_n is 1. And the other elements of the determinant H_n do not have constants. Therefore, $\boldsymbol{H_n(0)=1}$.

Thus we see that the Wronskian of the Hyperexponential functions of n-order is 1.

Therefore,we obtain the conclusion that the Hyper exponential functions of n-order are linearly independent one another.

Recent Publications:

1. Kumahara K, Saitoh S, Uchida K(2009) Normal solutions of linear ordinary differential equations of the second order,International Journal of Applied Mathematics,Volume 22 No.6 2009, 981-996.
2. Uchida K(2017) Introduction to Hyper exponential function and differential equation revised first edition, eBookland. (In Japanese). In this book, the method to generate the hyper exponential functions is described concretely.
3. Uchida K(2018) Hyper Exponential Function, Advances in Theoretical & Computational Physics